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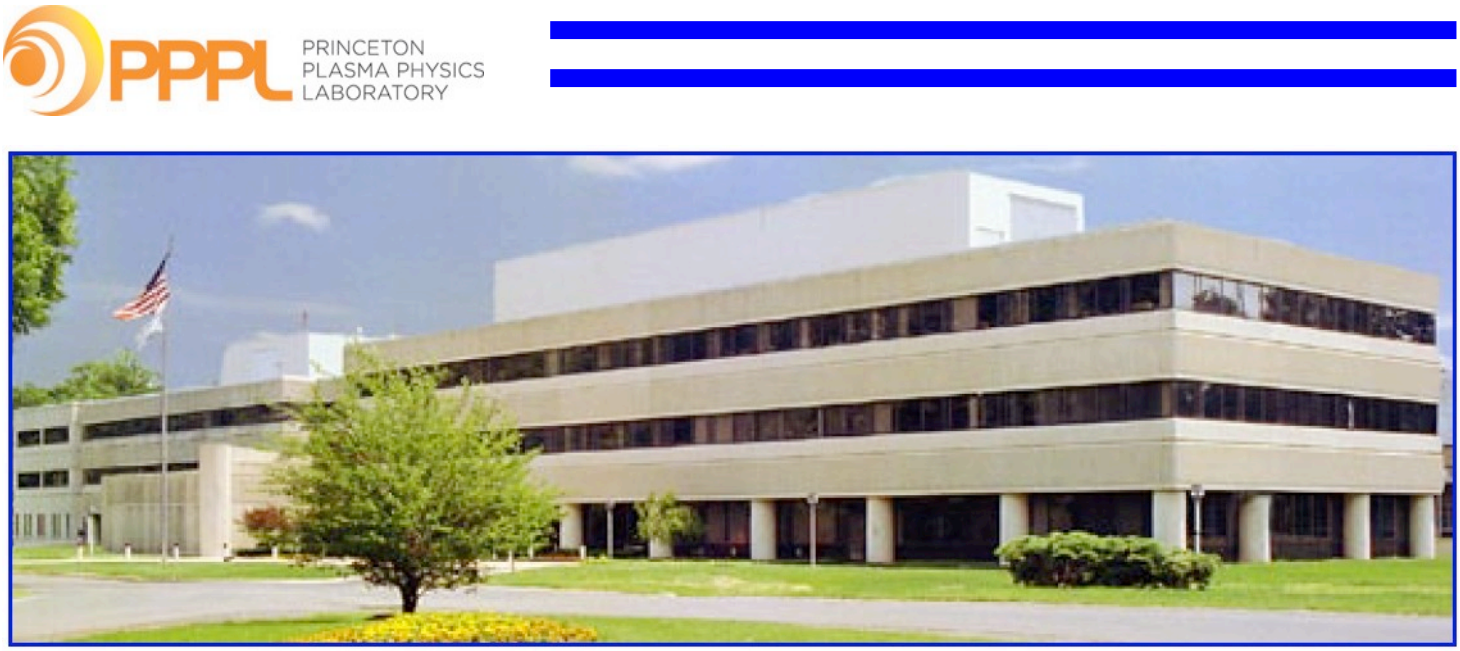

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# Self-consistent perturbed equilibrium with neoclassical toroidal torque in tokamaks

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## Abstract

Toroidal torque is one of the most important consequences of non-axisymmetric fields in tokamaks. The well-known neoclassical toroidal viscosity (NTV) is the second-order toroidal force by the anisotropic pressure tensor in the presence of these asymmetries. This work shows that the first-order force originating from the same anisotropic pressure tensor, despite having no flux surface average, can significantly modify the local perturbed force balance and thus must be included in perturbed equilibrium self-consistent with NTV. The force operator with an anisotropic pressure tensor is not self-adjoint when the second-order torque is finite, and thus is solved directly for each component. This approach yields a modified, non-self-adjoint Euler-Lagrange equation, that can be solved using a variety of common drift-kinetic models in generalized tokamak geometry. The resulting energy and torque integral provides a unique way to construct a torque response matrix, which contains all the information of self-consistent NTV torque profiles obtainable by applying non-axisymmetric fields to the plasma. This torque response matrix can then be used to systematically optimize non-axisymmetric field distributions for desired NTV profiles.

## I. INTRODUCTION

The calculation of perturbed equilibrium is an efficient approach to understanding the effects of non-axisymmetric magnetic perturbations in tokamaks, as the non-axisymmetric magnetic field $\delta\vec{B}$ is typically much smaller than the axisymmetric magnetic field $\vec{B}$. Although $|\delta\vec{B}/\vec{B}|$ is small, the effects of the perturbation can be significant to various channels in transport and consequently stability, from microscopic to macroscopic scales. One example, which is particularly relevant to this paper, is the non-ambipolar particle transport across non-axisymmetric magnetic flux surfaces that creates a toroidal torque.

Non-ambipolar transport on non-axisymmetric magnetic surfaces can be understood within the scope of neoclassical theory [1]. The resulting toroidal torque is pronounced in tokamaks, and known as neoclassical toroidal viscosity (NTV) [2–4]. The NTV torque comes from the toroidal co-variant component of anisotropic pressure tensor $\vec{e}_{\varphi_m}\cdot(\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}})$, where $\vec{e}_{\varphi_m}\equiv\partial\vec{x}/d\varphi_m$ and the toroidal angle $\varphi_m$ is defined on true magnetic surfaces including non-axisymmetric perturbations, i.e. in the Lagrangian frame, which is different from $\vec{e}_{\varphi}\cdot(\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}})$ where $\varphi$ is defined with unperturbed magnetic surfaces, i.e. in the Eulerian frame. This is an important distinction, as the prevailing transport theory is formulated in the Lagrangian frame but the information required to evaluate transport, such as $\delta\vec{B}$ or perturbed distribution function $\delta f$ is primarily calculated in the Eulerian frame.

Note that the NTV torque, $\vec{e}_{\varphi_m}\cdot(\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}})$, is second order in the perturbation when flux surface averaged, but that $\vec{e}_{\varphi}\cdot(\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}})$ is a first-order toroidal torque locally arising from anisotropic pressure tensor. There is no net torque in the first order, but its modulation amplitude is of course greater than the amplitudes of second-order quantities. It is a natural question to ask whether or not this first-order torque is important in equilibrium force balance when the second-order NTV torque is substantial. When both calculations are done together, perturbed equilibrium is consistent with the toroidal torque and the NTV torque is self-consistent up to the second order. This self-consistent torque is the main subject of this paper, and the complete formulation will be given in detail.

The calculation of perturbed equilibrium including $\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}$ generally requires solving each vector component in force balance directly, rather than the variational method adopted for ideal or kinetic energy principles. This is because the force operator is not self-adjoint any more when NTV torque is finite, that is, $\langle\vec{e}_{\varphi_m}\cdot(\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}})\rangle\neq 0$, where $\langle\ \rangle$ is the

flux-surface-average. If the force operator is self-adjoint, one can see easily that $\delta(\delta W) = (1/2)\int \left(\delta\vec{\xi}\cdot\vec{F}[\vec{\xi}] + \vec{\xi}\cdot\vec{F}[\delta\vec{\xi}]\right) = \int \delta\vec{\xi}\cdot\vec{F}[\vec{\xi}]$ in the zero-frequency limit and thus $\vec{F}[\vec{\xi}] = 0$ is the only solution for extremum $\delta(\delta W) = 0$ with arbitrary variation of displacements $\delta\vec{\xi}$. That is, the minimum energy state $\delta W$ in the zero-frequency limit becomes equivalent to the state of perturbed equilibrium [5].

The variational method for $\delta W$ was used by Newcomb to derive the cylindrical Newcomb equation in ideal MHD [6], which was then generalized by Glasser for axisymmetric tokamak geometry [7]. Various kinetic energy principles were also discussed with the variational method with the assumption that $\langle \vec{e}_{\varphi_m} \cdot (\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}})\rangle = 0$. This work will show the toroidal generalization of those kinetic energy principles with the explicit form of self-adjoint Euler-Lagrange equation giving minimized energy $\delta W$. It will also present a derivation of a new non-self-adjoint Euler-Lagrange equation giving perturbed energy and torque, in general when $\langle \vec{e}_{\varphi_m} \cdot (\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}})\rangle \neq 0$, by solving the force balance equation directly and performing the corresponding energy and torque integral. Strictly speaking, the force balance equation is not an Euler-Lagrange equation of a variational method when the force operator is not self-adjoint. This is just a convenient terminology adopted here to emphasize the similarity in appearance. It is important to note that the non-self-adjoint Euler-Lagrange equation does not give a minimum state of energy and the corresponding $\delta W$ is not a direct indicator of stability, despite accurately describing the energy of the perturbed equilibrium state.

The approach taken in this work is unique, but the MARS-K code [8] can also produce kinetic perturbed equilibrium solutions in the zero-frequency limit if the same kinetic model for $\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}$ is adopted. Both approaches are linear and use the single-fluid description with quasi-neutrality and a simplified Ohm's law $\vec{E} + \vec{v}\times\vec{B} = 0$. The derivation of the non-self-adjoint Euler-Lagrange equation in this work requires the analytic elimination of two vector components in force balance and thus is algebraically more intensive, and may not be easily extended when higher order physics are added. This is a disadvantage compared to MARS-K, but the greatest advantage is that this approach provides the full eigenmode structure of the solutions. It will be shown that the integration of the derived Euler-Lagrange equation leaves a non-Hermitian plasma response matrix, the anti-Hermitian part of which is a NTV torque response matrix. A torque response matrix describes all the self-consistent NTV profiles that can be produced by external 3D fields given an initial equilibrium. It thus enables the systematic optimization of NTV torque and, more generally, the optimization

of neoclassical 3D transport in perturbed tokamaks.

Note that the Ohm's law without resistivity maintains nested flux surfaces, which is an important assumption in most neoclassical models for $\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}$. The work presented here therefore excludes reconnected magnetic islands when a non-axisymmetric perturbation is applied, the inclusion of which remains for future work. In fact, the nested flux surface assumption generates a well-known problem in NTV calculations unless $\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}$ is self-consistently included. If only the isotropic pressure, $\vec{\nabla}\delta p$, is included in the perturbed equilibrium calculation as in the perturbative approach, the energy and NTV torque arising from $\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}$ becomes singular and non-integrable at the resonant surfaces. Although some of the special treatments across the resonant layer have been successful with perturbative approaches in reproducing experimentally observed kinetic stability [9] and NTV [10, 11], the self-consistent approach enables the elimination of those ad-hoc corrections.

The remaining paper is organized as follows. Section II will give the basic aspects of tensor pressure equilibrium, as well as the assumptions made in this work for the equations of perturbed equilibrium including $\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}$. In the Sec. III, it will be shown that the equations for ideal perturbed equilibrium can be derived by directly solving three components of perturbed force balance, leading to an identical toroidal Euler-Lagrange equation obtained by the $\delta W$ minimization originally used by Glasser. The non-self-adjoint Euler-Lagrange equation will be derived in Sec. IV, and the non-Hermitian plasma response matrix including NTV torque will be discussed in Sec. V. Sec. VI will briefly show how to incorporate an arbitrary driven force in the balance, which is followed by summary and discussions in Sec. VII.

## II. FORCE BALANCE WITH TENSOR PRESSURE

Equilibrium force balance in a single fluid description can be represented in tensor form by

$$\vec{\nabla}\cdot\overleftrightarrow{T} = \vec{\nabla}\cdot\overleftrightarrow{\mathcal{P}}, \tag{1}$$

where $\overleftrightarrow{T} = \vec{B}\vec{B} - B^2\overleftrightarrow{I}/2$ with quasi-neutrality, and also simply $\vec{\nabla}\cdot\overleftrightarrow{T} = \vec{j}\times\vec{B}$ by $\vec{j} = \vec{\nabla}\times\vec{B}$ in the equilibrium. Note $\mu_0 = 1$ for convenience, throughout this paper. The particle stress tensor has a diagonal form as $\overleftrightarrow{\mathcal{P}} = p_\perp\overleftrightarrow{I} + (p_\parallel - p_\perp)\hat{b}\hat{b}$ with $\hat{b} \equiv \vec{B}/B$, assuming the gyroradius is small enough $\rho/L \ll 1$. It is also assumed that the mean speed of fluid is generally small compared to the random velocity of particles, i.e., $|\vec{u}|^2 \ll |\vec{v}|^2$. Before introducing and

separating the non-axisymmetric component as a perturbation, it is worthwhile to take a brief look at the three components of Eq. (1):

$$\vec{B} \cdot (\vec{\nabla} \cdot \overleftrightarrow{\mathcal{P}}) = 0, \tag{2}$$

$$\vec{e}_{\varphi_m} \cdot (\vec{\nabla} \cdot \overleftrightarrow{\mathcal{P}}) = \vec{j} \cdot \vec{\nabla}\psi_p, \tag{3}$$

$$\vec{\nabla}_\perp \left( p_\perp + \frac{B^2}{2} \right) = \vec{\kappa} \left( B^2 + p_\perp - p_\parallel \right). \tag{4}$$

Here $\vec{\kappa} = \hat{b} \cdot \vec{\nabla}\hat{b}$ is the curvature vector. The decomposition is in a magnetic coordinate system $(\psi_m, \theta_m, \varphi_m)$, with the magnetic field $\vec{B} = \chi'(\vec{\nabla}\varphi_m \times \vec{\nabla}\psi_m + q\vec{\nabla}\psi_m \times \vec{\nabla}\theta_m)$, where $q$ is the safety factor, $\chi \equiv \psi_p$ is the poloidal flux and $\chi' \equiv d\chi/d\psi_m$.

