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Variational principles for dissipative (sub)systems, with applications to the theory of dispersion and geometrical optics

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Applications of variational methods are typically restricted to conservative systems. Some extensions to dissipative systems have been reported too but require *ad hoc* techniques such as artificial doubling of variables. Here, a different approach is proposed. We show that, for a broad class of dissipative systems of practical interest, variational principles can be formulated using *constant* Lagrange multipliers and Lagrangians nonlocal in time, which allow treating reversible and irreversible dynamics on the same footing. A general variational theory of linear dispersion is formulated as an example. In particular, we present a variational formulation for linear geometrical optics in a general dissipative medium, which is allowed to be nonstationary, inhomogeneous, nonisotropic, and exhibit both temporal and spatial dispersion simultaneously.

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I. INTRODUCTION

A. Motivation

Variational methods (VM) have long been known as powerful tools in theoretical physics, especially in the context of reduced modeling. As opposed to approximating differential equations, approximating the action functional guarantees that the corresponding model too has a variational structure by definition. This implies that conservation laws are also generally preserved, so a model inherits key features of the original system notwithstanding the reduction [1]. One of the areas that particularly benefit from this fact is wave theory [2, 3], where even crude approximations to an action functional often yield insightful reduced models [4–10]. The advantages of VM become especially obvious when one deals with media such as plasmas, which can be nonstationary, inhomogeneous, anisotropic, and exhibit both temporal and spatial dispersion simultaneously [11–17]. First-principle differential equations are often unrealistic to handle in this case, whereas VM allow for simple and intuitive modeling. Specifically, a wave theory can be constructed as an axiomatic field theory without even specifying the waves’ nature (electromagnetic, acoustic, etc.). Defining the action functional explicitly is needed only to connect the resulting general equations with specific quantities of interest [4–6]. For example, this provides a convenient alternative to using Maxwell’s equations for electromagnetic waves [4], which are generally complicated [18] (as opposed to the corresponding VM) even in the geometrical-optics (GO) limit [19].

But which exact variational principle does one begin

with? The microscopic least action principle (LAP) is typically available in the form $\delta A[q, \xi] = 0$, where q describes the wave field, ξ describes the medium, and A is a known functional. However, practical applications in the context described above require a different variational principle, where q is the *only* independent variable, while the medium’s degrees of freedom are “integrated out”, i.e., somehow expressed through q . If the medium response is adiabatic (e.g., if $\xi[q]$ can be approximated with a local function), an effective LAP for q can be formulated simply as $\delta A[q, \xi[q]] = 0$, as will also be discussed below. This principle has enjoyed insightful applications, e.g., in the theory of generalized ponderomotive forces [15–17] and nonlinear plasma waves [11–14]. However, when the medium response is not adiabatic, the dynamics of q becomes dissipative. Then, the mentioned effective LAP becomes inapplicable, and the problem requires a more subtle approach. This issue has been recently receiving attention in plasma physics [20] and is also important for advancing variational formulations of plasma and wave kinetics such as in Ref. [15]. The purpose of the present paper is to find the appropriate modification of the effective LAP that would incorporate dissipation under the above assumptions.

B. Historical context

Accommodating dissipative effects within the LAP has a long history (e.g., cf. Ref. [21]) both as a formal problem of finding a variational principle for a given equation [22–39] and also in modeling physical systems, for example, quantizing radiation in dissipative media [40–52]. It is hardly possible to overview the topic here, so the given references are intended as representative but not exhaustive. In any case, the subject of our paper is different from those discussed in literature, particularly, in the following aspects:

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- We do not address the problem of finding a LAP for a system with general dissipation. Instead, we consider only emergent dissipation in nonconservative subsystems of systems that are overall conservative.
- We avoid the standard approach (known under many different names [22–25, 53]) where additional independent functions are introduced that serve as Lagrange multipliers [54]. In our formulation, only time-independent Lagrange multipliers are used.
- We avoid variable transformations that are used (as in, e.g., Refs. [26–32]) to compensate the loss of the phase space volume caused by dissipation.
- In contrast to theories discussed in quantum contexts, our theory is not statistical. Instead, we search for a LAP that describes deterministic dynamics on a finite time interval for a given initial state of the medium. Thus, we cannot describe wave fields using the standard Fourier representation, and we also have to specify boundary conditions (BC) in time, which are often ignored. In fact, it is the attention to these BC that helps accommodate dissipation within a LAP.
- Instead of modeling media as collections of stationary oscillators (as in, e.g., Refs. [40, 46, 48]), we allow for media that are nonstationary, inhomogeneous, anisotropic, and have both temporal and spatial dispersion simultaneously. Moreover, our approach allows treating classical and quantum degrees of freedom on the same footing.
- Our theory is formulated within the standard calculus of variations. We avoid exotic constructs such as complex actions or fractional derivatives [33–35]. Feynman integrals, which are used in related quantum theories (e.g., see Ref. [46]), are also avoided.

C. Main results and outline

Our main results are as follows. We formulate a variational principle for a dissipative subsystem of an overall-conservative system by introducing constant Lagrange multipliers and Lagrangians nonlocal in time. We call it the variational principle for projected dynamics (VPPD). Although the work is originally motivated by needs of the plasma wave theory, our results are applicable to general dynamical systems just as well. In the first part of the paper (Secs. II-IV), we introduce the general approach where $\xi[q]$ is an arbitrary nonlinear functional and derive an exact effective LAP for q . In the second part (Sec. V), we elaborate on applications of this approach to the important special case of linear $\xi[q]$, which corresponds to linear media. Our formulation is equally applicable to both classical and quantum media. In the third part (Secs. VI and VII), we discuss various asymptotic approximations of the effective LAP for oscillations

and waves in linear media, including the quasistatic and GO approximation. To our knowledge, this is the *first variational formulation of dissipative GO* in a general linear medium, which is allowed to be nonstationary, inhomogeneous, nonisotropic, and exhibits both temporal and spatial dispersion simultaneously. Also notably, the “spectral representation” of the dielectric tensor $\epsilon_{(\omega, \mathbf{k})}$ that enters the GO equations is shown to be, strictly speaking, the Weyl symbol of the medium dielectric permittivity. (Previously, it has been known that Weyl symbols emerge naturally in wave kinetics [55–57].) This can be considered as an invariant definition of $\epsilon_{(\omega, \mathbf{k})}$, as opposed to *ad hoc* definitions used in literature that have been a source of a continuing debate [58–60].

Our work is also intended as a stepping stone for extending the variational theory of modulational stability in general wave ensembles [15] to dissipative systems. Details will be described in a separate publication.

II. NOTATION

The following notation is used throughout the paper. The symbol \doteq denotes definitions, and $(\dots)^T$ denotes transposition. For example, for a column vector q [Eq. (1)], one has $q^T = (q_1, q_2, \dots, q_N)$, which is a row vector; i.e., in this case, $(\dots)^T$ merely lowers the index. We assume $a_n = \delta_{nm} a^m = a^n$ for any a , where δ_{nm} is the Kronecker delta. (Summation over repeated indices is assumed unless specified otherwise.) Hence, in principle, a_n and a^n could be considered equivalent, but we prefer to retain the distinction because some quantities (such as coordinates) are naturally defined with upper indexes, while others (such as momenta) are naturally defined with lower indexes. Introducing a more fundamental geometrical interpretation will not be needed for our purposes. Note also that some indexes will be used *just* to introduce new symbols. This applies to all scalars (e.g., t_1 and t_2) and to some vectors (e.g., q_0 , q_T , ξ_0 , ξ'_0). We also define $a \cdot b \doteq a_n b^n$. The symbol ∂_a will denote $\partial/\partial a$. The notation “ δa :” will denote that the equation is obtained by extremizing a corresponding action functional with respect to a . The remaining notation is introduced within the main text. Finally, the abbreviations used in the text are summarized as follows:

BC	–	boundary condition(s),
ELE	–	Euler-Lagrange equation(s),
GO	–	geometrical optics,
LAP	–	least action principle,
VM	–	variational method(s),
VPPD	–	variational principle for projected dynamics.

III. LAGRANGIAN FORMULATION

We start by introducing the general formalism of Lagrangian mechanics. Although this formalism is com-

monly found in textbooks (e.g., see Ref. [61]), it is essential to restate it here, so we can explicitly reference specific definitions and equations later in the text.

A. Single system

Consider a Lagrangian system $\mathcal{S}^{(q)}$ characterized by some real coordinate vector

$$q \doteq \begin{pmatrix} q^1 \\ q^2 \\ \vdots \\ q^N \end{pmatrix}. \quad (1)$$

The dynamics of $\mathcal{S}^{(q)}$ on a time interval $t \in (t_1, t_2)$ is governed by the LAP. We call this interval T , and we also use the symbol T to denote its duration, $T \doteq t_2 - t_1$. We assume the action functional $A^{(q)}$ to have the form

$$A^{(q)}[q] = \int_{t_1}^{t_2} L^{(q)}(t, q, \dot{q}) dt, \quad (2)$$

where $L^{(q)}$ is called a Lagrangian. Then, the standard formulation of the LAP is as follows:

$$\delta A^{(q)}[q] = 0, \quad (3)$$

$$\delta q(t_{1,2}) = 0. \quad (4)$$

[Here and further, the notation $q(t_{1,2})$ means “both $q(t_1)$ and $q(t_2)$ ”. Note that Eq. (4) implies BC

$$q(t_1) = q_0, \quad q(t_2) = q_T, \quad (5)$$

where q_0 and q_T are some given constants.

From Eq. (2), one obtains

$$\delta A^{(q)} = (p \cdot \delta q) \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} [\partial_q L^{(q)} - \dot{p}] \delta q dt, \quad (6)$$

where we introduced the “canonical momentum”

$$p = (p_1, p_2, \dots, p_N) \quad (7)$$

with $p_n \doteq \partial L^{(q)}(t, q, \dot{q}) / \partial \dot{q}^n$, or, more compactly, $p \doteq \partial_{\dot{q}} L^{(q)}$. The first term on the right in Eq. (6) vanishes because $q(t_{1,2})$ are fixed. Hence, the LAP leads to the following Euler-Lagrange equations (ELE):

$$\delta q : \quad 0 = \partial_q L^{(q)} - \dot{p}. \quad (8)$$

This is a second-order equation for the N -dimensional coordinate q , so the $2N$ BC (5) (two per each component of q) provide just the right number of parameters to define a solution of Eq. (8). Alternatively, one can also approach Eq. (8) as an initial-value problem; then, the condition on $q(t = t_2)$ would need to be replaced with a condition on $\dot{q}(t = t_1)$. It is straightforward to extend this formulation to Lagrangians that depend also on higher-order derivatives, but we will not consider such extensions for the sake of brevity.

B. Coupled systems

Now suppose that $\mathcal{S}^{(q)}$ is coupled to another Lagrangian system $\mathcal{S}^{(\xi)}$, which we call the “medium”. Suppose also that $\mathcal{S}^{(\xi)}$ is characterized by some M -dimensional coordinate ξ . The dynamics of the resulting system \mathcal{S} is governed by the LAP $\delta A[q, \xi] = 0$, where A is the total action. For clarity, we adopt it in the form $A = \int_{t_1}^{t_2} L dt$, where L is the Lagrangian given by

$$L = \Lambda^{(q)}(t, q, \dot{q}) + \Lambda^{(\xi)}(t, \xi, \dot{\xi}) + \Lambda^{(\text{int})}(t, q, \dot{q}, \xi, \dot{\xi}). \quad (9)$$

The assumed BC are

$$\delta q(t_{1,2}) = 0, \quad \delta \xi(t_{1,2}) = 0, \quad (10)$$

which introduce $2(M+N)$ BC similar to Eq. (5), namely,

$$q(t_1) = q_0, \quad q(t_2) = q_T, \quad \xi(t_1) = \xi_0, \quad \xi(t_2) = \xi_T. \quad (11)$$

Like in Sec. III A, one then obtains

$$\begin{aligned} \delta A[q, \xi] &= (p \cdot \delta q) \Big|_{t_1}^{t_2} + (\eta \cdot \delta \xi) \Big|_{t_1}^{t_2} \\ &+ \int_{t_1}^{t_2} [(\partial_q L^{(q)} - \dot{p}) \delta q + (\partial_\xi L^{(\xi)} - \dot{\eta}) \delta \xi] dt, \end{aligned} \quad (12)$$

where we introduced

$$L^{(q)} \doteq \Lambda^{(q)} + \Lambda^{(\text{int})}, \quad L^{(\xi)} \doteq \Lambda^{(\xi)} + \Lambda^{(\text{int})}, \quad (13)$$

$$p \doteq \partial_{\dot{q}} L^{(q)}, \quad \eta \doteq \partial_{\dot{\xi}} L^{(\xi)}. \quad (14)$$

The first two terms on the right side of Eq. (12) vanish due to Eqs. (10), so the LAP yields the following ELE:

$$\delta q : \quad 0 = \partial_q L^{(q)} - \dot{p} \equiv F^{(q)}, \quad (15)$$

$$\delta \xi : \quad 0 = \partial_\xi L^{(\xi)} - \dot{\eta} \equiv F^{(\xi)}. \quad (16)$$

Equations (15) and (16) are second-order equations for $q(t)$ and $\xi(t)$, so the mentioned BC provide just enough parameters to define a solution.

C. Projected dynamics

Although Eqs. (15) and (16) form a self-consistent system, it is also convenient to approach Eq. (16) formally as if q were as a prescribed function. In that case, one can, in principle, solve for ξ and express it as some functional $\hat{\xi} = \xi[q]$, provided that one is given a $2M$ -dimensional integration constant C to determine a solution unambiguously. The resulting system is characterized by q alone and is not conservative. Our goal is to derive an effective variational principle for this subsystem, or, in other words, for the dynamics of \mathcal{S} “in projection” on $\mathcal{S}^{(q)}$.

First, consider the variation of A evaluated on $[q, \hat{\xi}]$:

$$\begin{aligned}\delta A[q, \hat{\xi}] &= (p \cdot \delta q) \Big|_{t_1}^{t_2} + (\hat{\eta} \cdot \delta \hat{\xi}) \Big|_{t_1}^{t_2} + \delta \mathcal{F}^{(q)} + \delta \mathcal{F}^{(\xi)}, \\ \delta \mathcal{F}^{(q)} &\doteq \int_{t_1}^{t_2} F^{(q)}(t, q, \dot{q}, \hat{\xi}, \dot{\hat{\xi}}) \delta q dt, \\ \delta \mathcal{F}^{(\xi)} &\doteq \int_{t_1}^{t_2} F^{(\xi)}(t, q, \dot{q}, \hat{\xi}, \dot{\hat{\xi}}) \delta \xi dt.\end{aligned}$$

By definition of $\hat{\xi}$, one has $\delta \mathcal{F}^{(\xi)} = 0$. We will also adopt $\delta q(t_{1,2}) = 0$, as usual. Then,

$$\delta A[q, \hat{\xi}] = (\hat{\eta} \cdot \delta \hat{\xi}) \Big|_{t_1}^{t_2} + \delta \mathcal{F}^{(q)}. \quad (17)$$

If C is a local function of (ξ_0, ξ_T) , then the first term on the right of Eq. (17) vanishes. In this case, by requiring that A is extremized, one arrives at the correct equation (15). However, such choice of C is inconvenient. It is more practical to choose C such that it characterizes the *initial* state of the medium, namely,

$$\hat{\xi}(t_1) = \xi_0, \quad \dot{\hat{\xi}}(t_1) = \xi'_0. \quad (18)$$

[For example, if q is electric field, and ξ is linear polarization, then $\hat{\xi} = X(t, \xi_0, \xi'_0) + \int_{t_1}^t \varkappa(t, t') q(t') dt'$, where X describes free oscillations independent of q , and \varkappa is some kernel (Secs. V-VII).] In this case, $\delta \hat{\xi}(t_1)$ is zero, but $\delta \hat{\xi}(t_2)$ is not even though $q(t_{1,2})$ are fixed. Hence,

$$\delta A[q, \hat{\xi}] = (\hat{\eta} \cdot \delta \hat{\xi}) \Big|_{t_1}^{t_2} + \delta \mathcal{F}^{(q)}, \quad (19)$$

where the first term on the right generally does not vanish. [An exception is the special case when $\hat{\xi}$ is a local function of q ; then, fixing $q(t_2)$ implies fixing $\hat{\xi}(t_2)$ too.] This means that satisfying Eq. (15) at fixed $q(t_{1,2})$ is *not* sufficient to extremize $A[q, \hat{\xi}]$. In order to ensure that the action is extremized at fixed $q(t_{1,2})$, we must constrain the value of $\hat{\xi}(t_2)$ separately.

The problem of extremizing A under an additional constraint $\hat{\xi}(t_2) = \text{const}$ is a standard isoperimetric problem that is solved by introducing a Lagrange multiplier [62, Sec. II.14]. In our case, it is an M -dimensional multiplier, which we will call λ . [The fact that λ itself needs to be found increases the number of independent variables in the system, which is why $\hat{\xi}(t_2)$ can be constrained independently from $q(t_{1,2})$.] Specifically, the dynamics “projected” on $\mathcal{S}^{(q)}$ is governed by the “effective” action

$$A^{(\text{eff})}[q] \doteq A[q, \hat{\xi}] + \lambda \cdot \hat{\xi}(t_2), \quad (20)$$

and the corresponding variational principle, VPPD, is

$$\delta A^{(\text{eff})}[q] = 0, \quad \delta q(t_{1,2}) = 0. \quad (21)$$

The value of λ can be readily found in the general case as follows. Under the BC (21), one has

$$\delta A^{(\text{eff})}[q] = \delta \mathcal{F}^{(q)} + (\eta_T + \lambda) \cdot \delta \hat{\xi}(t_2). \quad (22)$$

Here, η_T is the value of the functional $\hat{\eta}(t_2)$ corresponding to the physical trajectory for given q_0, q_T, ξ_0 , and ξ'_0 . The fact that the assumed fixed value of $\hat{\xi}(t_2)$ must be consistent also with the assumed $q(t_2)$ implies $\delta \mathcal{F}^{(q)} = 0$. Hence, to satisfy Eq. (21), we adopt

$$\lambda = -\eta_T. \quad (23)$$

This ensures that $\delta A^{(\text{eff})}[q] = 0$, so the variational principle (21) indeed leads to the correct equation (15).

The VPPD (21), combined with the solution for λ given by Eq. (23), is the main conceptual result of this paper. In what follows, we discuss various modifications of this principle and its applications to specific types of problems of practical interest.

IV. ROUTHIAN FORMULATION

A. Basic equations

First, consider the following alternative formulation. We start by introducing the Hamiltonian of $\mathcal{S}^{(\xi)}$,

$$H^{(\xi)} \doteq \eta \cdot \dot{\xi} - L^{(\xi)}, \quad (24)$$

as a function of $(t, q, \dot{q}, \xi, \eta)$. At fixed (t, q, \dot{q}) , one has

$$\begin{aligned}\delta H^{(\xi)} &= d\eta \cdot \dot{\xi} + \eta \cdot d\dot{\xi} - (\partial_\xi L^{(\xi)}) \cdot d\xi - (\partial_\xi L^{(\xi)}) \cdot d\dot{\xi} \\ &= d\eta \cdot \dot{\xi} + \eta \cdot d\dot{\xi} - \dot{\eta} \cdot d\xi - \eta \cdot d\dot{\xi} \\ &= d\eta \cdot \dot{\xi} - \dot{\eta} \cdot d\xi,\end{aligned} \quad (25)$$

where we substituted Eqs. (14) and (16). This leads to

$$\dot{\xi} = \partial_\eta H^{(\xi)}, \quad \dot{\eta} = -\partial_\xi H^{(\xi)}, \quad (26)$$

which are known as Hamilton’s equations. They are equivalent to the combination of Eqs. (14) and (16), as one can also recheck by direct calculation using Eq. (24).

Consider rewriting A in terms of these new variables:

$$A = \int_{t_1}^{t_2} [\Lambda^{(q)}(t, q, \dot{q}) + \eta \cdot \dot{\xi} - H^{(\xi)}(t, q, \dot{q}, \xi, \eta)] dt, \quad (27)$$

where $-\Lambda^{(q)} + H^{(\xi)}$ can be recognized as a Routhian [61, Sec. 41]. Then,

$$\delta A = p \cdot \delta q \Big|_{t_1}^{t_2} + \eta \cdot \delta \xi \Big|_{t_1}^{t_2} + \delta \mathcal{F}^{(q)} + \delta \mathcal{F}^{(\xi)}, \quad (28)$$

$$\delta \mathcal{F}^{(\xi)} \doteq \int_{t_1}^{t_2} [\delta \eta \cdot (\dot{\xi} - \partial_\eta H^{(\xi)}) - (\dot{\eta} + \partial_\eta H^{(\xi)}) \cdot \delta \xi] dt.$$

Let us assume that $q(t_{1,2})$ are kept fixed, as in Sec. III B. Then, the requirement $\delta A = 0$ leads to correct equations [Eqs. (15) and (26)] if (q, ξ, η) are treated as $N + 2M$ independent variables. For this reason, the LAP for our coupled systems can be reformulated as

$$\delta A[q, \xi, \eta] = 0 \quad (29)$$

with BC still given by Eqs. (10). (Note that, although η is now an independent variable too, it yet does *not* need to be fixed at the end points. Otherwise, the problem would be overdefined.) Accordingly, $A^{(\text{eff})}$ is defined as

$$A^{(\text{eff})}[q] \doteq A[q, \hat{\xi}, \hat{\eta}] + \lambda \cdot \hat{\xi}(t_2), \quad (30)$$

where λ is a constant Lagrange multiplier. Like earlier, one can show that the solution for λ is given by Eq. (23).

B. Symmetrized representation

Let us also introduce a ‘‘symmetrized’’ representation [63] of the above equations that we will need in Sec. V. Notice that the action (27) can be rewritten as

$$A[q, \xi, \eta] = \frac{1}{2} \eta \cdot \xi \Big|_{t_1}^{t_2} + \mathcal{A}[q, \xi, \eta], \quad (31)$$

$$\mathcal{A}[q, \xi, \eta] \doteq \int_{t_1}^{t_2} \left[\Lambda^{(q)} + \frac{1}{2} (\eta \cdot \dot{\xi} - \dot{\eta} \cdot \xi) - H^{(\xi)} \right] dt. \quad (32)$$

Using $\delta \mathcal{A} = \delta A - \delta(\eta \cdot \xi/2) \Big|_{t_1}^{t_2}$ and Eq. (28), one gets

$$\delta \mathcal{A} = \left[p \cdot \delta q + \frac{1}{2} (\eta \cdot \delta \xi - \delta \eta \cdot \xi) \right] \Big|_{t_1}^{t_2} + \delta \mathcal{F}^{(q)} + \delta \mathcal{F}^{(\xi)}. \quad (33)$$

As usual, we require $\delta q(t_{1,2}) = 0$. Then, we have M BC per end point, and the first term in the square brackets vanishes. The remaining BC can be defined, e.g., as

$$(\eta_m \delta \xi^m - \delta \eta_m \xi^m) \Big|_{t_{1,2}} = 0. \quad (34)$$

(Summation over indexes is *not* assumed here; also, the symbol $\Big|_{t_{1,2}}$ denotes that the equality must be satisfied independently at $t = t_1$ and $t = t_2$.) With obvious reservations, Eq. (34) is also equivalent to

$$0 = \delta(\xi^m/\eta_m) \Big|_{t_{1,2}}. \quad (35)$$

This provides precisely M BC per each end point, as needed, and eliminates the second term in the square brackets in Eq. (33). Hence, the requirement of extremal \mathcal{A} leads to correct equations [Eqs. (15) and (26)]. Therefore, the LAP can be reformulated as

$$\delta \mathcal{A}[q, \xi, \eta] = 0 \quad (36)$$

with BC given by Eqs. (4) and (34). Since the phase space variables ξ and η enter \mathcal{A} on the same footing, we call \mathcal{A} a phase-space action.