The first two equations (2) and (3) describe the force balance in surface, and can be obtained by taking the parallel and toroidal component of Eq. (1). It is interesting to compare these two relations with the neoclassical friction-flux relations [2],

$$\sum_s \left\langle \vec{B} \cdot (\vec{\nabla} \cdot \overleftrightarrow{\mathcal{P}}_s) \right\rangle = 0, \tag{5}$$

$$\left\langle \vec{e}_{\varphi_m} \cdot (\vec{\nabla} \cdot \overleftrightarrow{\mathcal{P}}_s) \right\rangle = q_s \Gamma^s_{NA}. \tag{6}$$

Eq. (5) means that the net parallel viscosity should vanish when summed over the two species $s$, for ions and electrons. The parallel force balance in Eq. (2) similarly means that the force along the field lines must vanish at every point of space, giving a stronger constraint than the neoclassical Eq. (5). Departure from this constraint means the violation of momentum conservation or quasi-neutrality, and requires an additional force $F_\parallel$ such as from an externally driven $E_\parallel$. Equation (6) is what is called neoclassical toroidal viscosity, giving the non-ambipolar component of particle flux $\Gamma^s_{NA} = \vec{v}_s \cdot \vec{\nabla}\psi_p$ across flux surfaces for each species of charge $q_s$. Note that the flux is independent for each species and the RHS of Eq. (6) need not vanish when summing over species.

The toroidal force balance in Eq. (3) gives a different point of view, showing the radial currents associated with the toroidal torque. Again, this relation should hold locally and the local distribution of radial currents can significantly alter the magnetic field in equilibrium through $\vec{j} = \vec{\nabla} \times \vec{B}$. The following sections will show the local toroidal torque and radial currents in Eq. (3) are finite and first order in the non-axisymmetric perturbations with either scalar or tensor pressure and in either collisionless or collisional limits. This differs from the flux-surface-averaged relation in Eq. (6), which is only finite to the second order

with tensor pressure and collisional or orbital resonant dissipation. As discussed in much literature [12–14], the equilibrium relation in Eqs. (2-4) holds from the fast MHD to the slower drift MHD scales $\geq \vec{k}\cdot\vec{v}_d$, unless the collision frequency is faster that that. Here, $\vec{k}$ is a wave vector typical for perturbations and $\vec{v}_d$ is the equilibrium drift velocity of particles. The transport in an equilibrium described by Eqs. (2-4) is smaller by an order in size and should occur from the drift MHD to momentum diffusion time scales.

The last equilibrium relation in Eq. (4) includes radial component of force, and the toroidal (or poloidal) component relation in Eq. (3) should be properly used to isolate the radial part. Then it describes how the total pressure, including thermal and magnetic pressure, is radially varied. The evaluation of tensor pressure requires a closure, and this work takes a kinetic approach with $p_\parallel = \int d^3v M v_\parallel f$ and $p_\perp = \int d^3v (M v_\perp/2) f$. The conservation of particle energy and magnetic moment leads to

$$p_\parallel = \sum_{s\sigma} \frac{2\pi B}{M^2} \int dE \int d\mu \frac{2(E-\mu B)}{|v_\parallel|} f_{s\sigma}, \tag{7}$$

$$p_\perp = \sum_{s\sigma} \frac{2\pi B}{M^2} \int dE \int d\mu \frac{\mu B}{|v_\parallel|} f_{s\sigma}, \tag{8}$$

where $\sigma \equiv sign(v_\parallel)$ denoting the co and counter rotating particles, respectively, and $E \equiv mv^2/2$ and $\mu \equiv mv_\perp^2/2B$. An important assumption in this work is that $f_s$ remains Maxwellian in the axisymmetric configuration before perturbation, i.e., $f_s = f_M = n/(\sqrt{\pi}v_t)^3 e^{-v^2/v_t^2}$, and simply then $p_\parallel = p_\perp = p = nT$ without non-axisymmetric perturbations. Therefore the unperturbed equilibrium is just a nominal scalar pressure equilibrium $\vec{j}\times\vec{B} = \vec{\nabla}p$.

With a non-axisymmetric perturbation $\delta\vec{B}$, the perturbed current is $\delta\vec{j} = \vec{\nabla}\times\delta\vec{B}$ by Ampere's law. The relation to Lagrangian displacement $\vec{\xi}(\vec{x})$ is given by Ohm's law, without rotation and resistivity, combined with Faraday's law as $\delta\vec{B} = \vec{\nabla}\times(\vec{\xi}\times\vec{B})$. Our goal is to find equations for $\vec{\xi}$, $\delta\vec{B}$, $\delta\vec{j}$ in force balance, and to describe these perturbed quantities on unperturbed magnetic coordinates, i.e., the Eulerian frame. That is, the magnetic coordinates $(\psi,\theta,\varphi)$ represent the unperturbed magnetic field $\vec{B} = \chi'(\vec{\nabla}\varphi\times\vec{\nabla}\psi + q\vec{\nabla}\psi\times\vec{\nabla}\theta)$. The basis vectors in this coordinate system can be the contravariant ones $(\vec{\nabla}\psi,\vec{\nabla}\theta,\vec{\nabla}\varphi)$ or covariant ones $(\vec{e}_\psi,\vec{e}_\theta,\vec{e}_\varphi)$ where each is defined by $\vec{e}_\psi \equiv \partial\vec{x}/\partial\psi = \mathcal{J}(\vec{\nabla}\theta\times\vec{\nabla}\varphi)$, etc. Any vector $\vec{A}$ can also be decomposed to the contravariant components such as $A^\psi \equiv \vec{A}\cdot\vec{\nabla}\psi$ or covariant ones such as $A_\psi \equiv \vec{A}\cdot\vec{e}_\psi$. Note that this Eulerian formulation requires an im-

portant transformation when combined with kinetic approaches formulated in Lagrangian frame, as will be shown later.

## III. IDEAL FORCE BALANCE AND GLASSER'S TOROIDAL NEWCOMB EQUATION

This section describes the three components of perturbed ideal force balance and the ideal, toroidal Euler-Lagrange equation derived by Glasser, which is the basic equation that DCON [7] solves and IPEC [15] uses. The formulation given here is expanded with tensor pressure in the next section.

The magnetic fields and currents in unperturbed equilibrium can be represented by

$$\vec{B} = B^\theta \vec{e}_\theta + B^\varphi \vec{e}_\varphi \tag{9}$$

$$\vec{j} = j^\theta \vec{e}_\theta + j^\varphi \vec{e}_\varphi, \tag{10}$$

where $B^\theta = \chi'/\mathcal{J}$ and $B^\varphi = q\chi'/\mathcal{J}$, and in the axisymmetric case $j^\theta = -2\pi f'/\mathcal{J}$ and $j^\varphi = -2\pi q f'/\mathcal{J} - p'/\chi'$, where $\mathcal{J} \equiv 1/(\vec{\nabla}\psi \cdot (\vec{\nabla}\theta \times \vec{\nabla}\varphi))$ is the coordinate Jacobian and $f \equiv B_\varphi$. The transformation between covariant and contravariant components can be done with the metric tensor, $g_{ij} \equiv \vec{e}_i \cdot \vec{e}_j$.

An ideally perturbed Maxwellian plasma satisfies force balance given by

$$\delta\vec{j} \times \vec{B} + \vec{j} \times \delta\vec{B} - \vec{\nabla}\delta p = 0. \tag{11}$$

The parallel component to $\vec{B}$ is simply

$$\vec{B} \cdot \vec{\nabla}\delta p + \vec{\nabla} p \cdot \delta\vec{B} = 0, \tag{12}$$

which means that the pressure is still constant along the perturbed field lines. Using $\delta\vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B})$, one can see that the parallel force balance gives the so-called *adiabatic* perturbed pressure $\delta p = -\vec{\xi} \cdot \vec{\nabla} p$. This adiabatic perturbed pressure can be naturally obtained in the kinetic description of pressure as shown in Sec. IV. The fluid description makes the use of ideal gas law $\delta p = -\vec{\xi} \cdot \vec{\nabla} p - \gamma(\vec{\nabla} \cdot \vec{\xi})$ with the incompressibility condition

$$\vec{\nabla} \cdot \vec{\xi} = 0, \tag{13}$$

to enforce the adiabatic perturbed pressure everywhere including the resonant surfaces. The incompressibility then relates $\xi_\parallel$ to the other two components for $\vec{\xi}_\perp$.

The two other components of force balance can be conveniently obtained by taking the covariant components of the force along $\vec{e}_\varphi$ and $\vec{e}_\psi$, defined on the unperturbed magnetic field,

$$\chi' \left( \frac{\partial \delta B_\varphi}{\partial \theta} - \frac{\partial \delta B_\theta}{\partial \varphi} \right) = \chi' \mathcal{J} j^\theta \left( \frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \varphi} \right) \xi^\psi - \mathcal{J} \frac{\partial}{\partial \varphi} \left( \vec{\xi} \cdot \vec{\nabla} p \right), \tag{14}$$

$$\begin{aligned} \frac{\partial}{\partial \psi} \left( p' \mathcal{J} \xi^\psi - \chi' \delta B_\theta - q \chi' \delta B_\varphi \right) &= \chi' \left( \frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \varphi} \right) \delta B_\psi + \left( \chi'' \delta B_\theta + (q\chi')' \delta B_\varphi \right) \\ &+ \mathcal{J} \chi' \left( j^\theta \frac{\partial \xi^\alpha}{\partial \theta} + j^\varphi \frac{\partial \xi^\alpha}{\partial \varphi} \right) + \mathcal{J} (j^\theta (q\chi')' - j^\varphi \chi'') \xi^\psi + p' \left( \mathcal{J}' \xi^\psi + \mathcal{J} \frac{\partial \xi^\psi}{\partial \psi} \right), \end{aligned} \tag{15}$$

where $\alpha \equiv q\theta - \varphi$, and the axisymmetric equilibrium relation $\chi'(q j^\theta - j^\varphi) = p'$ is used in Eq. (15).

The toroidal balance in Eq. (14) is equivalent to

$$\chi' \delta j^\psi = \vec{e}_\varphi \cdot \left( \vec{j} \times \delta \vec{B} - \vec{\nabla} \delta p \right), \tag{16}$$

which gives the local distribution of radial current across flux surfaces and the associated toroidal torque in ideal MHD. Besides the toroidal torque balance, note that there is no net first-order torque or radial current in flux-surface-average, which is also true on the perturbed magnetic surfaces in scalar pressure perturbed equilibrium. It is the anisotropic pressure tensor that provides the non-zero, second order torque. On the other hand, the radial force balance Eq. (15) is identical to

$$p' \mathcal{J} \xi^\psi - \chi' \delta B_\theta - q \chi' \delta B_\varphi = \mathcal{J} (\vec{\xi}_\perp \cdot \vec{\nabla} p - \vec{B} \cdot \delta \vec{B}), \tag{17}$$

which describes the perturbed thermal and magnetic pressure of the system. This is also proportional to the surface current representing the energy [15], and mathematically can be treated as a conjugate momentum $p$ in a Hamiltonian system with $t \to \psi$ and $x \to \xi_\psi$ [16].

The two balance Eqs. (14) and (15) have mixed representation across contravariant and covariant components of the perturbed quantities, which are simply related to each other by metric tensors. Using $\delta \vec{j} = \vec{\nabla} \times \delta \vec{B}$ and $\delta \vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B})$, as shown in the Appendix, one can obtain two coupled partial differential equations determining the two components of displacement $\xi^\alpha$ and $\xi^\psi$.

The partial differential equations are then transformable to ordinary vector equations by Fourier representation for poloidal and toroidal periodic variations;

$$\xi^{(\psi,\alpha)}(\psi,\theta,\varphi) = \sum_{m,n} \xi^{(\psi,\alpha)}_{mn}(\psi) e^{i(m\theta - n\varphi)}. \tag{18}$$

From here, two matrix vectors will be defined to represent the retained poloidal modes of displacement in their elements as

$$\Xi_{(\psi,\alpha)} \equiv \{\xi^{(\psi,\alpha)}_{mn} | m_{min} \leq m \leq m_{max}\} \tag{19}$$

for each $n$ separately. The scalar and differential operators become matrix operators. For example, an arbitrary matrix $\mathbf{X}$ is defined with elements

$$X_{mm'} \equiv \frac{1}{2\pi} \oint d\theta X(\theta) e^{i(m'-m)\theta}, \tag{20}$$

which represents the poloidal mode coupling in toroidal geometry. Note that the toroidal mode numbers are decoupled in perturbed tokamaks but that this formulation is easily expandable to stellarator geometry keeping the toroidal mode coupling. With the spectral representation, Eqs (14) and (15) become

$$\mathbf{A}_i \Xi_\alpha + \mathbf{B}_i \Xi'_\psi + \mathbf{C}_i \Xi_\psi = 0, \tag{21}$$

$$\left(\mathbf{D}_i \Xi'_\psi + \mathbf{E}_i \Xi_\psi + \mathbf{B}^\dagger_i \Xi_\alpha\right)' = \mathbf{E}^\dagger_i \Xi'_\psi + \mathbf{H}_i \Xi_\psi + \mathbf{C}^\dagger_i \Xi_\alpha. \tag{22}$$

The matrices $\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i$, $\mathbf{D}_i$, $\mathbf{E}_i$, and $\mathbf{H}_i$ are functions of the geometry, current and pressure gradient, as shown in the Appendix. These two relations and their matrices are identical to Glasser's derivation by energy minimization [7], providing explicit proof of the equivalence between the minimum energy state and perturbed equilibrium. The self-adjointness of the force operator is manifested here by $\mathbf{A}_i = \mathbf{A}^\dagger_i$, $\mathbf{D}_i = \mathbf{D}^\dagger_i$, $\mathbf{H}_i = \mathbf{H}^\dagger_i$, and $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{E}_i$ appearing twice through their adjoint matrices. These Hermitian properties are required for energy principles, but are not fundamental to the tensor pressure equilibrium calculation.