For \mathcal{A} , the analog of the constrained variational principle that we introduced in Sec. III C is

$$\delta \mathcal{A}^{(\text{eff})}[q] = 0, \quad \delta q(t_{1,2}) = 0, \quad (37)$$

where the action can be taken in the form

$$\mathcal{A}^{(\text{eff})}[q] \doteq \mathcal{A}[q, \hat{\xi}, \hat{\eta}] + \sum_m \lambda_m (\hat{\xi}^m/\hat{\eta}_m) \Big|_{t_2}. \quad (38)$$

It is easy to see that this VPPD leads to correct equations for q provided that λ_m satisfy

$$\lambda_m = -\eta_{m,T}^2, \quad (39)$$

where $\eta_{m,T}$ is the value of $\hat{\eta}_m(t_2)$ corresponding to the physical trajectory for given q_0, q_T, ξ_0 , and ξ_0' .

Let us also introduce an alternative form of $\mathcal{A}^{(\text{eff})}[q]$ that we will need further on. Instead of fixing $\hat{\xi}^m(t_2)/\hat{\eta}_m(t_2)$, one can fix $\hat{\eta}(t_2)$ and $\hat{\xi}(t_2)$ independently at the expense of doubling the number of unknown Lagrange multipliers. Specifically, one can introduce two M -dimensional Lagrange multipliers, $\lambda^{(\xi)}$ and $\lambda^{(\eta)}$, and define the effective action as follows:

$$\mathcal{A}^{(\text{eff})}[q] \doteq \mathcal{A}[q, \hat{\xi}, \hat{\eta}] + \hat{\eta}(t_2) \cdot \lambda^{(\eta)} + \lambda^{(\xi)} \cdot \hat{\xi}(t_2). \quad (40)$$

Then, one can show that Eq. (37) yields correct equations for q provided that

$$\lambda^{(\xi)} = -\eta_T/2, \quad \lambda^{(\eta)} = \xi_T/2, \quad (41)$$

where η_T and ξ_T are the values of $\hat{\eta}_m(t_2)$ and $\hat{\xi}_m(t_2)$ corresponding to the physical trajectory for given q_0, q_T, ξ_0 , and ξ_0' . These equations can also be presented in a more compact form, as discussed in Appendix A.

V. COUPLING TO LINEAR OSCILLATORS: GENERAL THEORY OF LINEAR DISPERSION

Let us now consider, as an example of practical interest, the case when $\mathcal{S}^{(\xi)}$ is an ensemble of linear oscillators. In this case, the above phase-space representation facilitates the formulation of a general theory of linear dispersion in a convenient ‘‘quantumlike’’ form that is introduced below. Note that truly quantum equations are also subsumed under this formulation. As a side note, the special case of *adiabatic* oscillators can be described by a simpler machinery discussed in Appendix B.

A. Quantumlike formalism

When $\mathcal{S}^{(\xi)}$ is an ensemble of linear oscillators, there exists [5] a linear transformation of the variables (ξ, η) to some variables (ψ_r, ψ_i) (Roman indexes are used to distinguish them from standard italic indexes that denote vector components) such that $\mathcal{A}^{(\xi)}[\psi_r, \psi_i] = \int_{t_1}^{t_2} \Lambda^{(\xi)} dt$,

$$\Lambda^{(\xi)} = \frac{i}{2} (\psi^\dagger \cdot \dot{\psi} - \dot{\psi}^\dagger \cdot \psi) - \psi^\dagger \cdot H_0 \cdot \psi + W(t, \psi, \psi) + W^\dagger(t, \psi^\dagger, \psi^\dagger), \quad (42)$$

where $\psi \doteq \psi_r + i\psi_i$ and $\psi^\dagger \doteq \psi_r^T - i\psi_i^T$. (Here, we assume that the underlying oscillators have positive energies for simplicity. Negative-energy oscillations [64], which are somewhat exotic and typically unstable yet

not impossible, could be included by generalizing the notation as explained in Ref. [5].) The matrix H_0 is Hermitian and serves as a Hamiltonian. Also, W is a bilinear form of ψ , and W^\dagger is the complex-conjugate bilinear form of ψ^\dagger . The terms W and W^\dagger are determined by the rate at which parameters of the medium evolve in time and vanish when those parameters are time-independent. Also [5],

$$\begin{aligned} (\eta \cdot d\xi - d\eta \cdot \xi)/2 &= (i/2)(\psi^\dagger \cdot d\psi - d\psi^\dagger \cdot \psi) \\ &= \psi_i \cdot d\psi_r - d\psi_i \cdot \psi_r, \end{aligned} \quad (43)$$

so the BC (34) can be expressed as

$$\begin{aligned} 0 &= \frac{i}{2} (\psi_m^\dagger \delta\psi^m - \delta\psi_m^\dagger \psi^m) \Big|^{t_{1,2}} \\ &= [(\psi_i^\dagger)_m \delta(\psi_r)^m - (\psi_r^\dagger)_m \delta(\psi_i)^m] \Big|^{t_{1,2}}, \end{aligned} \quad (44)$$

or, equivalently, as $\delta[(\psi_r)^m / (\psi_i)_m]_{t_{1,2}} = 0$ [65].

The corresponding ELE are as follows:

$$\frac{\delta\mathcal{A}(\xi)}{\delta\psi_r} = 0, \quad \frac{\delta\mathcal{A}(\xi)}{\delta\psi_i} = 0. \quad (45)$$

Yet ψ_r and ψ_i can be expressed as linear combinations of ψ and ψ^\dagger . Hence, one can also rewrite Eqs. (45) in an equivalent complex form

$$\frac{\delta\mathcal{A}(\xi)}{\delta\psi} = 0, \quad \frac{\delta\mathcal{A}(\xi)}{\delta\psi^\dagger} = 0, \quad (46)$$

where $\mathcal{A}(\xi)$ is considered a functional of ψ and ψ^\dagger . In this sense, ψ and ψ^\dagger can be formally treated as independent variables. Then, it is easy to see that, under the BC (44), one has

$$\begin{aligned} \delta\mathcal{A}(\xi) &= \int_{t_1}^{t_2} [\delta\psi^\dagger \cdot (i\dot{\psi} - H_0 \cdot \psi + \partial W^\dagger / \partial \psi^\dagger) \\ &\quad + (-i\dot{\psi}^\dagger - \psi^\dagger \cdot H_0 + \partial W / \partial \psi) \cdot \delta\psi] dt. \end{aligned} \quad (47)$$

Accordingly, the ELE can be cast as follows:

$$\delta\psi^\dagger : \quad i\dot{\psi} = H_0 \cdot \psi - \partial W^\dagger / \partial \psi^\dagger. \quad (48)$$

An additional equation is obtained for ψ^\dagger , but it is simply the adjoint of Eq. (48). (This is understood because we have only two independent real functions, ψ_r and ψ_i , so there can be only one independent complex equation.) Thus, the equation for ψ^\dagger will be omitted for brevity.

The terms W and W^\dagger describe effects caused by parametric resonances. (For example, if ψ were the complex probability amplitude of a free quantum particle, they would describe particle annihilation and production.) We ignore such effects for simplicity, so W and W^\dagger are henceforth omitted. Then, the Lagrangian of $\mathcal{S}(\xi)$ can be represented simply as follows:

$$\Lambda(\xi) = \frac{i}{2} (\psi^\dagger \cdot \dot{\psi} - \dot{\psi}^\dagger \cdot \psi) - \psi^\dagger \cdot H_0 \cdot \psi. \quad (49)$$

This leads to a quantumlike equation for ψ ,

$$\delta\psi^\dagger : \quad i\dot{\psi} = H_0 \cdot \psi, \quad (50)$$

which conserves $\psi^\dagger \cdot \psi$ even though H_0 may depend on time. The quantity $\psi^\dagger \cdot \psi$ is understood as the total number of ‘‘quanta’’, or ‘‘particles’’, in $\mathcal{S}(\xi)$ [5]. [Another name of this quantity is the total action of oscillations in $\mathcal{S}(\xi)$. We prefer not to use this term here to avoid confusion, since $\mathcal{A}(\xi)$ is also called the action of $\mathcal{S}(\xi)$.]

B. Interaction model

Now, let us introduce the interaction between $\mathcal{S}(\xi)$ and $\mathcal{S}^{(q)}$. We will describe it by an additional Hamiltonian $H^{(\text{int})}$. Assuming that the coupling is sufficiently weak, one can model $H^{(\text{int})}$ with a function linear in ξ and η . Since $H^{(\text{int})}$ is also real, its representation through ψ and ψ^\dagger must then have the form

$$H^{(\text{int})} = -\gamma^\dagger \cdot \psi - \psi^\dagger \cdot \gamma, \quad (51)$$

where γ is some complex vector and the minus sign is added for convenience. We will assume γ to depend on (t, q) for clarity, but the dependence on derivatives of q could also be included. As a side note, this particular model does not conserve $\psi^\dagger \cdot \psi$. A conservative model, which describes parametric interactions and quantum interactions in particular, is discussed in Appendix C.

Since H_0 in this model can depend only on time, it is convenient to eliminate it using a variable transformation. Specifically, consider a new variable Ψ defined via $\psi \doteq U \cdot \Psi$. Let us choose the operator U (a ‘‘propagator’’) such that $i\dot{U} = H_0 \cdot U$ and require that U be a unit matrix at $t = t_1$. This ensures that U remains unitary at all times, so it can be cast as $U = e^{-i\vartheta}$, where ϑ is some Hermitian operator. More specifically, the propagator can be expressed in terms of the following time-ordered exponential $\mathbb{T} \exp(\dots)$:

$$U(t, t_1) = e^{-i\vartheta} = \mathbb{T} \exp \left[-i \int_{t_1}^t H_0(t') dt' \right]. \quad (52)$$

Also importantly, for any t_a, t_b , and t , one has

$$U(t_b, t_a) = U(t_b, t) \cdot U(t, t_a). \quad (53)$$

In the new variables, the symmetrized action has the form $\mathcal{A}[q, \Psi, \Psi^\dagger] = \int_{t_1}^{t_2} L dt$, where

$$\begin{aligned} L &= \Lambda^{(q)}(t, q, \dot{q}) + \frac{i}{2} (\Psi^\dagger \cdot \dot{\Psi} - \dot{\Psi}^\dagger \cdot \Psi) \\ &\quad + \Gamma^\dagger(t, q) \cdot \Psi + \Psi^\dagger \cdot \Gamma(t, q), \end{aligned} \quad (54)$$

and $\Gamma \doteq U^\dagger \cdot \gamma$. The corresponding LAP is formulated as

$$\delta\mathcal{A}[q, \Psi, \Psi^\dagger] = 0 \quad (55)$$

with BC given by Eq. (4) and

$$[(\Psi_i)_m^\dagger \delta(\Psi_r)^m - (\Psi_r)_m^\dagger \delta(\Psi_i)^m] \Big|^{t_1, 2} = 0. \quad (56)$$

It is easy to see that the corresponding ELE are

$$\delta q: \quad \dot{p} = \partial_q \Lambda^{(q)} + (\partial_q \Gamma^\dagger) \cdot \Psi + \Psi^\dagger \cdot (\partial_q \Gamma), \quad (57)$$

$$\delta \Psi^\dagger: \quad i \dot{\Psi} = -\Gamma. \quad (58)$$

C. Projected dynamics

Now suppose that one has been able to express Ψ through q . We denote the resulting functional $\hat{\Psi}$ and assume that it is parameterized by some M complex constants (i.e., $2M$ real constants), say, $\hat{\Psi}(t_1) = \Psi_0$. Hence, we can introduce a VPPD for $\mathcal{S}^{(q)}$ as in Sec. IV B with

$$\begin{aligned} \mathcal{A}^{(\text{eff})}[q] &= \int_{t_1}^{t_2} \left\{ \Lambda^{(q)}(t, q, \dot{q}) \right. \\ &\quad \left. + \frac{1}{2} [\Gamma^\dagger(t, q) \cdot \hat{\Psi} + \hat{\Psi}^\dagger \cdot \Gamma(t, q)] \right\} dt + \mathcal{B}, \\ \mathcal{B} &= \frac{i}{2} [\nu^\dagger \cdot \hat{\Psi}(t_2) - \hat{\Psi}^\dagger(t_2) \cdot \nu]. \end{aligned} \quad (59)$$

Here, ν is a complex Lagrange multiplier introduced such that the action remains real. This representation implies that we seek to fix the values of $\hat{\Psi}_r(t_2)$ and $\hat{\Psi}_i(t_2)$ independently from $q(t_2)$ by introducing just the right number of Lagrange multipliers (namely, the $2M$ components of ν , where the real and imaginary parts are counted separately). The values of these multipliers are to be found from the condition that the values of $\hat{\Psi}_r(t_2)$ and $\hat{\Psi}_i(t_2)$ must be consistent with the value of $q(t_2)$. It is easily seen that choosing

$$\nu = -\Psi_T, \quad (60)$$

where Ψ_T is the value of $\hat{\Psi}(t_2)$ corresponding to the physical trajectory for given Ψ_0 , leads to a correct equation for q [Eq. (57)]. Hence, $\mathcal{A}^{(\text{eff})}$ is indeed the sought effective action for the projected dynamics of $\mathcal{S}^{(q)}$.

D. Polarizability

1. Response function

By integrating Eq. (58), one can express $\hat{\Psi}$ as follows:

$$\hat{\Psi} = \Psi_0 + \alpha_\Gamma \circ \Gamma. \quad (61)$$

Here, α_Γ is a linear integral operator,

$$(\alpha_\Gamma \circ \Gamma)(t) \doteq i \int_{t_1}^t \Gamma[t', q(t')] dt', \quad (62)$$

which can be understood as the polarizability of $\mathcal{S}^{(\xi)}$ with respect to Γ . (The symbol \circ means ‘‘applied to the expression on the right’’.) Hence, the effective Lagrangian can be cast as follows:

$$\mathcal{A}^{(\text{eff})}[q] = \int_{t_1}^{t_2} \Lambda_0^{(q)} dt + \Xi + \mathcal{B}, \quad (63)$$

$$\Xi \doteq \text{Re} \int_{t_1}^{t_2} \Gamma^\dagger \cdot (\alpha_\Gamma \circ \Gamma) dt. \quad (64)$$

The difference between $\Lambda_0^{(q)} \doteq \Lambda^{(q)} + \text{Re}(\Gamma^\dagger \cdot \Psi_0)$ and $\Lambda^{(q)}$ produces an effect similar to that of an external force, while Ξ can be described as follows.

Suppose a ‘‘small-amplitude approximation’’, namely, that the coordinate q is chosen such that $q = O(a)$, where a is a small parameter. Assuming that the interaction is weak, a typical interaction Hamiltonian will then be linear in q , so we assume $\gamma = v \cdot q/\sqrt{2}$, where $v = O(1)$ is an $M \times N$ complex matrix that may depend only on t , if at all. (The coefficient $\sqrt{2}$ is introduced to simplify the interpretation of the final result.) This gives

$$\Gamma(t, q) = \frac{1}{\sqrt{2}} V(t) \cdot q, \quad (65)$$

where V is another $M \times N$ complex matrix, namely,

$$V \doteq U^\dagger \cdot v = e^{i\vartheta} \cdot v \quad (66)$$

(so that, while v is typically slow, V is oscillatory). Then,

$$\hat{\Psi}(t) = \Psi_0 + \frac{i}{\sqrt{2}} \int_{t_1}^t V(t') \cdot q(t') dt', \quad (67)$$

and Ξ can be written as follows:

$$\Xi = \frac{1}{2} \int_{t_1}^{t_2} q^T \cdot (\alpha \circ q) dt, \quad (68)$$

where α is an integral operator defined as $\alpha \circ q \doteq \text{Re}[V^\dagger \cdot \alpha_\Gamma \circ (V \cdot q)]$. It is convenient to express this operator in terms of a response (or Green’s) function α as

$$\begin{aligned} (\alpha \circ q)(t) &= \int_{t_1}^t \alpha(t, t') \cdot q(t') dt' \\ &= \int_{t_1}^{t_2} \alpha(t, t') \Theta(t - t') \cdot q(t') dt'. \end{aligned} \quad (69)$$

Here, Θ is the Heaviside step function, and α is a real matrix function given by

$$\begin{aligned} \alpha(t, t') &\doteq \text{Re} [iV^\dagger(t) \cdot V(t')] \\ &= -\text{Im} [V^\dagger(t) \cdot V(t')] \\ &= (i/2) [V^\dagger(t) \cdot V(t') - V^T(t) \cdot V^*(t')], \end{aligned} \quad (70)$$

which satisfies

$$\alpha^T(t, t') = -\alpha(t', t). \quad (71)$$

Note that “Re” and “Im” above actually mean real and imaginary parts, *not* Hermitian and anti-Hermitian parts, and, in general, α is *not* Hermitian.

As a side remark, the polarizability can be understood as a pseudodifferential operator expressible through t and $i\partial_t$. The fact that the polarizability operator is not necessarily Hermitian can be related to the fact that $i\partial_t$ is not Hermitian on a finite time interval, as opposed to the whole time axis. A related discussion for spatial (rather than temporal) dispersion can be found in Refs. [5, 66].

Also note that, using Eq. (69), one can cast Ξ as

$$\Xi = \frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} q^T(t) \cdot \alpha(t, t') \Theta(t - t') \cdot q(t') dt dt', \quad (72)$$

or, equivalently, as

$$\Xi = \frac{1}{4} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' q^T(t) \cdot \alpha(t, t') \operatorname{sgn}(t - t') \cdot q(t'), \quad (73)$$

where we used Eq. (71) and $\operatorname{sgn}(t) = \Theta(t) - \Theta(-t)$. These formulas will be used below.

2. Covariant representation, or index lowering

If q is understood as a vector, then α serves as a tensor of rank (1, 1). Hence, it is convenient to introduce also the related covariant tensor of rank (0, 2), i.e., “lower the index” of the matrix α^m_n . We denote such index lowering by underlining. Using the corresponding matrix $\underline{\alpha}_{mn}$, one can write

$$a^T \cdot (\alpha \cdot b) \equiv a_n \alpha^n_m b^m = \underline{\alpha}_{nm} b^m a^n = (\underline{\alpha} \cdot b) \cdot a \quad (74)$$

for any a and b . Accordingly, we can lower the index also in α , i.e., define

$$(\underline{\alpha} \circ q)(t) \doteq \int_{t_1}^t \underline{\alpha}(t, t') \cdot q(t') dt'. \quad (75)$$

This allows one to write $(\alpha \circ q)^T = \underline{\alpha} \circ q$ (because transposing a vector is the same as lowering its index), and also Ξ can be expressed without involving q^T :

$$\Xi = \frac{1}{2} \int_{t_1}^{t_2} (\underline{\alpha} \circ q) \cdot q dt. \quad (76)$$

If one does not need to distinguish upper and lower indexes, the distinction between α and $\underline{\alpha}$ can be ignored.

E. Derivation of the ELE for q

It is instructive to explain at this point how the ELE (57) that we anticipate from the general theory can be derived also by a brute-force calculation using

$$\delta \mathcal{A}^{(\text{eff})}[q] = \int_{t_1}^{t_2} \left[\partial_q \Lambda^{(a)} - \dot{p} + \frac{\operatorname{Re}(V^\dagger \cdot \Psi_0)}{\sqrt{2}} \right] \cdot \delta q dt + \delta \Xi + \delta \mathcal{B}. \quad (77)$$

First, using Eq. (72) and the fact that $a^T = a$ for any scalar a , one gets

$$\begin{aligned} \delta \Xi &= \frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} [\delta q^T(t) \cdot \alpha(t, t') \Theta(t - t') \cdot q(t') + q^T(t) \cdot \alpha(t, t') \Theta(t - t') \cdot \delta q(t')] dt dt' \\ &= \frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} [q^T(t') \cdot \alpha^T(t, t') \Theta(t - t') \cdot \delta q(t) + q^T(t) \cdot \alpha(t, t') \Theta(t - t') \cdot \delta q(t')] dt dt' \\ &= \frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} [q^T(t') \cdot \alpha^T(t, t') \Theta(t - t') \cdot \delta q(t) + q^T(t') \cdot \alpha(t', t) \Theta(t' - t) \cdot \delta q(t)] dt dt' \\ &= \frac{1}{2} \int_{t_1}^{t_2} \left[\int_{t_1}^t q^T(t') \cdot \alpha^T(t, t') dt' + \int_t^{t_2} q^T(t') \cdot \alpha(t', t) dt' \right] \cdot \delta q(t) dt \\ &\equiv \int_{t_1}^{t_2} \Xi^{(q)T}(t) \cdot \delta q(t) dt. \end{aligned} \quad (78)$$

Using Eq. (71), we can also rewrite $\Xi^{(q)}$ as follows:

$$\begin{aligned} \Xi^{(q)}(t) &= \frac{1}{2} \int_{t_1}^t \alpha(t, t') \cdot q(t') dt' \\ &\quad - \frac{1}{2} \int_t^{t_2} \alpha(t, t') \cdot q(t') dt'. \end{aligned} \quad (79)$$

Next, one gets from Eq. (59) that

$$\delta \mathcal{B} = \frac{i}{2} [\nu^\dagger \cdot \delta \hat{\Psi}(t_2) - \delta \hat{\Psi}^\dagger(t_2) \cdot \nu], \quad (80)$$

where $\delta\hat{\Psi}(t_2)$ can be obtained from Eq. (67); namely,

$$\delta\hat{\Psi}(t_2) = \frac{i}{\sqrt{2}} \int_{t_1}^{t_2} V(t) \cdot \delta q(t) dt. \quad (81)$$

Let us substitute Eq. (60) and again use Eq. (67), now to express Ψ_T . This gives

$$\delta\mathcal{B} = \text{Re} \left\{ \left[\Psi_0^\dagger - \frac{i}{\sqrt{2}} \int_{t_1}^{t_2} q^T(t') \cdot V^\dagger(t') dt' \right] \cdot \frac{1}{\sqrt{2}} \int_{t_1}^{t_2} V(t) \cdot \delta q(t) dt \right\}, \quad (82)$$

or, equivalently,

$$\delta\mathcal{B} = \int_{t_1}^{t_2} \left\{ \frac{\text{Re}[\Psi_0^\dagger \cdot V(t)]}{\sqrt{2}} + \mathcal{B}^{(q)T} \right\} \cdot \delta q(t) dt, \quad (83)$$

$$\mathcal{B}^{(q)} \doteq \frac{1}{2} \int_{t_1}^{t_2} \alpha(t, t') \cdot q(t') dt', \quad (84)$$

and we also notice that

$$\begin{aligned} \mathcal{B}^{(q)}(t) &= \frac{1}{2} \int_{t_1}^t \alpha(t, t') \cdot q(t') dt' \\ &\quad + \frac{1}{2} \int_t^{t_2} \alpha(t, t') \cdot q(t') dt'. \end{aligned} \quad (85)$$

By combining Eqs. (78) and (83), we obtain

$$\begin{aligned} \delta\Xi + \delta\mathcal{B} &= \int_{t_1}^{t_2} \left\{ \frac{\text{Re}[\Psi_0^\dagger \cdot V(t)]}{\sqrt{2}} \right. \\ &\quad \left. + [\Xi^{(q)}(t) + \mathcal{B}^{(q)}(t)]^T \right\} \cdot \delta q(t) dt, \end{aligned} \quad (86)$$

and Eqs. (79) and (85) give $\Xi^{(q)} + \mathcal{B}^{(q)} = \alpha \circ q$. Thus,

$$\delta\Xi + \delta\mathcal{B} = \int_{t_1}^{t_2} \left\{ \frac{\text{Re}[\Psi_0^\dagger \cdot V(t)]}{\sqrt{2}} + (\underline{\alpha} \circ q)(t) \right\} \cdot \delta q(t) dt,$$

so Eq. (77) becomes

$$\begin{aligned} \delta\mathcal{A}^{(\text{eff})}[q] &= \int_{t_1}^{t_2} \left[\partial_q \Lambda^{(q)} - \dot{p} + \underline{\alpha} \circ q \right. \\ &\quad \left. + \frac{V^\dagger \cdot \Psi_0}{\sqrt{2}} + \frac{\Psi_0^\dagger \cdot V}{\sqrt{2}} \right] \cdot \delta q dt. \end{aligned} \quad (87)$$

Then, the requirement $\delta\mathcal{A}^{(\text{eff})}[q] = 0$ leads to

$$\delta q : \quad \dot{p} = \partial_q \Lambda^{(q)} + \underline{\alpha} \circ q + \frac{V^\dagger \cdot \Psi_0}{\sqrt{2}} + \frac{\Psi_0^\dagger \cdot V}{\sqrt{2}}. \quad (88)$$

Considering the assumed expression for Γ [Eq. (65)], this agrees with the general ELE [Eq. (57)], as expected.