Note that the toroidal balance, Eq. (21), algebraically relates the in-surface displacement to the radial displacement. Eliminating $\Xi_\alpha$ accordingly, one can obtain

$$(\mathbf{F}_i \Xi'_\psi + \mathbf{K}_i \Xi_\psi)' - (\mathbf{K}^\dagger_i \Xi'_\psi + \mathbf{G}_i \Xi_\psi) = 0, \tag{23}$$

with the ideal composite matrices

$$\mathbf{F}_i = \mathbf{D}_i - \mathbf{B}_i^\dagger \mathbf{A}_i^{-1} \mathbf{B}_i, \tag{24}$$

$$\mathbf{K}_i = \mathbf{E}_i - \mathbf{B}_i^\dagger \mathbf{A}_i^{-1} \mathbf{C}_i, \tag{25}$$

$$\mathbf{G}_i = \mathbf{H}_i - \mathbf{C}_i^\dagger \mathbf{A}_i^{-1} \mathbf{C}_i. \tag{26}$$

This is the Euler-Lagrange equation in toroidal geometry derived by Glasser. Compared to the cylindrical equation derived by Newcomb, $(f\xi^{\psi\prime})' - g\xi^{\psi} = 0$, $\mathbf{F}_i = \mathbf{F}_i^\dagger$ and $\mathbf{G}_i = \mathbf{G}_i^\dagger$ include coupling between poloidal modes, in addition to the contribution by new non-Hermitian matrix $\mathbf{K}_i$.

The ideal toroidal Euler-Lagrange equation contains regular singular points at every rational surface, which require additional layer physics for resolution. This can be seen from the structure of the composite matrices, $\mathbf{F}_i = \mathbf{Q}\bar{\mathbf{F}}_i\mathbf{Q}$ and $\mathbf{K}_i = \mathbf{Q}\bar{\mathbf{K}}_i$. Here the matrix $\mathbf{Q}$ is the singular factor defined as $Q_{mm'} \equiv (m - nq)\delta_{mm'}$, and $\bar{\mathbf{F}}_i$ and $\bar{\mathbf{K}}_i$ are non-singular everywhere as shown in the Appendix. The ideal solution is obtained by imposing the ideal jump condition $(\mathcal{J}\delta\vec{B} \cdot \vec{\nabla}\psi)_{mn} = 0$ at each surface. The ideal stability code DCON achieves this ideal constraint by equivalently eliminating large resonant solution in the asymptotic limit approaching each surface. The full details of the treatment are given in [7].

The results above are obtained by directly solving three components of force balance, and the question of stability of the system is a separate issue. The ideal perturbed equilibrium calculation above is valid in the presence of a conducting wall even if $\delta W_{no-wall} < 0$, as long as $\delta W_{wall} > 0$, that is, as long as the system is actually stable with the wall. The force balance calculation equivalent to $\delta W_{no-wall}$ above characterizes perturbed equilibrium since there is no current at the wall in the equilibrium state. Despite the conceptual validity of ideal perturbed equilibrium on either side of the no-wall limit, the solution is infinite at the no-wall limit. This is obviously problematic and should be either non-linearly saturated or corrected by non-ideal physics. Recently, it has been shown that the kinetic effect of a anisotropic pressure tensor is an important physics element in resolving this discrepancy [17].

## IV. FORCE BALANCE EQUATION WITH ANISOTROPIC PRESSURE

This section will extend the force balance with the anisotropic pressure tensor

$$\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}=\vec{\nabla}\cdot\left((\delta p_{\parallel}-\delta p_{\perp})\hat{b}\hat{b}+\delta p_{\perp}\overleftrightarrow{I}\right), \tag{27}$$

perturbed from a Maxwellian plasma $p_{\parallel}=p_{\perp}$ under the first gyroradius ordering. In this ordering and assumption, the perturbed anisotropic pressures $\delta p_{\parallel}=\int d^3v M v_{\parallel}^2\delta f$ and $\delta p_{\perp}=\int d^3v(Mv_{\perp}^2/2)\delta f$ relevant for $\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}$ become simply

$$\delta p_{\parallel}=\sum_{s\sigma}\frac{2\pi B}{M_s^2}\int dE\int d\mu\frac{2(E-\mu B)}{|v_{\parallel}|}\delta f_{s\sigma}, \tag{28}$$

$$\delta p_{\perp}=\sum_{s\sigma}\frac{2\pi B}{M_s^2}\int dE\int d\mu\frac{\mu B}{|v_{\parallel}|}\delta f_{s\sigma}. \tag{29}$$

The calculation of $\delta p_{\parallel}(\psi,\theta,\varphi)$ and $\delta p_{\perp}(\psi,\theta,\varphi)$ requires the evaluations of $B$ and $\delta f$ in the Eulerian frame, requiring a transformation for the Lagrangian $\delta f$ in transport theory.

### A. Eulerian vs. Lagrangian Magnetic Coordinates

In the presence of non-axisymmetry, the drift-kinetic equation for $\delta f$ is most conveniently formulated on the true magnetic coordinates $(\psi_m,\theta_m,\varphi_m)$, including the small non-axisymmetry, i.e. the Lagrangian frame. However, the force balance is obtained by introducing small perturbations upon the axisymmetric force balance and thus on the unperturbed magnetic coordinates $(\psi,\theta,\varphi)$, i.e. the Eulerian frame. This difference requires an important correction for the distribution function

$$\delta f(\vec{x})=\delta f(\vec{x}+\vec{\xi})-\vec{\xi}\cdot\vec{\nabla}f, \tag{30}$$

leading also to $\delta\overleftrightarrow{\mathcal{P}}(\vec{x})=\delta\overleftrightarrow{\mathcal{P}}(\vec{x}+\vec{\xi})-\vec{\xi}\cdot\vec{\nabla}\overleftrightarrow{\mathcal{P}}$, in the first order. Therefore for Maxwellian plasma $f=f_M$, the force balance $\delta\vec{j}\times\vec{B}+\vec{j}\times\delta\vec{B}-\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}(\vec{x})=0$ becomes

$$\delta\vec{j}\times\vec{B}+\vec{j}\times\delta\vec{B}+\vec{\nabla}\left(\vec{\xi}_{\perp}\cdot\vec{\nabla}p\right)-\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}(\vec{x}+\vec{\xi})=0. \tag{31}$$

The last term above, called the *non-adiabatic* part of perturbed pressure [14], is the anisotropic pressure tensor calculated using a Lagrangian perturbed distribution function typically found in transport theory. The *adiabatic* part, $\delta p=-\vec{\xi}_{\perp}\cdot\vec{\nabla}p$, requires only the

unperturbed pressure. Note neither the ideal gas law nor the incompressibility condition are necessary with this kinetic closure.

The field required to evaluate the perturbed distribution function $\delta f(\vec{B}, \vec{E})$ should also be in the Lagrangian frame and needs corrections as $\delta\vec{B}(\vec{x}+\vec{\xi}) = \delta\vec{B}(\vec{x})+\vec{\xi}\cdot\vec{\nabla}\vec{B}$, and $\delta\Phi(\vec{x}+\vec{\xi}) = \delta\Phi(\vec{x})+\vec{\xi}\cdot\vec{\nabla}\Phi$ if the perturbation on the radial electric field is also considered. The correction for the magnetic field is particularly important due to the strong inhomogeneity of $B \propto 1/R$ in tokamak geometry, as has been shown in attempts to accurately estimate NTV with non-axisymmetric variations in the field strength [18, 19].

This difference, i.e., transport formulated in the Lagrangian but equilibrium in the Eulerian frame, is the key to understand the identity

$$\left\langle \vec{e}_{\varphi_m} \cdot \vec{\nabla} \cdot \delta\overleftrightarrow{\mathcal{P}} \right\rangle_{(\psi_m,\theta_m,\varphi_m)} = -in \left\langle \vec{\xi} \cdot \delta\vec{F}[\vec{\xi}] \right\rangle_{(\psi,\theta,\varphi)}, \tag{32}$$

as proved in [20]. This equation shows the fundamental connection between the theory of non-axisymmetric neoclassical transport and that of kinetic stability in tokamaks. The torque is the reactive and imaginary energy, which has been ignored in collisionless kinetic energy principles to maintain self-adjointness. The torque becomes finite and important in the presence of collisions, and the treatment of collisions is what has made neoclassical theory differ significantly from kinetic stability theory.

### B. Parallel force balance and action variation

The parallel force balance with $\vec{\nabla} \cdot \delta\overleftrightarrow{\mathcal{P}}$ is determined by $\vec{B} \cdot (\vec{\nabla} \cdot \delta\overleftrightarrow{\mathcal{P}}) = 0$ since the ideal part is eliminated by $\vec{B} \cdot \vec{\nabla}\delta p + \vec{\nabla}p \cdot \delta\vec{B} = 0$ with $\delta p = -\vec{\xi}_\perp \cdot \vec{\nabla}p$ and $\delta\vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B})$. So one just has

$$\vec{B} \cdot \vec{\nabla}\delta p_\parallel - (\delta p_\parallel - \delta p_\perp)(\hat{b} \cdot \vec{\nabla}B) = 0. \tag{33}$$

Interestingly, it is automatically satisfied since

$$\begin{aligned}
\vec{B} \cdot \vec{\nabla}\delta p_\parallel &= \sum_s \frac{2\pi B}{M_s} \int dE d\mu \delta f_s \left( \hat{b} \cdot \vec{\nabla}(B|v_\parallel|) \right) \\
&= \sum_s \frac{2\pi B}{M_s} \int dE d\mu \delta f_s \left( |v_\parallel| - \frac{v_\perp^2}{2|v_\parallel|} \right) \hat{b} \cdot \vec{\nabla}\vec{B} \\
&= (\delta p_\parallel - \delta p_\perp)(\hat{b} \cdot \vec{\nabla}B),
\end{aligned} \tag{34}$$

provided that

$$\sum_s \frac{1}{M_s} \int dE d\mu (B|v_\parallel|)(\hat{b} \cdot \vec{\nabla} \delta f_s) = 0. \tag{35}$$

This condition holds sufficiently if $\hat{b} \cdot \vec{\nabla} \delta f_s = 0$, that is, if the perturbed distribution function is constant along the field lines as $\delta f_s = \delta f_s(E, \mu, \psi, \alpha)$. It is also consistent with the typical assumptions of non-axisymmetric neoclassical transport theory in the long mean-free-path regime, where $\delta f_s$ is solved by orbit averaging [1, 2, 21]. This leaves only two components in the force balance to determine $\vec{\xi}$ and thus $\vec{\xi}$ is seemingly underdetermined. However, the orbit averaging process also automatically eliminates the parallel displacement $\xi_\parallel$.

Note that no parallel component in the anisotropic pressure tensor force also implies

$$\vec{\nabla} \cdot \Big( (\delta p_\parallel - \delta p_\perp) \hat{b} \Big) + \hat{b} \cdot \vec{\nabla} \delta p_\perp = 0, \tag{36}$$

and gives a convenient form for the anisotropic pressure tensor

$$\vec{\nabla} \cdot \delta \overleftrightarrow{\mathcal{P}} = (\delta p_\parallel - \delta p_\perp) \vec{\kappa} + \vec{\nabla} \delta p_\perp - \hat{b} (\hat{b} \cdot \vec{\nabla} \delta p_\perp), \tag{37}$$

where the curvature vector $\vec{\kappa} B^2 = \vec{\nabla}_\perp (p + B^2/2)$. Using Eq. (37), one can formulate the other two covariant components of the force balance, i.e. $\vec{e}_\varphi \cdot \delta \vec{F}$ and $\vec{e}_\psi \cdot \delta \vec{F}$.