VI. ASYMPTOTIC APPROXIMATIONS

In this section, we make approximations to the VPPD and derive reduced theories for certain limiting cases. Although the calculations below may seem cumbersome, producing equivalent results by a brute-force reduction of Eq. (88) (as opposed to the underlying functional) would still have been harder to do in the general case.

A. Basic concepts

1. "Symmetrized" polarizability

Consider a representation of the response function in terms of the "symmetrized" time coordinates

$$\bar{t} \doteq (t + t')/2, \quad \tau \doteq t - t'. \quad (89)$$

Specifically, we define

$$\bar{\alpha}(\bar{t}, \tau) \doteq \alpha(\bar{t} + \tau/2, \bar{t} - \tau/2) \equiv \alpha(t, t'), \quad (90)$$

which has the following important property:

$$\bar{\alpha}(\bar{t}, -\tau) = -\bar{\alpha}^T(\bar{t}, \tau). \quad (91)$$

Using Eqs. (70)-(90) along with Eq. (66), one can write

$$\bar{\alpha}(\bar{t}, \tau) = -\text{Im} [v^\dagger(\bar{t} + \tau/2) \cdot \mathcal{U}(\bar{t}, \tau) \cdot v(\bar{t} - \tau/2)], \quad (92)$$

$$\mathcal{U}(\bar{t}, \tau) \doteq U(\bar{t} + \tau/2, t_1) \cdot U^\dagger(\bar{t} - \tau/2, t_1). \quad (93)$$

Since U is unitary, one has $U^\dagger = U^{-1}$. Using this along with Eq. (53), we can rewrite \mathcal{U} as follows:

$$\begin{aligned} \mathcal{U}(\bar{t}, \tau) &= U(\bar{t} + \tau/2, t_1) \cdot U^{-1}(\bar{t} - \tau/2, t_1) \\ &= U(\bar{t} + \tau/2, t_1) \cdot U(t_1, \bar{t} - \tau/2) \\ &= U(\bar{t} + \tau/2, \bar{t} - \tau/2), \end{aligned} \quad (94)$$

or, in other words,

$$\mathcal{U}(\bar{t}, \tau) = \mathbb{T} \exp \left[-i \int_{\bar{t}-\tau/2}^{\bar{t}+\tau/2} H_0(t') dt' \right], \quad (95)$$

where $\mathbb{T} \exp(\dots)$ is a time-ordered exponential.

2. Phase mixing and the associated small parameters

First, consider a stationary medium, i.e., a medium where v and H_0 are constant. Then, Eq. (95) gives $\mathcal{U} = \exp(-iH_0\tau)$, and Eq. (92) gives

$$\bar{\alpha}(\tau) = -\text{Im}(v^\dagger \cdot e^{-iH_0\tau} \cdot v). \quad (96)$$

For simplicity, assume the basis in which H_0 is diagonal ("energy basis"); i.e., $H_0 = \text{diag}(\Omega_1, \Omega_2, \dots, \Omega_M)$. Then, matrix elements of $\bar{\alpha}$ can be written as

$$\bar{\alpha}^m_n(\tau) = -\text{Im} \sum_s v_s^{m*} v_s^n e^{-i\Omega_s \tau}. \quad (97)$$

Provided that the set of Ω_s is dense enough, the sum over s can be replaced with an integral [$s \mapsto s(\Omega)$]:

$$\bar{\alpha}(\tau) \approx -\text{Im} \int e^{-i\Omega\tau} f(\Omega) d\Omega, \quad (98)$$

where $f_n^m(\Omega) \doteq (ds/d\Omega)[v_{s(\Omega)}^{m*} v^{s(\Omega)}_n]$. Since $\bar{\alpha}(\tau)$ is thereby a Fourier image of $f(\Omega)$ (or vice versa), it approaches zero at $\tau \rightarrow \infty$, at least if $f(\Omega)$ is continuous. If the characteristic gap $\Delta\Omega$ between neighboring eigenfrequencies is nonzero (and $M \gg 1$), the same conclusion applies at $\tau \ll \Delta\Omega^{-1}$, when the spectrum discreteness is inessential. Thus, as long as larger τ are not of interest (as will be assumed below), there is no need to distinguish continuous and discrete spectra.

The effect $\bar{\alpha}(\tau \rightarrow \infty) \rightarrow 0$ is known as phase mixing. Although introduced here for a stationary medium, it extends also to media evolving at large enough time scales \mathcal{T} , namely,

$$\frac{\tau_{\text{pm}}}{\mathcal{T}} \ll 1, \quad \bar{\Omega}\mathcal{T} \left(\frac{\tau_{\text{pm}}}{\mathcal{T}} \right)^2 \ll 1, \quad (99)$$

where $\bar{\Omega}$ is the characteristic eigenvalue of H_0 . The former requirement ensures that v and H_0 be approximately constant on the time scale τ_{pm} . The latter requirement allows approximating Eq. (95) much like in the case of a stationary medium:

$$\mathcal{U}(\tau, \bar{t}) \approx \exp[-iH_0(\bar{t})\tau]. \quad (100)$$

Our intention is to discuss approximations to the VPPD that become possible in this limit. In order to do that, let us introduce some definitions first.

3. Polarizability in the spectral representation

Let us introduce an auxiliary matrix function

$$\tilde{\alpha}_\omega(\bar{t}) \doteq \int_0^{\tau_m} \bar{\alpha}(\bar{t}, \tau) e^{i\omega\tau} d\tau, \quad (101)$$

where ω is assumed real, $\bar{\alpha}$ is given by Eq. (90), and τ_m is some constant. (In later sections, we will treat τ_m as a function of \bar{t} , but this distinction is not important here.) Let us also introduce the Hermitian and anti-Hermitian parts of $\tilde{\alpha}_\omega$, namely,

$$\tilde{\alpha}_{\omega,H} \doteq \frac{\tilde{\alpha}_\omega + \tilde{\alpha}_\omega^\dagger}{2}, \quad \tilde{\alpha}_{\omega,A} \doteq \frac{\tilde{\alpha}_\omega - \tilde{\alpha}_\omega^\dagger}{2i}, \quad (102)$$

such that $\tilde{\alpha}_\omega = \tilde{\alpha}_{\omega,H} + i\tilde{\alpha}_{\omega,A}$. (In this notation, both $\tilde{\alpha}_{\omega,H}$ and $\tilde{\alpha}_{\omega,A}$ are Hermitian, so $i\tilde{\alpha}_{\omega,H}$ and $i\tilde{\alpha}_{\omega,A}$ are anti-Hermitian.) As shown in Appendix D,

$$\tilde{\alpha}_{\omega,H}(\bar{t}) = \frac{1}{2} \int_{-\tau_m}^{\tau_m} \bar{\alpha}(\bar{t}, \tau) \text{sgn}(\tau) e^{i\omega\tau} d\tau, \quad (103)$$

$$\tilde{\alpha}_{\omega,A}(\bar{t}) = \frac{1}{2i} \int_{-\tau_m}^{\tau_m} \bar{\alpha}(\bar{t}, \tau) e^{i\omega\tau} d\tau. \quad (104)$$

Provided efficient phase mixing, the function $\tilde{\alpha}_\omega$ has a well-defined limit at large τ_m . We call this limit α_ω ; i.e.,

$$\begin{aligned} \alpha_\omega(\bar{t}) &\doteq \int_0^\infty \bar{\alpha}(\bar{t}, \tau) e^{i\omega\tau} d\tau \\ &\equiv \int_{-\infty}^\infty \alpha(\bar{t} + \tau/2, \bar{t} - \tau/2) \Theta(\tau) e^{i\omega\tau} d\tau. \end{aligned} \quad (105)$$

The function α_ω is recognized as the Laplace image of $\bar{\alpha}$ evaluated at real ω and also, more generally, as the *Weyl image of the polarizability operator* [67]. Like for $\tilde{\alpha}_\omega$, the Hermitian and anti-Hermitian parts of α_ω satisfy

$$\alpha_{\omega,H}(\bar{t}) = \frac{1}{2} \int_{-\infty}^\infty \bar{\alpha}(\bar{t}, \tau) \text{sgn}(\tau) e^{i\omega\tau} d\tau, \quad (106)$$

$$\alpha_{\omega,A}(\bar{t}) = \frac{1}{2i} \int_{-\infty}^\infty \bar{\alpha}(\bar{t}, \tau) e^{i\omega\tau} d\tau. \quad (107)$$

Approximate formulas for $\tilde{\alpha}_\omega$ and α_ω in the limit (99) can be found in Appendix D. It is also shown there that $\alpha_{\omega,H}$ and $\alpha_{\omega,A}$ satisfy the Kramers-Kronig relations.

B. Quasistatic limit

First, let us consider the case when q and, possibly, the medium evolve on time scales \mathcal{T} that satisfy

$$\epsilon_0 \doteq \max \left\{ \frac{\tau_{\text{pm}}}{\mathcal{T}}, \bar{\Omega}\mathcal{T} \left(\frac{\tau_{\text{pm}}}{\mathcal{T}} \right)^2, \frac{1}{\bar{\Omega}\mathcal{T}} \right\} \ll 1. \quad (108)$$

(To simplify the notation, we will assume all the three small parameters to be comparable to each other, but this is not essential.) Below, we derive an asymptotic expression for $\mathcal{A}^{(\text{eff})}$ using ϵ_0 as a small parameter.

1. Expression for Ξ

To approximate the term Ξ as given by Eq. (73), let us first map the integration variables (t, t') in the double integral to (\bar{t}, τ) using Eqs. (89). The integration domain is then mapped as shown in Figs. 1(a) and (b), namely, to $(\bar{t}, \tau) \in (t_1, t_2) \times (-\tau_m(\bar{t}), \tau_m(\bar{t}))$, where $\tau_m(\bar{t})$ is a piecewise-linear function. This gives

$$\Xi = \frac{1}{4} \text{Tr} \int_{t_1}^{t_2} d\bar{t} \int_{-\tau_m(\bar{t})}^{\tau_m(\bar{t})} d\tau w(\bar{t}, \tau) \cdot \bar{\alpha}(\bar{t}, \tau) \text{sgn}(\tau), \quad (109)$$

where Tr stands for ‘‘trace’’, and w is a matrix given by

$$w(\bar{t}, \tau) \doteq q(\bar{t} - \tau/2) q^T(\bar{t} + \tau/2) = q(\bar{t}) q^T(\bar{t}) + O(\epsilon_0).$$

To the lowest (zeroth) order in ϵ_0 , one has

$$\begin{aligned} \Xi &\approx \frac{1}{4} \text{Tr} \int_{t_1}^{t_2} d\bar{t} w(\bar{t}, 0) \cdot \int_{-\tau_m(\bar{t})}^{\tau_m(\bar{t})} d\tau \bar{\alpha}(\bar{t}, \tau) \text{sgn}(\tau) \\ &= \frac{1}{2} \int_{t_1}^{t_2} q^T(\bar{t}) \cdot \tilde{\alpha}_{0,H}(\bar{t}) \cdot q(\bar{t}) d\bar{t}. \end{aligned} \quad (110)$$

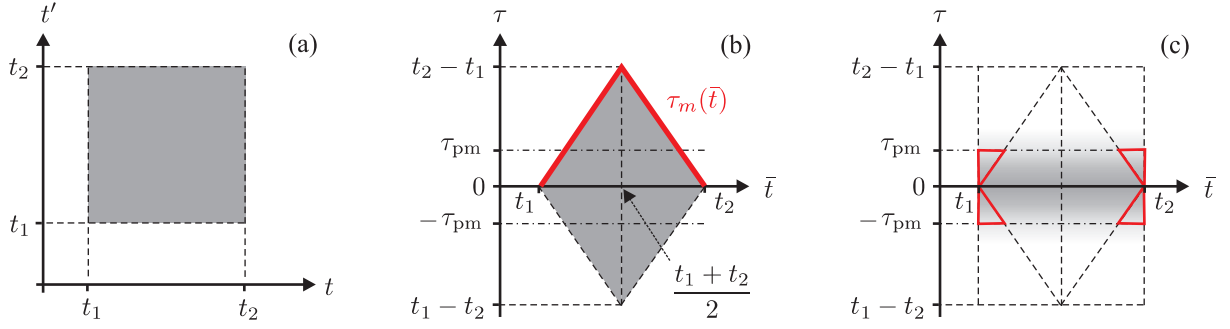


FIG. 1: Integration domains (shaded): (a) The exact domain in (t, t') space; namely, $(t, t') \in (t_1, t_2) \times (t_1, t_2)$. (b) The exact domain in (\bar{t}, τ) space; namely, $(\bar{t}, \tau) \in (t_1, t_2) \times (-\tau_m(\bar{t}), \tau_m(\bar{t}))$. The function $\tau_m(\bar{t})$ is plotted in bold red. (c) The approximate domain in (\bar{t}, τ) space; namely, $(\bar{t}, \tau) \in (t_1, t_2) \times (-\infty, \infty)$. This approximation is justified by the fact that the integrand is negligible at $|\tau| \gtrsim \tau_{\text{pm}}$, as indicated by a gradient fill. The red triangles indicate the error introduced by this approximation.

The function $\tilde{\alpha}_{0,H}$ [given by Eq. (103) at $\omega = 0$] can be approximated with $\alpha_{0,H}$ everywhere except for an $O(\epsilon_0)$ part of the integration domain. Neglecting this $O(\epsilon_0)$ correction, we get

$$\begin{aligned} \Xi &\approx \frac{1}{2} \int_{t_1}^{t_2} q^T(\bar{t}) \cdot \alpha_0(\bar{t}) \cdot q(\bar{t}) d\bar{t} \\ &\equiv \int_{t_1}^{t_2} [\underline{\alpha}_0(\bar{t}) \cdot q(\bar{t})] \cdot q(\bar{t}) d\bar{t}, \end{aligned} \quad (111)$$

which corresponds to the approximation illustrated in Fig. 1(c). We have omitted the index H in $\alpha_{0,H}$ because, in the zero-frequency limit, the polarizability's Weyl symbol (105) becomes Hermitian (Appendix D); i.e., $\alpha_0 \approx \alpha_{0,H}$. Specifically,

$$\alpha_0 \approx v^\dagger \cdot H_0^{-1} \cdot v \quad (112)$$

as seen from Eq. (D4) in the limit when the oscillating term is eliminated by phase mixing.

2. Expression for \mathcal{B}

Using Eq. (67), let us express $\hat{\Psi}$ as

$$\hat{\Psi} = \Psi_0 + \frac{i}{\sqrt{2}} \int_{t_1}^t e^{i\vartheta(t')} \cdot v(t') \cdot q(t') dt'. \quad (113)$$

The function ϑ is such that, at zero ϵ_0 , it has a constant derivative equal to H_0 . Hence, at small nonvanishing ϵ_0 , one can expect $\partial_t \vartheta$ to be slow and close to H_0 . Then, the integrand in Eq. (113) is a rapidly oscillating exponent times a slow function $v \cdot q$. Such integral is entirely determined by values of the integrand at the ends of the integration domain and is of order $\|v\| \|q\| / \bar{\Omega}$ [68]. This gives the following estimate for the term \mathcal{B} in Eq. (59):

$$\mathcal{B} \sim \|v\|^2 \|q\|^2 / \bar{\Omega}^2, \quad (114)$$

assuming the effect of Ψ_0 vanishes due to phase mixing. Hence, $\mathcal{B}/\Xi \sim \epsilon_0$.

3. Final result

By combining the above results and neglecting corrections $O(\epsilon_0)$ and smaller, one obtains

$$\mathcal{A}^{(\text{eff})}[q] = \int_{t_1}^{t_2} [\Lambda_0^{(q)}(t, q, \dot{q}) - \Phi_0(t, q)] dt, \quad (115)$$

$$\Phi_0(t, q) \doteq -\frac{1}{2} [\underline{\alpha}_0(t) \cdot q] \cdot q. \quad (116)$$

Clearly, the corresponding ELE is as follows:

$$\delta q : \quad \dot{p} = \partial_q \Lambda_0^{(q)} + \underline{\alpha}_0 \cdot q. \quad (117)$$

These results indicate that the interaction of a quasistatic system with a linear medium $\mathcal{S}^{(\xi)}$ is equivalent to the interaction with an effective potential Φ_0 given by Eq. (116). The symmetric matrix $\underline{\alpha}_0$ represents the covariant form of α_0 given by Eq. (112) and serves as the polarizability of $\mathcal{S}^{(\xi)}$ with respect to q . Also note that, for positive-definite Hamiltonian H_0 , the polarizability α_0 is positive-definite too, and thus $\Phi_0 < 0$.

C. Limit of fast oscillations

Here, we derive an asymptotic variational principle for the case when q oscillates rapidly compared to the evolution of the medium. The method is akin to Whitham's averaging [2], but the averaging procedure is justified differently. Specifically, this is done as follows.

1. Basic equations

Suppose $\Lambda^{(q)}$ is bilinear in (q, \dot{q}) . Then, Eq. (88) is linear and allows complex solutions. Let us consider a complex solution q_+ of the form $q_+ = Q(t) e^{i\theta(t)}$, where Q is a complex vector and θ is a real phase. Provided that the time scale \mathcal{T} at which the medium evolves is large

enough, in principle it is possible to construct asymptotic Q and θ such that $\omega(t) \doteq -\dot{\theta}(t)$ is slow compared to θ at all t ; i.e., q is quasimonochromatic. (The effect of Ψ_0 will be assumed negligible, like in Sec. VI B.) More specifically, harmonics of ω remain suppressed with exponential accuracy in the small parameter $(\omega\mathcal{T})^{-1}$. (If the system supports more than one mode, we assume that only one mode is excited, and the spectral distance to other modes is $\Delta\omega \gtrsim \omega$.) Here, we assume that \mathcal{T} is also the time scale of the evolution of Q and ω , and $\mathcal{T} \sim T$.

Clearly, if $q_+ = Q(t)e^{i\theta(t)}$ is a solution, then $q_- = Q^*(t)e^{-i\theta(t)}$ is a solution too, and so is $(q_+ + q_-)/2$. Since the latter is also real, it can be adopted as an *exponentially accurate asymptotic model* of a realizable physical process, which we can also express as follows:

$$q = \text{Re}[Q(t)e^{i\theta(t)}]. \quad (118)$$

Let us substitute this model into Eq. (54). As usual [68], the rapidly oscillating terms that couple q_+ and q_- can be eliminated, which is also expected from the definition of q_+ and q_- . (Such “ θ -averaging” is denoted below with angular brackets, $\langle \dots \rangle$.) This implies that the medium responds to q_+ and q_- independently, as if there were two independent subsystems $\mathcal{S}^{(\xi)}$ that interact with q_+ and q_- , respectively. The corresponding responses will be denoted Ψ_{\pm} . Hence, instead of the Lagrange multipliers ν and ν^\dagger that we introduced before, now we must introduce twice as much Lagrange multipliers. We denote them ν_{\pm} and ν_{\pm}^\dagger . Hence, like in Sec. V C, the effective action (63) can be represented as follows:

$$\mathcal{A}^{(\text{eff})} = \langle A^{(q)} \rangle + \langle \Xi \rangle + \langle \mathcal{B} \rangle, \quad (119)$$

$$\langle A^{(q)} \rangle = \int_{t_1}^{t_2} \langle \Lambda^{(q)} \rangle dt, \quad (120)$$

$$\begin{aligned} \langle \Xi \rangle &= \frac{1}{8} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' e^{i\theta(t')-i\theta(t)} \\ &\quad \times Q^\dagger(t) \cdot \alpha(t, t') \text{sgn}(\tau) \cdot Q(t') \end{aligned} \quad (121)$$

[where we used Eq. (73)], and $\langle \mathcal{B} \rangle = \mathcal{B}_+ + \mathcal{B}_-$, where

$$\mathcal{B}_{\pm} = -\frac{1}{\sqrt{2}} \text{Re} \sum_{\pm} \left[\nu_{\pm}^\dagger \cdot \int_{t_1}^{t_2} V(t') \cdot Q_{\pm}(t') e^{\pm i\theta(t')} dt' \right], \quad (122)$$

assuming the notation $Q_+ \doteq Q$ and $Q_- \doteq Q^*$. Also like in Sec. V C, one finds

$$\nu_{\pm} = -\frac{\Psi_{\pm, T}}{4} = -\frac{i}{4\sqrt{2}} \int_{t_1}^{t_2} V(t) \cdot Q_{\pm}(t) e^{\pm i\theta(t)} dt. \quad (123)$$

[The quantity ν_{\pm} must not be confused with ν given by Eq. (60). In particular, note the extra factor 1/4.]

As usual, the functions θ , Q , and Q^\dagger are considered independent. Then,

$$\begin{aligned} \delta\mathcal{A}^{(\text{eff})} &= \int_{t_1}^{t_2} \left[\frac{\delta\mathcal{A}^{(\text{eff})}}{\delta\theta} \delta\theta \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left(\frac{\delta\mathcal{A}^{(\text{eff})}}{\delta Q} \cdot \delta Q + Q^\dagger \cdot \frac{\delta\mathcal{A}^{(\text{eff})}}{\delta Q^\dagger} \right) \right] dt. \end{aligned}$$

We seek to approximate $\delta\mathcal{A}^{(\text{eff})}$ to the zeroth order in the small parameter

$$\epsilon_\omega \doteq \max \left\{ \frac{\tau_{\text{pm}}}{\mathcal{T}}, \omega\mathcal{T} \left(\frac{\tau_{\text{pm}}}{\mathcal{T}} \right)^2, \frac{1}{\omega\mathcal{T}} \right\} \ll 1, \quad (124)$$

where τ_{pm} is the phase-mixing time (Sec. VI A 2). (As in Sec. VI B, we adopt for clarity that all the three parameters are comparable to each other.) We assume $\bar{\Omega} \sim \omega$ and also the standard ordering for α_ω (Sec. VI A 3), which will be verified *a posteriori*:

$$\alpha_{\omega, H} = O(1), \quad \alpha_{\omega, A} \lesssim O(\epsilon_\omega). \quad (125)$$

Then, the individual terms on the right side of Eq. (119) can be calculated as follows.