As shall be seen, these two components are closely related to the variation in the field strength and the action. The non-axisymmetric variation in the action is defined and is given by

$$\begin{aligned} \delta \mathcal{J} \equiv & \delta \left( \oint M v_\parallel dl \right) = \frac{M}{\chi'} \oint d\theta \delta \left( \mathcal{J} B v_\parallel \right) \\ = & \oint d\theta \frac{\mathcal{J} B}{v_\parallel \chi'} \Big( (3\mu B - 2E)(\vec{\xi}_\perp \cdot \vec{\kappa}) + \mu B (\vec{\nabla} \cdot \vec{\xi}_\perp) \Big) \end{aligned} \tag{38}$$

on the coordinates $(\psi, \theta, \alpha = q\theta - \varphi)$. This includes the displacement of magnetic field lines through $\delta \mathcal{J}$, which is often neglected in NTV theories but is not ignorable. Even in Hamada coordinates, this effect gives $\mathcal{J} \to 1 + (\vec{\nabla} \cdot \vec{\xi})$. The subscript for each species $s$ is omitted here for simplification and will be omitted throughout this paper unless it is necessary. As shown in [20], the parallel component of displacement $\vec{\xi}_\parallel$ does not have any contribution and thus can be dropped.

### C. Toroidal and radial force balance

The toroidal and radial balance can be obtained by evaluating $-\vec{e}_\varphi\cdot(\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}})$ and $-\vec{e}_\psi\cdot(\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}})$, and combining with the ideal force balance. The extended toroidal balance becomes

$$\chi'\delta j^\psi = \vec{e}_\varphi\cdot\left(\vec{j}\times\delta\vec{B}-\vec{\nabla}\delta p-\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}\right), \tag{39}$$

and gives the distribution of the first-order toroidal torque by each term and non-ambipolar currents. The radial balance is also extended, and the LHS of the Eq. (15), i.e. the term requiring the radial derivative, becomes $\mathcal{J}(\vec{\xi}_\perp\cdot\vec{\nabla}p-\delta p_\perp-\vec{B}\cdot\delta\vec{B})$, representing the perturbed thermal and magnetic pressure on the flux surface. The perturbed thermal pressure now includes both adiabatic and non-adiabatic contributions in the isotropic pressure.

To form a matrix representation of the remaining equations, we define linear operators

$$\hat{\mathcal{S}} = \frac{1}{B^2}\frac{\partial}{\partial\psi}\left(p+\frac{B^2}{2}\right)-\frac{\chi'\vec{B}\cdot(\vec{\nabla}\theta\times\vec{\nabla}\varphi)}{B^3}\frac{\partial B}{\partial\theta}, \tag{40}$$

$$\hat{\mathcal{T}} = \frac{\chi'\vec{B}\cdot(\vec{\nabla}\psi\times\vec{\nabla}\theta)}{B^3}\frac{\partial B}{\partial\theta}, \tag{41}$$

$$\begin{aligned}\hat{\mathcal{X}} = {}& \frac{\partial}{\partial\psi}+\frac{\partial\ln\mathcal{J}}{\partial\psi}-\left[\frac{\chi'\vec{B}\cdot(\vec{\nabla}\theta\times\vec{\nabla}\varphi)}{B^2}\right]\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\varphi}\right)\\ & -\frac{1}{\mathcal{J}}\frac{\partial}{\partial\theta}\left[\frac{\chi'\mathcal{J}\vec{B}\cdot(\vec{\nabla}\theta\times\vec{\nabla}\varphi)}{B^2}\right],\end{aligned} \tag{42}$$

$$\begin{aligned}\hat{\mathcal{Z}} = {}& \left[\frac{\chi'\vec{B}\cdot(\vec{\nabla}\psi\times\vec{\nabla}\theta)}{B^2}\right]\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\varphi}\right)\\ & +\frac{1}{\mathcal{J}}\frac{\partial}{\partial\theta}\left[\frac{\chi'\mathcal{J}\vec{B}\cdot(\vec{\nabla}\psi\times\vec{\nabla}\theta)}{B^2}\right]-\frac{\partial}{\partial\varphi},\end{aligned} \tag{43}$$

and two kinetic multipliers

$$w_\parallel \equiv \frac{\mathcal{J}B(2E-2\mu B)}{v_\parallel\chi'}, \quad w_\perp \equiv \frac{\mathcal{J}\mu B^2}{v_\parallel\chi'}. \tag{44}$$

These operators are related to the action integral. One can show

$$\begin{bmatrix}\vec{\xi}\cdot\vec{\kappa}\\ \vec{\nabla}\cdot\vec{\xi}_\perp\end{bmatrix} = \begin{bmatrix}\hat{\mathcal{S}} & \hat{\mathcal{T}}\\ \hat{\mathcal{X}} & \hat{\mathcal{Z}}\end{bmatrix}\begin{bmatrix}\xi^\psi\\ \xi^\alpha\end{bmatrix}, \tag{45}$$

and thus

$$\delta\mathcal{J}\left[\vec{\xi}\right] = \oint d\theta\begin{bmatrix}w_\perp-w_\parallel\\ w_\perp\end{bmatrix}^T\begin{bmatrix}\hat{\mathcal{S}} & \hat{\mathcal{T}}\\ \hat{\mathcal{X}} & \hat{\mathcal{Z}}\end{bmatrix}\begin{bmatrix}\xi^\psi\\ \xi^\alpha\end{bmatrix}. \tag{46}$$

Using these operators, Eqs. (28-29) and Eq. (37), the radial and toroidal force balance become

$$\begin{bmatrix} \mathcal{J}\vec{e}_\psi \cdot (\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}) \\ -\mathcal{J}\vec{e}_\varphi \cdot (\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}) \end{bmatrix} = -\begin{bmatrix} \hat{\mathcal{S}} & \hat{\mathcal{T}} \\ \hat{\mathcal{X}} & \hat{\mathcal{Z}} \end{bmatrix}^* \begin{bmatrix} \mathcal{J}(\delta p_\perp - \delta p_\parallel) \\ \mathcal{J}\delta p_\perp \end{bmatrix} = -\frac{2\pi\chi'}{M^2}\int dEd\mu \begin{bmatrix} \hat{\mathcal{S}} & \hat{\mathcal{T}} \\ \hat{\mathcal{X}} & \hat{\mathcal{Z}} \end{bmatrix}^* \Big[w_\perp\Big]\,[\delta f], \tag{47}$$

where $[\ ]^*$ indicates the transpose matrix with the adjoint operation for each element, i.e. $\hat{\mathcal{S}}^*, \hat{\mathcal{T}}^*, \hat{\mathcal{X}}^*, \hat{\mathcal{Z}}^*$. Note again that the sum of each species $s$ and sign of $v_\parallel$ is omitted for simplicity. In the orbit averaged formulation, formally one has $\delta f = \mathcal{L}[\delta\mathcal{J}]$ depending on the model of collisions, where $\mathcal{L}$ is a linear operator independent of $\theta$. Then

$$\begin{bmatrix} \mathcal{J}\vec{e}_\psi \cdot (\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}) \\ -\mathcal{J}\vec{e}_\varphi \cdot (\vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}) \end{bmatrix} = -\frac{2\pi\chi'}{M^2}\int dEd\mu \begin{bmatrix} \hat{\mathcal{S}} & \hat{\mathcal{T}} \\ \hat{\mathcal{X}} & \hat{\mathcal{Z}} \end{bmatrix}^* \begin{bmatrix} w_\perp - w_\parallel \\ w_\perp \end{bmatrix} \oint d\theta\mathcal{L} \left[ \begin{bmatrix} w_\perp - w_\parallel \\ w_\perp \end{bmatrix}^T \begin{bmatrix} \hat{\mathcal{S}} & \hat{\mathcal{T}} \\ \hat{\mathcal{X}} & \hat{\mathcal{Z}} \end{bmatrix} \begin{bmatrix} \xi^\psi \\ \xi^\alpha \end{bmatrix} \right]. \tag{48}$$

The extra minus sign for the toroidal balance is used for the difference between the coordinate $\alpha$ and $\varphi$. One can see that the symmetric self-adjoint structure will hold in the force balance if the operator $\mathcal{L}$ is self-adjoint. It is the collisional process that breaks the self-adjointness in the operator $\mathcal{L}$ and force operator $\delta\vec{F}$, as will become clear in the energy integral shown later.

The two balance equations are three-dimensional partial differential equations, but can be reduced to ordinary vector equations using the spectral analysis for periodic coordinates $(\theta, \varphi)$. It is convenient to separate the first order derivative for $\xi_\psi$ as in the ideal case using

$$\hat{\mathcal{X}} = \frac{\partial}{\partial\psi} + \hat{\mathcal{Y}}. \tag{49}$$

Then each term for the variation in the field strength becomes

$$\vec{\xi}_\perp \cdot \vec{\kappa} = \mathbf{S}\Xi_\psi + \mathbf{T}\Xi_\alpha \tag{50}$$

$$\vec{\nabla}\cdot\vec{\xi}_\perp = \Xi'_\psi + \mathbf{Y}\Xi_\psi + \mathbf{Z}\Xi_\alpha, \tag{51}$$

where the matrices $\mathbf{S}, \mathbf{T}, \mathbf{Y}, \mathbf{Z}$ are obtained using Eq. (20) for each operator $\hat{\mathcal{S}}, \hat{\mathcal{T}}, \hat{\mathcal{Y}}, \hat{\mathcal{Z}}$. The action variation in Eq. (46) becomes

$$\delta\mathcal{J} = W^\partial\Xi'_\psi + W^\psi\Xi_\psi + W^\alpha\Xi_\alpha, \tag{52}$$

where the row matrix vectors are defined as

$$W^{\partial} \equiv W_{\perp} \tag{53}$$

$$W^{\psi} \equiv (W_{\perp} - W_{\parallel})\mathbf{S} + W_{\perp}\mathbf{Y} \tag{54}$$

$$W^{\alpha} \equiv (W_{\perp} - W_{\parallel})\mathbf{T} + W_{\perp}\mathbf{Z}. \tag{55}$$

The elements of the sub-matrices $W_{\parallel}$ and $W_{\perp}$ provides the orbit integration in velocity space

$$W_{\parallel,mn} = \oint d\theta \frac{\mathcal{J}B(2E - 2\mu B)}{v_{\parallel}\chi'} e^{i(m-nq)\theta} \tag{56}$$

$$W_{\perp,mn} = \oint d\theta \frac{\mathcal{J}\mu B^2}{v_{\parallel}\chi'} e^{i(m-nq)\theta}. \tag{57}$$

The extra factor $e^{-inq\theta}$ is required since the action integral should be done with fixed $\alpha$. In tokamaks, Eq. (52) represents each element of $\delta\mathcal{I}_n$ in $\delta\mathcal{I} = \delta\mathcal{I}_n e^{in\alpha}$. Using $\delta f = \delta f(E, \mu, \psi, \alpha) = \delta f(E, \mu, \psi)e^{in\alpha}$, each Fourier element of the toroidal and radial tensor forces become

$$\left[\mathcal{J}\vec{e}_{\varphi} \cdot (\vec{\nabla} \cdot \overleftrightarrow{\delta P})\right]_{mn} = \frac{\chi'}{M^2} \int dE d\mu W^{\alpha\dagger}_{mn} \delta f_n \tag{58}$$

$$\begin{aligned}\left[\mathcal{J}\vec{e}_{\psi} \cdot (\vec{\nabla} \cdot \overleftrightarrow{\delta P})\right]_{mn} &= \frac{\chi'}{M^2}\frac{\partial}{\partial\psi}\left(\int dE d\mu W^{\partial\dagger}_{mn}\delta f_n\right) \\ &\quad - \frac{\chi'}{M^2}\int dE d\mu (W^{\psi\dagger}_{mn}\delta f_n).\end{aligned} \tag{59}$$

further one needs a kinetic model to have the perturbed distribution function $\delta f$, which will be discussed in the next section. Nonetheless considering $\delta\mathcal{I} = W^{\partial}\Xi'_{\psi} + W^{\psi}\Xi_{\psi} + W^{\alpha}\Xi_{\alpha}$ and $\delta f = \mathcal{L}[\delta\mathcal{I}]$, one can always reduce two force balances to

$$\mathbf{A}_k\Xi_{\alpha} + \mathbf{B}_u\Xi'_{\psi} + \mathbf{C}_u\Xi_{\psi} = 0, \tag{60}$$

$$\left(\mathbf{D}_k\Xi'_{\psi} + \mathbf{E}_u\Xi_{\psi} + \mathbf{B}^{\dagger}_l\Xi_{\alpha}\right)' = \mathbf{E}^{\dagger}_l\Xi'_{\psi} + \mathbf{H}_k\Xi_{\psi} + \mathbf{C}^{\dagger}_l\Xi_{\alpha}, \tag{61}$$

unless $\mathcal{L}$ involves the radial derivatives of $\delta\mathcal{I}$. The above equations look similar to the ideal force balance in Eqs. (21-22), but there are important differences. The matrices now have kinetic contributions with a quadratic form related to the matrices $W^{\partial}$, $W^{\psi}$, $W^{\alpha}$, in addition to the ideal contributions. Due to the collision term, the matrices $\mathbf{A}_k, \mathbf{D}_k, \mathbf{H}_k$ are not Hermitian any more, and $\mathbf{B}_u \neq \mathbf{B}_l$, $\mathbf{C}_u \neq \mathbf{C}_l$, $\mathbf{E}_u \neq \mathbf{E}_l$.