2. Approximation of $\langle A^{(q)} \rangle$

Since $\Lambda^{(q)}$ is a local function of (q, \dot{q}) , the function $\langle \Lambda^{(q)} \rangle$ cannot depend on θ other than through ω . Hence,

$$\begin{aligned} \delta\langle A^{(q)} \rangle &= \int_{t_1}^{t_2} \left[\frac{\partial}{\partial t} \left(\frac{\delta\langle A^{(q)} \rangle}{\delta\omega} \right) \delta\theta \right. \\ &\quad \left. + \frac{\delta\langle A^{(q)} \rangle}{\delta Q} \cdot \delta Q + \delta Q^\dagger \cdot \frac{\delta\langle A^{(q)} \rangle}{\delta Q^\dagger} \right] dt, \end{aligned}$$

where we used $\delta\omega = -\partial_t(\delta\theta)$ and integrated by parts with $\delta\theta(t_{1,2}) = 0$. Since

$$\delta\theta = O(\epsilon_\omega^{-1}), \quad \delta Q = O(1), \quad (126)$$

while $\partial_t(\delta\omega\langle A^{(q)} \rangle) = O(\epsilon_\omega)$, $\delta_Q\langle A^{(q)} \rangle = O(1)$, and $\delta_{Q^\dagger}\langle A^{(q)} \rangle = O(1)$, all the three terms in the square brackets are $O(1)$. Hence, within the accuracy of our model, it is enough to approximate $\langle A^{(q)} \rangle$ to the zeroth order in ϵ_ω . Since $\Lambda^{(q)}$ was assumed to be a bilinear function of (q, \dot{q}) , the functional $\langle A^{(q)} \rangle$ then has the following form:

$$\langle A^{(q)} \rangle \approx \frac{1}{4} \int_{t_1}^{t_2} \text{Re} [Q^\dagger(t) \cdot D_{\omega(t)}^{(q)}(t) \cdot Q(t)] dt, \quad (127)$$

where the factor 1/4 is introduced for convenience, and $D_{\omega(t)}^{(q)}$ is some matrix that is determined by the local frequency $\omega(t)$ and, in nonstationary systems, also t . [Below, we also refer to this matrix using a shortened notation $D_{\omega}^{(q)}$.] The anti-Hermitian part of this matrix provides zero contribution to $\text{Re}[Q^\dagger \cdot D_{\omega}^{(q)} \cdot Q]$, so we can assume that $D_{\omega}^{(q)}$ is Hermitian for all real ω without loss of generality. Then, “Re” in Eq. (127) can be omitted.

3. Approximation of $\langle \Xi \rangle$, dressed action

Let us now approximate $\langle \Xi \rangle$ given by Eq. (121). First, note that, due to Eq. (89), we have

$$\begin{aligned}\theta(t') &= \theta(\bar{t}) - \frac{\tau}{2} \partial_t \theta(\bar{t}) + \frac{\tau^2}{8} \partial_t^2 \theta(\bar{t}) - \frac{\tau^3}{48} \partial_t^3 \theta(\bar{t}) + \dots, \\ \theta(t) &= \theta(\bar{t}) + \frac{\tau}{2} \partial_t \theta(\bar{t}) + \frac{\tau^2}{8} \partial_t^2 \theta(\bar{t}) + \frac{\tau^3}{48} \partial_t^3 \theta(\bar{t}) + \dots\end{aligned}$$

Hence,

$$\begin{aligned}\theta(t') - \theta(t) &= -\tau \partial_t \theta(\bar{t}) - \frac{\tau^3}{24} \partial_t^3 \theta(\bar{t}) + \dots \\ &= \tau \omega(\bar{t}) + \frac{\tau^3}{24} \partial_t^2 \omega(\bar{t}) + \dots,\end{aligned}\quad (128)$$

where the second term on the right is estimated as

$$\tau^3 \partial_t^2 \omega \sim \frac{\tau_{\text{pm}}}{\mathcal{T}} \left[\omega \mathcal{T} \left(\frac{\tau_{\text{pm}}}{\mathcal{T}} \right)^2 \right] = O(\epsilon_\omega^2) \ll 1. \quad (129)$$

This gives

$$e^{i\theta(t') - i\theta(t)} = e^{i\omega(\bar{t})\tau + O(\epsilon_\omega^2)} = e^{i\omega(\bar{t})\tau} + O(\epsilon_\omega^2). \quad (130)$$

Omitting $O(\epsilon_\omega^2)$, one can rewrite Eq. (121) as follows:

$$\begin{aligned}\langle \Xi \rangle &\approx \frac{1}{8} \text{Re Tr} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' W(\bar{t}, \tau) \\ &\quad \cdot \bar{\alpha}(\bar{t}, \tau) \text{sgn}(\tau) e^{i\omega(\bar{t})\tau}.\end{aligned}\quad (131)$$

Here, the matrix W is given by

$$W(\bar{t}, \tau) \doteq Q(t') Q^\dagger(t) = Q(\bar{t}) Q^\dagger(\bar{t}) + O(\epsilon_\omega). \quad (132)$$

Since the integrand in Eq. (131) does not depend on θ explicitly, the correction $O(\epsilon_\omega)$ can be omitted for the same reason as in Sec. VI C 2. Hence, we switch from integration in (t, t') to the integration in (\bar{t}, τ) and get

$$\langle \Xi \rangle \approx \frac{1}{4} \text{Tr} \int_{t_1}^{t_2} W(\bar{t}, 0) \cdot \tilde{\alpha}_{\omega, H}(\bar{t}) d\bar{t}. \quad (133)$$

Note that $\tilde{\alpha}_{\omega, H} \sim \alpha_{\omega, H} = O(1)$ (Sec. VI C 1). This function can be approximated with $\alpha_{\omega, H}$ everywhere except for an $O(\epsilon_\omega)$ part of the integration domain. Since corrections $O(\epsilon_\omega)$ in $\langle \Xi \rangle$ are deemed negligible, we obtain

$$\langle \Xi \rangle \approx \frac{1}{4} \int_{t_1}^{t_2} Q^\dagger \cdot \alpha_{\omega, H} \cdot Q dt. \quad (134)$$

As a side note, one can also rewrite Eq. (134) as $\langle \Xi \rangle \approx -\int_{t_1}^{t_2} \Phi_\omega(t) dt$, so $\langle \Xi \rangle$ is understood as a contribution of an effective potential

$$\Phi_\omega \doteq -\frac{1}{4} Q^\dagger \cdot \alpha_{\omega, H} \cdot Q \quad (135)$$

that the subsystem $\mathcal{S}^{(q)}$ experiences due to the interaction with the subsystem $\mathcal{S}^{(\xi)}$. The potential Φ_ω represents the ponderomotive energy (or, more precisely, the ponderomotive term in the Lagrangian) of $\mathcal{S}^{(\xi)}$. In the context of particle dynamics with Hermitian α_ω , the obtained linear relation between Φ_ω and α_ω is also known as the K - χ theorem [15, 69–71].

Since $\langle \Xi \rangle$ has the same form as $\langle A^{(q)} \rangle$ [Eq. (127)], let us combine them into a ‘‘dressed action’’ $A_{\text{dr}}^{(q)} \doteq \langle A^{(q)} \rangle + \langle \Xi \rangle$. For that, let us introduce $D_\omega \doteq D_\omega^{(q)} + \alpha_\omega$, whose Hermitian and anti-Hermitian parts are

$$D_{\omega, H} = D_\omega^{(q)} + \alpha_{\omega, H}, \quad D_{\omega, A} = \alpha_{\omega, A}. \quad (136)$$

Then,

$$A_{\text{dr}}^{(q)} = \frac{1}{4} \int_{t_1}^{t_2} Q^\dagger \cdot D_{\omega, H} \cdot Q dt. \quad (137)$$

This can also be written as $A_{\text{dr}}^{(q)} = \int_{t_1}^{t_2} \mathcal{L} dt$, where

$$\mathcal{L}(t, \omega, Q, Q^\dagger) \doteq \frac{1}{4} Q^\dagger \cdot D_{\omega, H}(t) \cdot Q \quad (138)$$

is understood as the dressed Lagrangian of $\mathcal{S}^{(q)}$. Accordingly, one gets

$$\begin{aligned}\delta A_{\text{dr}}^{(q)} &= \int_{t_1}^{t_2} \left[(\partial_t I) \delta\theta + \frac{1}{4} \delta Q^\dagger \cdot (D_{\omega, H} \cdot Q) \right. \\ &\quad \left. + \frac{1}{4} (Q^\dagger \cdot D_{\omega, H}) \cdot \delta Q \right] dt,\end{aligned}\quad (139)$$

where

$$I \doteq \partial_\omega \mathcal{L}(t, \omega, Q, Q^\dagger) = Q^\dagger \cdot \partial_\omega D_{\omega, H} \cdot Q / 4 \quad (140)$$

is understood as the number of quanta of the oscillation mode, or the action of this mode [4, 5] [again, not to be confused with the action such as $\mathcal{A}^{(\text{eff})}$].

4. Approximation of $\delta \mathcal{B}_\pm$

A straightforward calculation (Appendix E) gives

$$\delta \langle \mathcal{B} \rangle = \delta \mathcal{B}_\theta + \delta \mathcal{B}_Q + O(\epsilon_\omega^2), \quad (141)$$

$$\delta \mathcal{B}_\theta \doteq -\frac{i}{4} \text{Tr} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' e^{i\omega\tau} \delta\theta(\bar{t}) W(\bar{t}, \tau) \cdot \bar{\alpha}(\bar{t}, \tau), \quad (142)$$

$$\delta \mathcal{B}_Q \doteq -\frac{i}{4} \text{Tr} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' e^{i\omega\tau} \delta Y(\bar{t}, \tau) \cdot \bar{\alpha}(\bar{t}, \tau), \quad (143)$$

$$\delta Y(\bar{t}, \tau) \doteq \frac{i}{2} [Q(t') \cdot \delta Q^\dagger(t) - \delta Q(t') \cdot Q^\dagger(t)]. \quad (144)$$

Using $W(\bar{t}, \tau) \approx W(\bar{t}, 0) + \tau \partial_\tau W(\bar{t}, 0)$, one further gets

$$\begin{aligned}\delta \mathcal{B}_\theta &= \frac{1}{2} \text{Tr} \int_{t_1}^{t_2} d\bar{t} \delta\theta(\bar{t}) \\ &\quad \times \{ W(\bar{t}, 0) - i[\partial_\tau W(\bar{t}, 0)] \partial_\omega \} \cdot \tilde{\alpha}_{\omega, A}.\end{aligned}\quad (145)$$

Note that $\tilde{\alpha}_{\omega,A} \sim \alpha_{\omega,A} = O(\epsilon_\omega)$ (Sec. VIC 1) and $\delta\theta = O(\epsilon_\omega^{-1})$ [Eq. (126)]. Since we need to calculate $\delta\mathcal{B}_\theta$ to the zeroth order in ϵ_ω , the contribution of $\partial_\tau W \sim W/\mathcal{T}$ can then be neglected. Hence,

$$\delta\mathcal{B}_\theta \approx \frac{1}{2} \int_{t_1}^{t_2} [Q^\dagger(\bar{t}) \cdot \tilde{\alpha}_{\omega,A}(\bar{t}) \cdot Q(\bar{t})] \delta\theta(\bar{t}) d\bar{t}. \quad (146)$$

Similarly, one obtains that $\delta\mathcal{B}_Q = O(\epsilon_\omega)$, so \mathcal{B}_Q is entirely negligible. In summary, substituting this result into Eq. (141) gives

$$\delta\langle\mathcal{B}\rangle \approx \delta\mathcal{B}_\theta \approx \frac{1}{2} \int_{t_1}^{t_2} (Q^\dagger \cdot \alpha_{\omega,A} \cdot Q) \delta\theta dt, \quad (147)$$

where we omitted corrections $O(\epsilon_\omega)$ and, for the same reason, replaced $\tilde{\alpha}_{\omega,A}$ with $\alpha_{\omega,A}$.