The detailed form of each matrix will be discussed in Sec. V. To shorten the description of the matrices, define the $\mathbf{M}^{ij}$ as

$$\begin{bmatrix} \mathbf{M}^{\alpha\alpha} & \mathbf{M}^{\alpha\partial} & \mathbf{M}^{\alpha\psi} \\ \mathbf{M}^{\partial\alpha} & \mathbf{M}^{\partial\partial} & \mathbf{M}^{d\psi} \\ \mathbf{M}^{\psi\alpha} & \mathbf{M}^{\psi\partial} & \mathbf{M}^{\psi\psi} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{A}_k & \mathbf{B}_u & \mathbf{C}_u \\ \mathbf{B}_l^\dagger & \mathbf{D}_k & \mathbf{E}_u \\ \mathbf{C}_l^\dagger & \mathbf{E}_l^\dagger & \mathbf{H}_k \end{bmatrix}, \tag{62}$$

and also the $\mathbf{M}_I^{ij}$ as

$$\begin{bmatrix} \mathbf{M}_I^{\alpha\alpha} & \mathbf{M}_I^{\alpha\partial} & \mathbf{M}_I^{\alpha\psi} \\ \mathbf{M}_I^{\partial\alpha} & \mathbf{M}_I^{\partial\partial} & \mathbf{M}_I^{\partial\psi} \\ \mathbf{M}_I^{\psi\alpha} & \mathbf{M}_I^{\psi\partial} & \mathbf{M}_I^{\psi\psi} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i & \mathbf{C}_i \\ \mathbf{B}_i^\dagger & \mathbf{D}_i & \mathbf{E}_i \\ \mathbf{C}_i^\dagger & \mathbf{E}_i^\dagger & \mathbf{H}_i \end{bmatrix}, \tag{63}$$

for the ideal matrices, which give $\mathbf{M}_I = \mathbf{M}_I^\dagger$. Then $\delta f = \mathcal{L}[\delta \mathcal{J}]$ leads to

$$\mathbf{M}^{ij} = \mathbf{M}_I^{ij} - \frac{\chi'}{M^2} \int dE d\mu \left( W^{i\dagger} \mathcal{L} \left[ W^j \right] \right), \tag{64}$$

when it is combined with the ideal force balance. The kinetic correction is to the ideal matrices $\mathbf{M}_I$ is small in low $\beta$, but the importance increases in high $\beta$ plasmas. In any case, the kinetic correction gives the resolution of the singularity at the rational surfaces whenever the torque is finite.

### D. Non-self-adjoint Euler-Lagrange Equation

Equations (60-61) can be combined similar to the ideal equations, giving a new Euler-Lagrange equation

$$(\mathbf{F}_k \Xi_\psi' + \mathbf{K}_u \Xi_\psi)' - (\mathbf{K}_l^\dagger \Xi_\psi' + \mathbf{G}_k \Xi_\psi) = 0, \tag{65}$$

but with non-Hermitian composite matrices

$$\mathbf{F}_k = \mathbf{D}_k - \mathbf{B}_l^\dagger \mathbf{A}_k^{-1} \mathbf{B}_u, \tag{66}$$

$$\mathbf{K}_u = \mathbf{E}_u - \mathbf{B}_l^\dagger \mathbf{A}_k^{-1} \mathbf{C}_u, \tag{67}$$

$$\mathbf{K}_l = \mathbf{E}_l - \mathbf{C}_l^\dagger \mathbf{A}_k^{-1} \mathbf{B}_u, \tag{68}$$

$$\mathbf{G}_k = \mathbf{H}_k - \mathbf{C}_l^\dagger \mathbf{A}_k^{-1} \mathbf{C}_u. \tag{69}$$

The nature of singularity of the equation is also changed. The composite matrices can be decomposed

$$\mathbf{F}_k = \mathbf{Q}\bar{\mathbf{F}}_k\mathbf{Q} - \mathbf{P}_l^\dagger\mathbf{Q} - \mathbf{Q}\mathbf{P}_u + \mathbf{R}_1, \tag{70}$$

$$\mathbf{K}_u = \mathbf{Q}\bar{\mathbf{K}}_u + \mathbf{R}_2, \tag{71}$$

$$\mathbf{K}_l = \bar{\mathbf{K}}_l\mathbf{Q} + \mathbf{R}_3, \tag{72}$$

where the sub-matrices $\bar{\mathbf{F}}_k$, $\mathbf{P}_l$, $\mathbf{P}_u$, $\bar{\mathbf{K}}_l$, $\bar{\mathbf{K}}_u$, $\mathbf{R}_{1,2,3}$ are all non-singular as given in the Appendix. In the ideal case, the matrix $\mathbf{F}_i$ is semi-positive definite Hermitian and $\det(\mathbf{F}) = 0$ at the rational surfaces $\psi_r$ where $q(\psi_r) = m/n$. With the kinetic correction without the torque, as found in collisionless kinetic energy principles, $\mathbf{F}$ is still Hermitian but $\det(\mathbf{F}) = 0$ can occur at $\psi = \psi_r - \sigma_L, \psi_r + \sigma_R$ on the either side of the rational surfaces or $\det(\mathbf{F})$ becomes positive definite. When the singular surface is split, the singularity becomes logarithmic rather than power-like and is integrable. The split of the singular surface occurs typically when the correction is locally destabilizing, as can be shown analytically in a cylindrical limit. When the torque is finite, $\det(\mathbf{F})$ becomes complex and the singularity is removed in the solution as well as in the torque integral. This is a consequence of the self-consistent treatment of torque.

In the presence of torque, the Euler-Lagrange equation is a regular second-order vector differential equation and thus can be integrated throughout $\psi$ from the magnetic axis to the edge of the plasma. Assigning the regular condition at the magnetic axis, i.e. $\Xi_\psi = \vec{0}$, there are $M$ linearly independent solutions when $M$ number of poloidal modes are retained. The $M$ coefficients of a perturbed equilibrium are determined by the prescribed boundary deformation in a fixed boundary problem, or by the applied external field in a free boundary problem where the boundary deformation can be determined by virtual casing principle as used for IPEC.

## V. DRIFT-KINETIC SOLUTIONS FOR ANISOTROPIC PRESSURE FORCE

Formation of the matrices described in Sec. IV, $\mathbf{M}$ and all other composite matrices in the non-Hermitian Euler-Lagrange equation (65), requires the calculation of $\delta f$. The relevant kinetic model in the first-order gyroradius ordering is the drift-kinetic equation

$$\frac{\partial f}{\partial t} + (\vec{v}_\parallel + \vec{v}_d)\cdot\vec{\nabla} f + \dot{U}\frac{\partial f}{\partial U} = \hat{C}[f], \tag{73}$$

where $U \equiv E + q_s\phi$ is the total guiding-center energy. Note that $q_s$ is the charge of species, i. e, $+Z$ for ions and $-1$ for electrons, but the subscript $s$ will be omitted hereafter for simplicity. Taking the parallel force balance $\hat{b}\cdot\vec{\nabla}\delta f = 0$ for each species up the first order, the drift-kinetic equation can be averaged over the bounce motion of trapped particles and over the cyclic motion of passing particles in terms of poloidal angle $\theta$. By linearizing the equation from the axisymmetric Maxwellian equilibrium,

$$\frac{\partial \delta f}{\partial t} + \langle \vec{v}_d \cdot \vec{\nabla}\alpha \rangle_b \frac{\partial \delta f}{\partial \alpha} + \langle \vec{v}_d \cdot \vec{\nabla}\psi \rangle_b \frac{\partial f_M}{\partial \psi} + \langle \delta \dot{U} \rangle_b \frac{\partial f_M}{\partial U} = \hat{C}_b[\delta f], \tag{74}$$

where the subscript $b$ denotes the orbit averaging. This is equivalent to the Hastie's form [13, 22],

$$\frac{\partial \delta f}{\partial t} - \frac{1}{q\chi'}\left(\frac{\mathcal{J}_\psi}{\mathcal{J}_U}\frac{\partial \delta f}{\partial \alpha} - \frac{\delta \mathcal{J}_\alpha}{\mathcal{J}_U}\frac{\partial f_M}{\partial \psi}\right) - \frac{\delta \mathcal{J}_t}{\mathcal{J}_U}\frac{\partial f_M}{\partial U} = \hat{C}_b[\delta f], \tag{75}$$

where the subscript in $\mathcal{J}$ denotes the partial derivatives with respect to each variable. This equation becomes analytically tractable if additional ordering assumptions are made. The treatments presented below are used and published by various authors in works on the kinetic energy principle and neoclassical non-ambipolar transport, but here are further generalized without geometric simplifications. When the solution for $\delta f$ is used to evaluate the matrix $\mathbf{M}$, the force balance Eqs. (60-61) give the eigenfunctions minimizing kinetic energy integral, or perturbed equilibrim self-consistent with NTV.

### A. Force balance in fast MHD - Kruskal-Oberman

The collisionless kinetic energy principle originated from the early work by Kruskal-Oberman (KO) [12]. The KO limit essentially describes the energy associated with kinetic motions of particles frozen to the magnetic lines of force, in addition to ideal MHD in the fast MHD time scales, by strictly neglecting the particle drift and collisions;

$$\frac{\partial}{\partial t}\left(\delta f - \frac{\delta \mathcal{J}}{\partial \mathcal{J}/\partial U}\frac{\partial f_M}{\partial U}\right) = 0. \tag{76}$$

The perturbed distribution function here is identical to ones obtained with the Lagrange multiplier in KO approach. The solution simply becomes

$$\delta f_{ko} = -\frac{\omega_b}{2\pi T} f_M \delta \mathcal{J}, \tag{77}$$

where $\omega_b$ is the bounce frequency and $T$ is the temperature of the species. The matrix $\mathbf{M}$ in the KO limit can then be obtained by

$$\mathbf{M}^{ij}_{ko} = \mathbf{M}^{ij}_{I} + \frac{\chi'}{2\pi M^2 T}\int dE d\mu \left(\omega_b f_M W^{i\dagger} W^j\right). \tag{78}$$

It is obvious that $\mathbf{M}_{ko} = \mathbf{M}^{\dagger}_{ko}$ as in ideal MHD, which is expected from the collisionless kinetic energy principle. This KO limit describes the kinetic perturbed equilibrium state accessible in the fast MHD time scale, but the equilibrium should evolve further due to the drift motions of particles. Note that in the KO limit, the kinetic contributions from both ions and electrons become identical if $T_i = T_e$, $n_i = n_e$;

$$\mathbf{M}_{ko,ion} = \mathbf{M}_{ko,electron}. \tag{79}$$

This can be seen from the action integral, and thus $(\omega_b f_M W^{i\dagger} W^j)/M^2 \propto n e^{-E/T} f(E,\mu)$ other than geometry and field dependency.

### B. Force balance in drift MHD - Krook, SBP

On the slower time scale of the particle drift motion, the time derivatives in the drift kinetic equation (74) can be ignored in perturbed equilibrium. This gives a drift kinetic equation

$$\langle \vec{v}_d \cdot \vec{\nabla}\alpha\rangle_b \frac{\partial \delta f}{\partial \alpha} + \langle \vec{v}_d \cdot \vec{\nabla}\psi\rangle_b \frac{\partial f_M}{\partial \psi} = \hat{C}_b[\delta f]. \tag{80}$$

The corresponding kinetic energy principle can be developed by neglecting collisions as shown in [13, 14, 23]. Collisions break the energy conservation in the temporal evolution of the perturbation and thus the energy principle can not be used to assess the stability of the system with collisions. The collisions also generate non-ambipolar diffusion and toroidal torque, which is an important phenomenon known as neoclassical toroidal viscosity (NTV) transport by Shaing et al [2]. If the collision operator is approximated by the Krook form $\hat{C}_b[\delta f] = -\nu(E)\delta f$, the $\delta f$ can be obtained in integral form and can reproduce the drift-MHD kinetic energy principle [14], NTV in the superbanana-plateau (SBP) regime [24, 25], and combined-regime NTV approximation [18, 26, 27]. In this case, the perturbed distribution function is

$$\delta f_{kr} = -\frac{1}{2\pi q}\frac{in\omega_b}{in\omega_p + \nu}\frac{\partial f_M}{\partial \chi}\delta \mathcal{J}, \tag{81}$$

where the orbit-averaged precession frequency $\omega_p \equiv \omega_E + \omega_B(E,\mu)$ is the sum of the electric and magnetic precession frequencies respectively,

$$\omega_E = \frac{d\Phi}{d\chi}, \tag{82}$$

$$\omega_B = \frac{1}{q}\left\langle \mu\frac{dB}{d\chi} - (2E - 2\mu B)\frac{d}{d\chi}\ln(\mathcal{J}B)\right\rangle_b. \tag{83}$$

The matrix $\mathbf{M}$ in the Krook model then becomes

$$\mathbf{M}^{ij}_{kr} = \mathbf{M}^{ij}_{I} + \frac{\chi'}{2\pi q M^2}\int dE d\mu \frac{in\omega_b}{in\omega_p + \nu}\frac{\partial f_M}{\partial\chi}W^{i\dagger}W^j. \tag{84}$$

The KO and SBP limits can be addressed by

$$\delta f_{ko} = \lim_{\omega_E\to\infty}\delta f_{kr}, \quad \mathbf{M}^{ij}_{ko} = \lim_{\omega_E\to\infty}\mathbf{M}^{ij}_{kr}, \tag{85}$$

$$\delta f_{sbp} = \lim_{\substack{\omega_E\to 0\\ \nu\to 0}}\delta f_{kr}, \quad \mathbf{M}^{ij}_{sbp} = \lim_{\substack{\omega_E\to 0\\ \nu\to 0}}\mathbf{M}^{ij}_{kr}. \tag{86}$$

Note the SBP limit also holds $\mathbf{M}_{sbp} = \mathbf{M}^{\dagger}_{sbp}$, in two species plasmas with $T_i = T_e$ and $n_i = n_e$. This is due to the cancelation of the imaginary part between ions and electrons. Using $\lim_{\nu\to 0}\nu/(x^2+\nu^2) = \pi\delta(x)$,

$$\Im(\mathbf{M}^{ij}_{sbp}) = \frac{\chi'}{2qM^2}\int dE d\mu\delta(\omega_B)\left(\omega_b\frac{\partial f_M}{\partial\chi}W^{i\dagger}W^j\right). \tag{87}$$

Similar to the KO limit, the quantity $(\omega_b(\partial f_M/\partial\chi)W^{i\dagger}W^j)/M^2$ is identical for ions and electrons. The integral with the delta function is finite through the point $\omega_B(E,\mu) = 0$ and is identical for both species despite the sign dependence in the magnetic precession since $\delta(x) = \delta(-x)$. Therefore, the sign remaining in $q$ makes ion and electron contributions canceled each other in summation. This cancelation occurs, of course, only when $\nu$ is low enough for both ions and electrons to be in the SBP regime and only when $\omega_E = 0$. Nonetheless a local region near the pedestal in thermonuclear tokamaks can satisfy both conditions and thus may enter this zero torque regime.

### C. Force balance in drift MHD - Lorentz

A more frequently used collision model in the drift-kinetic equation is the Lorentz pitch-angle operator

$$C[\delta f] = \nu\left(\frac{v_\parallel}{B}\right)\frac{\partial}{\partial\mu}\left(Mv_\parallel\mu\frac{\partial\delta f}{\partial\mu}\right). \tag{88}$$

The orbit averaged form in Eq. (80) then gives

$$\delta \mathcal{J} = -\frac{\mathcal{q}}{\frac{\partial f_M}{\partial \chi}} \left[ \frac{2\pi\omega_p}{\omega_b}\delta f + i\frac{\nu}{n}\frac{\partial}{\partial \mu}\left( \left( \int dl \frac{M v_\parallel}{B} \right) \mu \frac{\partial \delta f}{\partial \mu} \right) \right]. \tag{89}$$

The solution for $\delta f$ with the pitch-angle operator is analytically intractable and thus, in general, requires solving coupled differential equations with force balance. One way to examine the structure of the solution is to make asymptotic evaluations, leading to the popular $1/\nu$ regime or $\nu - \sqrt{\nu}$ regime [21, 28].

### 1. $1/\nu$ regime

The $1/\nu$ regime is a characteristic transport process that can be found only in non-axisymmetric configuration. If the precession is slow compared to the collision rates, i.e. $\omega_p \ll \nu$, one can ignore the first term in the RHS of Eq. (89) and obtain the solution by integration. Integration by parts gives,

$$\mathbf{M}^{ij}_{1/\nu} = \mathbf{M}^{ij}_I + i\frac{n\chi'}{\mathcal{q}M^2}\int dE d\mu \frac{1}{\nu}\frac{\partial f_M}{\partial \chi}\frac{\int d\mu W^{i\dagger}\int d\mu W^j}{\mathcal{K}}, \tag{90}$$

where $\mathcal{K} \equiv \mu \int dl \frac{M v_\parallel}{B}$ and complete generality in tokamak geometry has been retained. The $1/\nu$ approximation results in a purely anti-Hermitian addition to the ideal force, which breaks the self-adjointness of the force operator. There is no kinetic contribution to the Hermitian part from the first order solution in the expansion of $\omega_p/\nu$.

### 2. $\nu - \sqrt{\nu}$ regime

The opposite limit is analytically solvable, as can be seen by ignoring the second term in the RHS of Eq. (89). The solution then simply gives what can be obtained by Krook operator in the limit $\nu \to 0$, i.e. $\mathbf{M}^{ij}_\nu = \lim_{\nu\to 0}\mathbf{M}_{kr}$. The anti-Hermitian part, or toroidal torque, appears from the next order correction as described in [21]. One can show

$$\begin{aligned}\mathbf{M}^{ij}_\nu = \mathbf{M}^{ij}_I + \frac{\chi'}{2\pi \mathcal{q} M^2}\int dE d\mu \frac{\omega_b}{\omega_p}\frac{\partial f_M}{\partial \chi} W^{i\dagger} W^j + \\ + i\frac{\chi'}{4\pi^2 n \mathcal{q} M^2}\int dE d\mu \nu \mathcal{K}\frac{\partial f_M}{\partial \chi}\frac{\partial}{\partial \mu}\left(\frac{\omega_b W^{i\dagger}}{\omega_p}\right)\frac{\partial}{\partial \mu}\left(\frac{\omega_b W^j}{\omega_p}\right).\end{aligned} \tag{91}$$

The anti-Hermitian part above is unfortunately non-integrable in $\mu$ space, due to the $\partial/\partial\mu$ steepened singularity. This mathematical singularity can be removed by restoring the collision operator in a thin layer at the trapped-passing boundary as shown in [28], resulting in a more complicated formulation and in $\sqrt{\nu}$ regime.

The purpose here is to simply demonstrate the quadratic dependency of force balance on the action integral components with the Lorentz model. It should be emphasized that a model with a Krook operator is more practical than regime based approaches due to several important reasons described in [18]; (1) It is difficult to choose a dominant regime due to significant overlapping and rapid variations of $\omega_E$ vs. $\nu$, in radial and also in energy space. (2) The connection of different regimes in velocity space that addresses this has not been extended to include transport through orbit resonances [29]. (3) Orbit resonances, such as bounce-harmonic and transit-harmonic resonances, are what typically dominante transport since particles in orbit resonance do not undergo phase-mixing and are effectively in the $1/\nu$ regime [30].

### D. Force balance in drift MHD - Krook with orbit resonances

The formulation given previously for a Krook model can be easily extended for trapped particle bounce-harmonic resonances and passing particle transit-harmonic resonances when $\omega_E \sim \mathcal{O}(\omega_b, \omega_t)$. One can expand $\delta f_{k\ell s} = \sum_{\sigma\ell} \delta f_{\sigma\ell} \mathcal{P}_{\sigma\ell}$, where the phase factor $\mathcal{P}_{\sigma\ell} = e^{-i2\pi(\ell-\gamma nq)\sigma h(\theta)}$ with $h(\theta) \equiv \left(\int_{-\theta_t}^{\theta} d\theta \mathcal{J} B v_D^\alpha / v_\parallel\right) / \oint d\theta \mathcal{J} B v_D^\alpha / v_\parallel$. Here $\gamma = 1(0)$ for passing (trapped) particles.

With the expansion above, the orbit-averaged distribution function is

$$\delta f_{k\ell s} = \sum_{\sigma\ell} \frac{1}{2\pi q} \frac{in\omega_{b,t}}{i[(\ell-\gamma nq)\omega_{b,t} - n\omega_p] - \nu} \frac{\partial f_M}{\partial \chi} \delta \mathcal{J}_{\sigma\ell}, \tag{92}$$

where $\omega_{b,t}$ is the bounce frequency for trapped and passing frequency for passing particles. The action integration $\delta \mathcal{J}_{\sigma\ell}$ can be obtained simply by modifying Eqs. (56-57);

$$W_{\parallel,\sigma\ell mn} = \oint d\theta \frac{\mathcal{J} B (2E - 2\mu B)}{v_\parallel \chi'} \mathcal{P}^{\sigma\ell} e^{i(m-nq)\theta}, \tag{93}$$

$$W_{\perp,\sigma\ell mn} = \oint d\theta \frac{\mathcal{J} \mu B^2}{v_\parallel \chi'} \mathcal{P}^{\sigma\ell} e^{i(m-nq)\theta}. \tag{94}$$

Note the bounce integral $\oint d\theta$ includes each half (co and counter) for trapped particles in the present notation. Also, $\omega_t = \sigma|\omega_t|$ while $\omega_b$ is always positive. The matrix $\mathbf{M}$ is then

$$\mathbf{M}^{ij}_{k\ell s} = \mathbf{M}^{ij}_{I} - \frac{\chi'}{2\pi q M^2} \times \sum_{\sigma\sigma'\ell} \int dE d\mu \frac{in\omega_{b,t}}{i[(\ell - \alpha n q)\omega_{b,t} - n\omega_p] - \nu} \frac{\partial f_M}{\partial \chi} W^{i\dagger}_{\sigma'\ell} W^{j}_{\sigma\ell}. \tag{95}$$

The asymptotic behaviors of this formulation recover a number of interesting regimes. This formulation of the force balance recovers the collisionless kinetic energy principle derived by Porcelli [14];

$$\delta f_{po} = \lim_{\nu\to 0} \delta f_{k\ell s}, \quad \mathbf{M}^{ij}_{po} = \lim_{\nu\to 0} \mathbf{M}^{ij}_{k\ell s}. \tag{96}$$

Another interesting limit is Chew-Goldberger-Lee (CGL) double adiabatic limit, and as it has been shown that $\omega_E \to \infty$ limit of $\delta W_k$ with all $\ell$ summation is identical to $\delta W_{cgl}$ [9]. Thus,

$$\delta f_{cgl} = \lim_{\omega_E\to\infty} \delta f_{k\ell s}, \quad \mathbf{M}^{ij}_{cgl} = \lim_{\omega_E\to\infty} \mathbf{M}^{ij}_{k\ell s}. \tag{97}$$

The orbit resonance is a critical process to enhance transport and is essential to describe kinetic stabilization and neoclassical toroidal viscosity. This is true even if the plasma rotation is generally low, as expected in ITER, since local $\omega_E$ can still be large enough to resonate with $\omega_b$, $\omega_t$, or $\omega_B$ of some fraction of particles. In the self-consistent force balance, however, a subtlety arises since $\delta f_{k\ell s} = \delta f_{k\ell s}(\psi, \theta, \alpha)$, that is, the perturbed distribution function has a gradient along the field line. First this can break the parallel force balance Eq. (35) since obviously now $\hat{b}\cdot\vec{\nabla}\delta f_{k\ell s} \neq 0$, and next, the variation in the action Eq. (38) is no longer independent of $\xi_\parallel$. The parallel force balance can hold in principle if $\xi_\parallel$ is maintained to balance $\hat{b}\cdot\vec{\nabla}\delta f_{k\ell s} \neq 0$ for each species and between co and counter-rotating particles, although the meaning of $\xi_\parallel$ is ambiguous in kinetic theory. Within the scope of this paper, the force balance with orbit resonances is only an approximation. Indeed one can show $\hat{b}\cdot\delta f_{k\ell s}$ is cancelled between co and counter-rotating particles at the turning points, which typically dominate transport. More rigorous treatment of the parallel force balance with strong precession and orbit resonances will be discussed in a separate paper.

## VI. ENERGY AND TORQUE INTEGRAL

The energy and toroidal torque associated with perturbations can be obtained by integrating $\int dx^3 \left(\vec{\xi}\cdot\delta\vec{F}\right)$, where the perturbed force $\delta\vec{F} \equiv \delta\vec{j}\times\vec{B} + \vec{j}\times\delta\vec{B} - \vec{\nabla}\cdot\delta\overleftrightarrow{\mathcal{P}}$. In force balance, $\delta\vec{F} = 0$ and thus perturbations have no total energy and torque of their own. This implies that an external system, such as non-axisymmetric coils, must provide energy and torque to the plasma. If only the energy and torque of the plasma volume are considered, i.e., $\int_p dx^3 \left(\vec{\xi}\cdot\delta\vec{F}\right)$, the integral becomes

$$2\delta W + i\frac{\tau_\varphi}{n} = 2\pi\int d\psi d\theta d\varphi \mathcal{J}\left(\xi^{\psi *}\delta F_\psi - \xi^{\alpha *}\delta F_\varphi\right), \tag{98}$$

when $\delta F_\parallel = 0$. Here all the quantities are complex due to toroidal Fourier decomposition. Then Eq. (47) implies

$$2\delta W + i\frac{\tau_\varphi}{n} = 2\delta W_I - \frac{2\pi\chi'}{M^2}\int d\psi d\varphi dE d\mu \left(\delta\mathcal{J}^*\delta f\right), \tag{99}$$

where $\delta W_I$ is the perturbed energy in ideal MHD and Eq. (48) implies

$$\begin{aligned} 2\delta W + i\frac{\tau_\varphi}{n} = &-\frac{2\pi\chi'}{M^2}\int d\psi d\varphi dE d\mu \oint d\theta \times \\ &\begin{bmatrix}\xi^\psi \\ \xi^\alpha\end{bmatrix}^{*T}\begin{bmatrix}\hat{\mathcal{S}} & \hat{\mathcal{T}} \\ \hat{\mathcal{X}} & \hat{\mathcal{Z}}\end{bmatrix}^*\begin{bmatrix}w_\perp - w_\parallel \\ w_\perp\end{bmatrix}\oint d\theta\mathcal{L}\left[\begin{bmatrix}w_\perp - w_\parallel \\ w_\perp\end{bmatrix}^T\begin{bmatrix}\hat{\mathcal{S}} & \hat{\mathcal{T}} \\ \hat{\mathcal{X}} & \hat{\mathcal{Z}}\end{bmatrix}\begin{bmatrix}\xi^\psi \\ \xi^\alpha\end{bmatrix}\right]. \end{aligned} \tag{100}$$

Clearly all the operations are self-adjoint and the torque $\tau_\varphi = 0$ if $\delta f = \mathcal{L}[\delta\mathcal{J}]$ is self-adjoint without collisions. In the matrix representations used in Sec. IV, one can also show

$$\begin{aligned} 2\delta W + i\frac{\tau_\varphi}{n} = &4\pi^2\int d\psi \left(\Xi_\alpha^\dagger \mathbf{A}_k \Xi_\alpha + \Xi_\alpha^\dagger \mathbf{B}_u \Xi'_\psi + \Xi_\alpha^\dagger \mathbf{C}_u \Xi_\psi + \Xi'^\dagger_\psi \mathbf{B}_l^\dagger \Xi_\alpha + \Xi_\psi^\dagger \mathbf{C}_l^\dagger \Xi_\alpha \right. \\ &\left. + \Xi'^\dagger_\psi \mathbf{D}_k \Xi'_\psi + \Xi'^\dagger_\psi \mathbf{E}_u \Xi_\psi + \Xi_\psi^\dagger \mathbf{E}_l^\dagger \Xi'_\psi + \Xi_\psi^\dagger \mathbf{H}_l \Xi_\psi\right) \equiv 4\pi^2\int d\psi \left(\mathbf{\Xi}^\dagger\cdot\mathbf{M}\cdot\mathbf{\Xi}\right), \end{aligned} \tag{101}$$

where $\mathbf{\Xi}^T = [\Xi_\alpha, \Xi'_\psi, \Xi_\psi]$. Eliminating $\Xi_\alpha$ and using Eq. (60) on the force balance, the energy and torque integration becomes

$$\begin{aligned} 2\delta W + i\frac{\tau_\varphi}{n} &= 4\pi^2\int d\psi \left[\Xi_\psi^\dagger\left(\mathbf{F}_k\Xi'_\psi + \mathbf{K}_u\Xi_\psi\right)\right]' - 4\pi^2\int d\psi\left[\Xi_\psi^\dagger\left((\mathbf{F}_k\Xi'_\psi + \mathbf{K}_u\Xi_\psi)' - (\mathbf{K}_l^\dagger\Xi'_\psi + \mathbf{G}_k\Xi_\psi)\right)\right] \\ &= 4\pi^2\Xi_\psi^\dagger\left(\mathbf{F}_k\Xi'_\psi + \mathbf{K}_u\Xi_\psi\right), \end{aligned} \tag{102}$$

since the volumetric term vanishes by Euler-Lagrange equation (65).

### A. Plasma response matrix

The equation above can be rewritten as

$$2\delta W + i\frac{\tau_\varphi}{n} = \Xi_\psi^\dagger \mathbf{R}_P \Xi_\psi, \tag{103}$$

where $\mathbf{R}_P$ is the plasma response matrix consistent with force balance. $\mathbf{R}_P$ can be obtained if one solves general solutions of the Euler-Lagrange equation. Let $\mathbf{\Xi}_\psi$ be a $M \times M$ matrix containing $M$ linearly independent $\Xi_\psi$ solutions in each column, then

$$\mathbf{R}_P = 4\pi^2 \left(\mathbf{F}_k \mathbf{\Xi}_\psi' + \mathbf{K}_u \mathbf{\Xi}_\psi\right) \mathbf{\Xi}_\psi^{-1}. \tag{104}$$

The plasma response matrix $\mathbf{R}_P(\psi)$ is non-Hermitian, containing information about both energy and torque driven in the plasma region $(0, \psi]$ associated with the plasma displacement $\Xi_\psi(\psi)$. The loss of Hermiticity in the plasma response matrix is the manifestation of non-self-adjointness. In this case, the stability of the system cannot be determined merely by the sign of the minimum eigenvalue of $\mathbf{R}_P$, but requires the dispersion relation with kinetic inertia and appropriate boundary conditions such as a resistive wall. In terms of perturbed equilibrium, however, one can still address how much the system will gain or lose energy and torque through each eigenmode using the eigenvalues of separate eigendecompositions of the Hermitian and non-Hermitian parts of $\mathbf{R}_P$ respectively.

### B. Torque response matrix

The anti-Hermitian part of $\mathbf{R}_P$ provides the torque associated with the plasma displacement for each surface $\psi$. It is more practical, however, to relate the the torque and its profile the the driving, external non-axisymmetric perturbations. For this, one can relate $\Xi_\psi(\psi)$ to the total perturbed field measured at the plasma boundary $\mathbf{\Phi}$ at $\psi = \psi_b$, and then to the external perturbed field using the permeability matrix $\mathbf{P}$ through the virtual casing principle, $\mathbf{\Phi} = \mathbf{P}\mathbf{\Phi}^x$[15]. The total field at the boundary is related to the plasma displacement at the boundary $\Xi_{\psi b} \equiv \Xi_\psi(\psi_b)$ with $\mathbf{\Phi} = \chi'\mathbf{Q}\Xi_{\psi b}$. All together,

$$\begin{aligned}
2\delta W + i\frac{\tau_\varphi}{n} &= \Xi_{\psi b}^\dagger (\mathbf{\Xi}_\psi \mathbf{\Xi}_{\psi b}^{-1})^\dagger \mathbf{R}_P (\mathbf{\Xi}_\psi \mathbf{\Xi}_{\psi b}^{-1}) \Xi_{\psi b}, \\
&= \mathbf{\Phi}^{x\dagger} \left(\mathbf{\Xi}_\psi \mathbf{\Xi}_{\psi b}^{-1} \mathbf{Q}^{-1} \mathbf{P}/\chi'\right)^\dagger \mathbf{R}_P \left(\mathbf{\Xi}_\psi \mathbf{\Xi}_{\psi b}^{-1} \mathbf{Q}^{-1} \mathbf{P}/\chi'\right) \mathbf{\Phi}^x, \\
&= \mathbf{\Phi}^{x\dagger} \mathbf{\Lambda}^{-1}(\psi) \mathbf{\Phi}^x,
\end{aligned} \tag{105}$$

where the last expression is similar to one by Boozer [31], but $\mathbf{\Lambda}(\psi)$ is the inductance function of plasma for $(0, \psi]$. Taking only the Hermitian part, one can show

$$\delta W = \mathbf{\Phi}^{x\dagger} \frac{\left(\mathbf{\Lambda}^{-1} + \mathbf{\Lambda}^{-1\dagger}\right)}{4} \mathbf{\Phi}^{x} \equiv \mathbf{\Phi}^{x\dagger} \mathbf{W}_X(\psi) \mathbf{\Phi}^{x}, \tag{106}$$

where $\mathbf{W}_X$ is the energy response matrix function relating the energy inside $\psi$ to the external field applied on the plasma boundary. Similarly, taking the anti-Hermitian part gives

$$\tau_\varphi = \mathbf{\Phi}^{x\dagger} \frac{n\left(\mathbf{\Lambda}^{-1} - \mathbf{\Lambda}^{-1\dagger}\right)}{2i} \mathbf{\Phi}^{x} \equiv \mathbf{\Phi}^{x\dagger} \mathbf{T}_X(\psi) \mathbf{\Phi}^{x}, \tag{107}$$

and $\mathbf{T}_X$ is the torque response matrix function for external fields at the boundary.

The torque response matrix function is a unique and significant result of the formulation presented in this paper. $\mathbf{T}_X(\psi)$ contains all the information about any possible torques from external fields, for a given axisymmetric equilibrium. The torque here is identical to what is called NTV torque, and is self-consistent with the first order perturbed equilibrium force balance. Therefore optimization of external fields for NTV applications is just a matter of examining $\mathbf{T}_X(\psi)$. In fact NTV represents neoclassical transport driven by non-axisymmetric fields in tokamaks and thus $\mathbf{T}_X(\psi)$ provides a method of systematic 3D neoclassical optimization in tokamaks, which can also be extended to stellarators with the relevant drift-kinetic model. Note that in the past, NTV or neoclassical optimization of the external 3D field has been considered a complicated non-linear problem requiring the applications of various non-linear optimizers such as STELLOPT [32].

The torque response matrix function itself is obviously Hermitian by Eq. (107), having eigenvalues and eigenvectors. For example, the maximum eigenvalue is the maximum torque inducible inside a given radius $(0, \psi]$ with unit normalized field or power, and its eigenvector corresponds to the external field required to generate that maximum torque. Similarly, the minimum eigenvalue and eigenvector correspond the minimum torque that any unit external field can produce and so on. More complex optimizations are possible as well, such as localized NTV optimization between $(\psi_1, \psi_2)$, when total torque is fixed or power of field at the boundary is fixed. The $\mathbf{T}_X(\psi)$ may be required to be positive definite for some of these problems, as otherwise constraints such as fixed total torque can make the problem singular. However, as long as the questions and constraints are well-posed, $\mathbf{T}_X(\psi)$ turns the very complicated non-linear neoclassical 3D optimizations into simple quadratic matrix optimization problems.

## VII. DRIVEN NON-AXISYMMETRIC FORCE BALANCE

The direct treatment of non-axisymmetric force balance described in previous sections enables the simple addition of any arbitrary force $\delta\vec{\mathcal{F}} = \delta\mathcal{F}_\psi\vec{\nabla}\psi + \delta\mathcal{F}_\theta\vec{\nabla}\theta + \delta\mathcal{F}_\varphi\vec{\nabla}\varphi$. The parallel force balance in Sec. IV.B indicates $\hat{b}\cdot\delta\vec{\mathcal{F}} = 0$ to have a solution, giving a constraint $\vec{\nabla}\delta\mathcal{F}_\theta = -q\vec{\nabla}\delta\mathcal{F}_\varphi$ and thus $\delta\vec{\mathcal{F}} = \delta\mathcal{F}_\psi\vec{\nabla}\psi - \delta\mathcal{F}_\varphi\vec{\nabla}\alpha$. The toroidal and radial components of force balance are then simply

$$\mathbf{A}_k\Xi_\alpha + \mathbf{B}_u\Xi'_\psi + \mathbf{C}_u\Xi_\psi = \mathcal{F}_\varphi, \tag{108}$$

$$\left(\mathbf{D}_k\Xi'_\psi + \mathbf{E}_u\Xi_\psi + \mathbf{B}_l^\dagger\Xi_\alpha\right)' = \mathbf{E}_l^\dagger\Xi'_\psi + \mathbf{H}_k\Xi_\psi + \mathbf{C}_l^\dagger\Xi_\alpha + \mathcal{F}_\psi, \tag{109}$$

where $\mathcal{F}_{(\psi,\varphi)}$ represents a vector with Fourier elements of each $\delta\mathcal{F}_{(\psi,\varphi)}$. The Euler-Lagrange equation becomes an inhomogeneous vector differential equation

$$(\mathbf{F}_k\Xi'_\psi + \mathbf{K}_u\Xi_\psi)' - (\mathbf{K}_l^\dagger\Xi'_\psi + \mathbf{G}_k\Xi_\psi) = \mathcal{F}_d, \tag{110}$$

where the driving force $\mathcal{F}_d$ is given by

$$\mathcal{F}_d = -(\mathbf{B}_l\mathbf{A}_k^{-1}\mathcal{F}_\varphi)' + \mathbf{C}_l\mathbf{A}_k^{-1}\mathcal{F}_\varphi + \mathcal{F}_\psi. \tag{111}$$

The particular solution of this equation will change the internal structure of the perturbed magnetic field and displacement as well as the coupling to vacuum region. It will also change the energy and torque associated with the perturbation. Eliminating $\Xi_\alpha$ by Eq. (108) and using Eq. (111) to eliminate the volumetric term, one can obtain

$$2\delta W + i\frac{\tau_\varphi}{n} = 4\pi^2\Xi_\psi^\dagger\left(\mathbf{F}_k\Xi'_\psi + \mathbf{K}_u\Xi_\psi + \mathbf{B}_l^\dagger\mathbf{A}_k^{-1}\mathcal{F}_\varphi\right), \tag{112}$$

which is similar to Eq. (102).

## VIII. SUMMARY AND DISCUSSION

This paper presents a new Euler-Lagrange equation derived from force balance with first order anisotropic pressure driven by non-axisymmetric fields. Like Glasser's ideal Euler-Lagrange equation, the new form is toroidally generalized using magnetic coordinates. As just discussed, an arbitrary driven force $\delta\vec{F}$ can also be added as an inhomogeneous term of Euler-Lagrange equation. Comparing the Newcomb cylindrical equation, Glasser toroidal equation, the toroidal anisotropic equation and the toroidal driven force balance equation,

- $(f\xi')' - g\xi = 0,$
- $(\mathbf{F}_i \Xi'_\psi + \mathbf{K}_i \Xi_\psi)' - (\mathbf{K}_i^\dagger \Xi'_\psi + \mathbf{G}_i \Xi_\psi) = 0,$
- $(\mathbf{F}_k \Xi'_\psi + \mathbf{K}_u \Xi_\psi)' - (\mathbf{K}_l^\dagger \Xi'_\psi + \mathbf{G}_k \Xi_\psi) = 0,$
- $(\mathbf{F}_k \Xi'_\psi + \mathbf{K}_u \Xi_\psi)' - (\mathbf{K}_l^\dagger \Xi'_\psi + \mathbf{G}_k \Xi_\psi) = \mathcal{F}_d.$

The cylindrical Newcomb equation is a scalar differential equation for each $(m, n)$, but the toroidally generalized version by Glasser is subjected to poloidal mode coupling, leading to a vector differential equation. It is quite straightforward to extend Glasser's equation to full 3D geometry, e. g. stellarator geometry, with a poloidally and toroidally coupled vector differential equation. This is future work, and an important issue will be how to properly treat ideal constraints near resonant surfaces when magnetic surfaces may be intrinsically abscent in the unperturbed state. Including the anisotropic pressure tensor, the force balance equation is generally not self-adjoint, which is manifested in the non-Hermitian matrices $\mathbf{F}_k, \mathbf{K}_{(u,l)}, \mathbf{G}_k$. The resulting equation is called a non-Hermitian Euler-Lagrange equation for $\delta W$, although it is not a result of a variational method but derived directly from the three components of force balance. Finally, the inhomogeneous non-Hermitian Euler-Lagrange equation can be constructed if an arbitrary force is driven in the plasma volume.

The new matrices $\mathbf{F}_k, \mathbf{K}_{(u,l)}, \mathbf{G}_k$ are composite matrices with 9 modified matrices $\mathbf{M}$, as shown in Eqs. (66-69). The matrices in $\mathbf{M}$ contain action integrals in general geometry and can be calculated in integral form if a perturbed distribution function is given, as presented in Sec. IV C. As shown in Sec. V, the matrices $\mathbf{M}$ can be obtained in various drift-kinetic models, including collisionless Kruskal-Oberman, CGL, Pocelli, or collisional $1/\nu$-regime, $\nu$-regime, SBP-regime, and combined formulation for orbit resonances in general tokamak geometry. When the collisional effects are accounted for, the method yields force balance self-consistent with neoclassical torque by $\vec{\nabla} \cdot \delta \overleftrightarrow{\mathcal{P}}$.

The energy and torque integral with this force balance represents physical quantities of the second order in perturbations as shown in Eq. (102). When the Euler-Lagrange equation is solved for $M$ linearly independent solutions, one can construct the general plasma response matrix, which is non-Hermitian with energy and torque. Changing basis from displacements to external fields and taking the anti-Hermitian part, one can derive the torque response matrix, Eq. (107). The torque response matrix function $\mathbf{T}_X(\psi)$ provides all the information

for self-consistent NTV torque profile variations that external magnetic perturbations can possibly generate. It can thus be used to systematically optimize fields for desired torque profiles.

The numerical implementation of the formulations presented here are straightforward when $\mathbf{M}$ is supplied by subroutines, by integrating the new Euler-Lagrange equation with $M$ linearly independent boundary conditions, and by coupling the solutions to external systems. As this procedure has already been successfully implemented in DCON and IPEC, these codes have been extended to build a general perturbed equilibrium code (GPEC). GPEC is not a stability code unless a Hermitian kinetic limit is taken (e. g. Kruskal-Oberman), but it does calculate the kinetic force balance as well as self-consistent NTV torque. The numerical implementation and applications of GPEC will be presented in separate works.

The addition of an arbitrary driven force gives the inhomogeneous Euler-Lagrange equation for $\delta W$, which will have important future applications. For example, perturbed equilibrium could be calculated consistent with non-axisymmetric neutral beam injection torque. Another interesting example is the incorporation of NTV torque calculations by first-principle transport codes such as XGC0 [33], POCA [34], FORTEC-3D [35] into the general perturbed equilibrium code through this inhomogeneous term. When a transport code supplies the perturbed distribution function and first-order neoclassical torque as a function of space based on a given $\delta\vec{B}$ structure, one can solve the inhomogeneous Euler-Lagrange equation to update $\delta\vec{B}$. This iterative process provides a unique path to integrating perturbed equilibrium and computationally demanding transport codes.

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### Appendix A: Matrix operators on general coordinates

The ideal matrices used in Eqs. (21-22), or $\mathbf{M}_i$ in Eq. (63), are functions of geometry, $p'$ and $q$. Following Glasser's notations for metric tensors, define the index $1,2,3$ for $(\psi,\theta,\varphi)$ and the geometric matrix

$$(\mathbf{G}_{ab})_{mm'} \equiv \frac{1}{2\pi}\oint d\theta e^{i(m'-m)\theta}\frac{\vec{e}_a\cdot\vec{e}_b}{\mathcal{J}}, \tag{A1}$$

where $\vec{e}_{(a,b)}$ are covariant basis vectors. Also define $\mathbf{M}\equiv m\delta_{mm'}$, $\mathbf{Q}\equiv(m-nq)\delta_{mm'}$, jacobian matrix

$$\mathbf{J}\equiv\frac{1}{2\pi}\oint d\theta e^{i(m'-m)\theta}\mathcal{J}. \tag{A2}$$

Then $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$, $\mathbf{E}_i$, $\mathbf{H}_i$ matrices are given by

$$\mathbf{A}_i = \chi'^2\left[n(n\mathbf{G}_{22}+\mathbf{G}_{23}\mathbf{M})+\mathbf{M}(n\mathbf{G}_{32}+\mathbf{G}_{33}\mathbf{M})\right] \tag{A3}$$

$$\mathbf{B}_i = -i\chi'^2\left[n(\mathbf{G}_{22}+q\mathbf{G}_{23})+\mathbf{M}(\mathbf{G}_{32}+q\mathbf{G}_{33})\right] \tag{A4}$$

$$\begin{aligned}\mathbf{C}_i = &-i\chi'\left[\chi''(\mathbf{M}\mathbf{G}_{32}+n\mathbf{G}_{22})+(q\chi')'(\mathbf{M}\mathbf{G}_{33}+n\mathbf{G}_{23})\right]\\ &-\chi'^2(\mathbf{M}\mathbf{G}_{31}\mathbf{Q}+n\mathbf{G}_{21}\mathbf{Q})+i(2\pi\chi' f'\mathbf{Q}-np'\mathbf{J})\end{aligned} \tag{A5}$$

$$\mathbf{D}_i = \chi'^2(\mathbf{G}_{22}+q\mathbf{G}_{23}+q\mathbf{G}_{32}+q^2\mathbf{G}_{33}) \tag{A6}$$

$$\begin{aligned}\mathbf{E}_i = &\chi'\left[\chi''(\mathbf{G}_{22}+q\mathbf{G}_{23})+(q\chi')'(\mathbf{G}_{32}+q\mathbf{G}_{33})\right]\\ &-i\chi'^2(\mathbf{G}_{21}+q\mathbf{G}_{31})\mathbf{Q}+p'\mathbf{J}\end{aligned} \tag{A7}$$

$$\begin{aligned}\mathbf{H}_i = &\chi''(\chi''\mathbf{G}_{22}+(q\chi')'\mathbf{G}_{23})+(q\chi')'(\chi''\mathbf{G}_{23}+(q\chi')'\mathbf{G}_{33})\\ &+i\chi'\left[\chi''(\mathbf{M}\mathbf{G}_{12}-\mathbf{G}_{21}\mathbf{M})+q(\chi')'(\mathbf{M}\mathbf{G}_{13}-\mathbf{G}_{31}\mathbf{M})\right]\\ &+\chi'^2\mathbf{Q}\mathbf{G}_{11}\mathbf{Q}+p'(\chi''\mathbf{J}/\chi'+\mathbf{J}')-2\pi f'q'\chi'\mathbf{I}.\end{aligned} \tag{A8}$$

These matrices are identical to (A6) in [7], except the differences in normalization due to $\Xi_s=\chi'\Xi_\alpha$ and $(\theta,\varphi)$ defined in $(0,2\pi)$ rather than $(0,1)$, concluding the equivalence between the minimum energy state and force balance. The composite matrices in Eqs. (24-26) in the toroidal Newcomb equation can be decomposed further with respect to the singular factor $\mathbf{Q}$. To do this, define $\mathbf{b}\equiv i(\chi'^2/n)(n\mathbf{G}_{23}+\mathbf{M}\mathbf{G}_{33})$ and rewrites $\mathbf{B}_i=-(i/n)\mathbf{A}_i+\mathbf{b}\mathbf{Q}$. Then one can easily show $\mathbf{F}_i=\mathbf{Q}\bar{\mathbf{F}}_i\mathbf{Q}$ and $\mathbf{K}_i=\mathbf{Q}\bar{\mathbf{K}}_i$ with

$$\bar{\mathbf{F}}_i = (\chi'/n)^2\mathbf{G}_{33}-\mathbf{b}^\dagger\mathbf{A}_i^{-1}\mathbf{b}, \tag{A9}$$

$$\begin{aligned}\bar{\mathbf{K}}_i = &-(\chi'/n)(\chi''\mathbf{G}_{23}+(q\chi')'\mathbf{G}_{33}-i\chi'\mathbf{G}_{31}\mathbf{Q}-2\pi f'\mathbf{I})\\ &-\mathbf{b}^\dagger\mathbf{A}_i^{-1}\mathbf{C}_i,\end{aligned} \tag{A10}$$

which are again identical to the definitions in [7].

The non-Hermitian composite matrices in Eqs. (66-69) for the new Euler-Lagrange equation can also be decomposed with respect to $\mathbf{Q}$, as presented in Eqs. (70-72). Define the kinetic correction as $\mathbf{M} = \mathbf{M}_i + \mathbf{M}_a$, for example, $\mathbf{A}_k = \mathbf{A}_i + \mathbf{A}_a$, $\mathbf{B}_u = \mathbf{B}_i + \mathbf{B}_{au}$, and $\mathbf{B}_l = \mathbf{B}_i + \mathbf{B}_{al}$ and also write $\mathbf{b}_{au} \equiv \mathbf{B}_{au} + (i/n)\mathbf{A}_a$ and $\mathbf{b}_{al} \equiv \mathbf{B}_{al} + (i/n)\mathbf{A}_a$. Then one can show

$$\bar{\mathbf{F}}_k = (\chi'/n)^2 \mathbf{G}_{33} - \mathbf{b}^\dagger \mathbf{A}_k \mathbf{b} \tag{A11}$$

$$\begin{aligned}\bar{\mathbf{K}}_u = &-\mathbf{b}^\dagger \mathbf{A}_k^{-1} \mathbf{C}_u \\ &- (\chi'/n)(\chi'' \mathbf{G}_{23} + (q\chi')' \mathbf{G}_{33} - i\chi' \mathbf{G}_{31}\mathbf{Q} - 2\pi f' \mathbf{I})\end{aligned} \tag{A12}$$

$$\begin{aligned}\bar{\mathbf{K}}_l = &-\mathbf{C}_l^\dagger \mathbf{A}_k^{-1} \mathbf{b} \\ &- (\chi'/n)(\chi'' \mathbf{G}_{23} + (q\chi')' \mathbf{G}_{33} - i\chi' \mathbf{G}_{31}\mathbf{Q} - 2\pi f' \mathbf{I})\end{aligned} \tag{A13}$$

$$\mathbf{P}_u = \mathbf{b}^\dagger \mathbf{A}_k^{-1} \mathbf{b}_{au} \tag{A14}$$

$$\mathbf{P}_l = \mathbf{b}^\dagger \mathbf{A}_k^{\dagger -1} \mathbf{b}_{al} + (i/n)\mathbf{b}^\dagger (\mathbf{I} - \mathbf{A}_k^{\dagger -1} \mathbf{A}_k) \tag{A15}$$

$$\begin{aligned}\mathbf{R}_1 = &\mathbf{D}_a - \mathbf{A}_a^\dagger / n^2 + (i/n)\mathbf{b}_{al}^\dagger \\ &- (i/n)\mathbf{A}_k^\dagger \mathbf{A}_k^{-1} \mathbf{b}_{au} - \mathbf{b}_{al}^\dagger \mathbf{A}_k^{-1} \mathbf{b}_{au}\end{aligned} \tag{A16}$$

$$\begin{aligned}\mathbf{R}_2 = &\mathbf{E}_{au} - (i/n)\mathbf{C}_{au} \\ &+ (i/n)(\mathbf{I} - \mathbf{A}_k^\dagger \mathbf{A}_k^{-1})\mathbf{C}_u - \mathbf{b}_{al}^\dagger \mathbf{A}_k^{-1} \mathbf{C}_u\end{aligned} \tag{A17}$$

$$\mathbf{R}_3 = \mathbf{E}_{al}^\dagger + (i/n)\mathbf{C}_{al}^\dagger - \mathbf{C}_l^\dagger \mathbf{A}_k^{-1} \mathbf{b}_{au}. \tag{A18}$$

The matrices $\bar{\mathbf{F}}_k$ and $\bar{\mathbf{K}}_{u,l}$ are similar to the ideal ones but with kinetic corrections through $\mathbf{A}_k$ and $\mathbf{C}_{u,l}$. Assuming the kinetic corrections are small, i.e. $|\mathbf{M}_I| \gg |\mathbf{M}_a|$ one can see the matrices $\mathbf{P}_{u,l}$ and $\mathbf{R}_{1,2,3}$ are all small in size. Thus, despite the absence of the singularity, it can still be important to separate the $\mathbf{Q}$ factor in numerical implementations.


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