5. Euler-Lagrange equations

Using $\mathcal{A}^{(\text{eff})} \approx A_{\text{dr}}^{(q)} + \langle\mathcal{B}\rangle$ along with Eqs. (139) and (147) and also requiring $\delta\mathcal{A}^{(\text{eff})} = 0$, we get the following set of ELE:

$$\delta\theta : \quad \partial_t(Q^\dagger \cdot \partial_\omega D_{\omega,H} \cdot Q) = -2Q^\dagger \cdot D_{\omega,A} \cdot Q, \quad (148)$$

$$\delta Q^\dagger : \quad D_{\omega,H} \cdot Q = 0, \quad (149)$$

plus the conjugate equation for Q^\dagger . Note that Eq. (148) implies $\alpha_{\omega,A} \lesssim \alpha_{\omega,H}/(\omega\mathcal{T})$, which is in agreement with the original assumption (125).

It is also instructive to rewrite these in terms of the unit polarization vector \mathbf{e}_ω and the scalar amplitude a , assuming $Q = \mathbf{e}_\omega a$:

$$D_{\omega,H} \cdot \mathbf{e}_\omega = 0, \quad (150)$$

$$\partial_t[(\mathbf{e}_\omega^\dagger \cdot \partial_\omega D_{\omega,H} \cdot \mathbf{e}_\omega) a^2] = -2(\mathbf{e}_\omega^\dagger \cdot D_{\omega,A} \cdot \mathbf{e}_\omega) a^2. \quad (151)$$

Equation (150) can be understood as a local dispersion relation, which determines $\mathbf{e}_\omega(t)$ and the local frequency $\omega(t)$. Equation (151) can be understood as an amplitude equation. In a dissipationless medium, which we *define* as a medium with $\alpha_{\omega,A} = 0$, this equation becomes $\partial_t I = 0$, which is known as the action conservation theorem [72].

For example, consider a one-dimensional stationary system. In this case, Eq. (150) becomes

$$D_{\omega,H} = 0, \quad (152)$$

and the amplitude equation gives $a = e^{-\gamma t} \times \text{const}$, where

$$\gamma \doteq D_{\omega,A}/\partial_\omega D_{\omega,H}. \quad (153)$$

(This quantity is not to be confused with γ that was introduced in Sec. VB.) One hence obtains that q has a well-defined complex frequency; i.e., $q(t) = \text{Re}(e^{-i\omega_c t} \times \text{const})$, where $\omega_c \doteq \omega - i\gamma = \omega - iD_{\omega,A}/\partial_\omega D_{\omega,H}$. Within

the accuracy of our theory, this can be understood as a solution of the complex dispersion relation

$$D_{\omega_c} = 0, \quad (154)$$

as seen from the fact that $D_{\omega_c} = D_{\omega-i\gamma,H} + iD_{\omega-i\gamma,A} \approx D_{\omega,H} + i(-\gamma\partial_\omega D_{\omega,H} + D_{\omega,A})$ combined with Eqs. (152) and (153). Hence, D_{ω_c} can be understood as the complex dispersion function.

VII. WAVES IN DISTRIBUTED SYSTEMS

A. Basic definitions

Although the above results were derived for discrete oscillators, they can be readily extended to waves in continuous media. A continuous medium is understood as a system where the scalar product can be expressed as an integral over some continuous space \mathcal{X} (we assume that \mathcal{X} is Euclidean for clarity, but see Refs. [4, 5] for a more general case), specifically as

$$a \cdot b = \int_{\mathcal{X}} \mathbf{a}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x}) d^D x. \quad (155)$$

Here, $D = \dim \mathcal{X}$, and $\mathbf{a}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ are fields of some finite dimension D' . For example, they can be electromagnetic fields in physical space; then $D' = 3$. (We use bold symbols to distinguish D -dimensional coordinates and D' -dimensional vectors from N - and M -dimensional coordinates and vectors that we introduced earlier, since N and M are infinite in the continuous limit.) Accordingly, the polarizability operator can be rewritten as

$$(\boldsymbol{\alpha} \circ \mathbf{q})(t, \mathbf{x}) = \int_{t_1}^t dt' \int_{\mathcal{X}} d^D x' \boldsymbol{\alpha}(t, t'; \mathbf{x}, \mathbf{x}') \cdot \mathbf{q}(t', \mathbf{x}').$$

The symmetrized kernel is introduced as $\bar{\boldsymbol{\alpha}}(\bar{t}, \tau; \bar{\mathbf{x}}, \boldsymbol{\chi}) \doteq \boldsymbol{\alpha}(t, t'; \mathbf{x}, \mathbf{x}')$, where \bar{t} and τ are defined as usual [Eq. (89)], and

$$\bar{\mathbf{x}} \doteq (\mathbf{x} + \mathbf{x}')/2, \quad \boldsymbol{\chi} \doteq \mathbf{x} - \mathbf{x}'. \quad (156)$$

We also introduce the corresponding Weyl symbol (the term is used with the same reservations as earlier [67]),

$$\boldsymbol{\alpha}_{\omega,\mathbf{k}}(\bar{t}, \bar{\mathbf{x}}) \doteq \int_0^\infty d\tau \int_{\mathcal{X}} d^D \chi \bar{\boldsymbol{\alpha}}(\bar{t}, \tau; \bar{\mathbf{x}}, \boldsymbol{\chi}) e^{i\omega\tau - i\mathbf{k}\cdot\boldsymbol{\chi}}.$$

The modification of the remaining notation is obvious, so it will not be presented in detail.

B. Geometrical optics

Let us consider a special case of practical interest where the medium is weakly inhomogeneous in both time and space. (Anisotropy and general dispersion, including

both temporal and spatial dispersion, are implied.) Unlike in Sec. VI C, where all oscillators were assumed to have the same ω at given t , we now allow the frequency to be spatially inhomogeneous. Specifically, we adopt

$$\mathbf{q}(t, \mathbf{x}) = \text{Re}[\mathbf{Q}(t, \mathbf{x}) e^{i\theta(t, \mathbf{x})}] \quad (157)$$

and assume that the frequency and the wave vector,

$$\omega \doteq -\partial_t \theta, \quad \mathbf{k} \doteq \nabla \theta, \quad (158)$$

are slow functions of (t, \mathbf{x}) . Explicitly, we require

$$\max\{\epsilon_\omega, \epsilon_{\mathbf{k}}\} \ll 1, \quad (159)$$

where ϵ_ω is defined in Eq. (124), and $\epsilon_{\mathbf{k}}$ is the analogous parameter that involves spatial scales instead of temporal scales. Then, one obtains $\mathcal{A}^{(\text{eff})} \approx A_{\text{dr}}^{(q)} + \langle \mathcal{B} \rangle$, where $A_{\text{dr}}^{(q)} = \int \mathcal{L} d^D x$, and \mathcal{L} is the Lagrangian density given by

$$\mathcal{L}(t, \mathbf{x}, \mathbf{Q}, \mathbf{Q}^\dagger, \omega, \mathbf{k}) = \frac{1}{4} \mathbf{Q}^\dagger \cdot \mathbf{D}_{(\omega, \mathbf{k}), H}(t, \mathbf{x}) \cdot \mathbf{Q}. \quad (160)$$

This leads to [4]

$$\begin{aligned} \delta A_{\text{dr}}^{(q)} = & \int_{t_1}^{t_2} dt \int_{\mathcal{X}} d^D x \left\{ (\partial_t \mathcal{I} + \nabla \cdot \mathcal{J}) \delta \theta \right. \\ & \left. + \frac{1}{4} \delta \mathbf{Q}^\dagger \cdot [\mathbf{D}_{(\omega, \mathbf{k}), H} \cdot \mathbf{Q}] + \frac{1}{4} [\mathbf{Q}^\dagger \cdot \mathbf{D}_{(\omega, \mathbf{k}), H}] \cdot \delta \mathbf{Q} \right\}, \end{aligned}$$

where $\mathcal{I} \doteq \partial_\omega \mathcal{L}$ and $\mathcal{J} \doteq -\partial_{\mathbf{k}} \mathcal{L}$. The expression for $\delta \langle \mathcal{B} \rangle$ is obtained like in Sec. VI C and can be written as

$$\delta \langle \mathcal{B} \rangle \approx \frac{1}{2} \int_{t_1}^{t_2} dt \int d^D x [\mathbf{Q}^\dagger \cdot \boldsymbol{\alpha}_{(\omega, \mathbf{k}), A} \cdot \mathbf{Q}] \delta \theta. \quad (161)$$

Hence, one obtains the following ELE:

$$\delta \theta : \quad \partial_t \mathcal{I} + \nabla \cdot \mathcal{J} = -2[\mathbf{Q}^\dagger \cdot \mathbf{D}_{(\omega, \mathbf{k}), A} \cdot \mathbf{Q}], \quad (162)$$

$$\delta \mathbf{Q}^\dagger : \quad \mathbf{D}_{(\omega, \mathbf{k}), H} \cdot \mathbf{Q} = 0, \quad (163)$$

plus the conjugate equation for \mathbf{Q}^\dagger .

It is also instructive to rewrite these equations as

$$\mathbf{D}_{(\omega, \mathbf{k}), H} \cdot \mathbf{e}_{\omega, \mathbf{k}} = 0, \quad (164)$$

$$\partial_t \mathbf{k} = -\nabla \omega, \quad (165)$$

$$\partial_t \mathcal{I} + \nabla \cdot (\mathbf{v}_g \mathcal{I}) = -2\gamma \mathcal{I}. \quad (166)$$

Here, $\mathbf{e}_{\omega, \mathbf{k}}$ is the unit polarization vector. We also added an equation for \mathbf{k} that flows from its definition [Eq. (158)], introduced $\mathbf{v}_g \doteq -\partial_{\mathbf{k}} \mathcal{L} / \partial_\omega \mathcal{L}$, which represents the group velocity, and also introduced

$$\gamma \doteq \frac{\mathbf{e}_{\omega, \mathbf{k}}^\dagger \cdot \mathbf{D}_{(\omega, \mathbf{k}), A} \cdot \mathbf{e}_{\omega, \mathbf{k}}}{\mathbf{e}_{\omega, \mathbf{k}}^\dagger \cdot \partial_\omega \mathbf{D}_{(\omega, \mathbf{k}), H} \cdot \mathbf{e}_{\omega, \mathbf{k}}}, \quad (167)$$

which represents the local dissipation rate. Equations (164)-(167) form a complete set of equations that describe dissipative GO waves in inhomogeneous nonstationary media. They are in agreement with results of

Ref. [4], where dissipative effects were described as an addition to the variational formulation rather than as a part of it. (The variational formulation of *dissipationless* wave dynamics is also discussed in Refs. [2, 3].)

Applications of these general equations to electromagnetic waves are discussed in detail in Ref. [4]. Here, we only point out that one can choose \mathbf{Q} to be the complex amplitude of the electric field, \mathbf{E}_c ; then,

$$\mathcal{L} = \frac{1}{16\pi} \mathbf{E}_c^* \cdot \boldsymbol{\varepsilon}_{(\omega, \mathbf{k}), H} \cdot \mathbf{E}_c - \frac{c^2 |\mathbf{k} \times \mathbf{E}_c|^2}{16\pi \omega^2}, \quad (168)$$

$$\mathbf{D}_{(\omega, \mathbf{k}), A} = \frac{\boldsymbol{\varepsilon}_{(\omega, \mathbf{k}), A}}{16\pi}. \quad (169)$$

The second term in Eq. (168) is understood as $|\mathbf{B}_c|^2/16\pi$, where we used Faraday's law $\mathbf{B}_c \approx (c\mathbf{k}/\omega) \times \mathbf{E}_c$ to express the complex amplitude of the magnetic field \mathbf{B}_c , and c is the speed of light. Also, $\boldsymbol{\varepsilon}_{(\omega, \mathbf{k})}$ is the dielectric tensor in the spectral representation (or, more specifically, the Weyl symbol of the dielectric permittivity operator), and the indexes H and A denote its Hermitian and anti-Hermitian parts, as usual. Accordingly, Eq. (162) can be understood as a restatement of Poynting's theorem [4].

As a side note, an even simpler variational derivation of the results reported in this section is possible by means of the Weyl calculus combined with a technique *à la* Ref. [28]. We leave details to future publications.

VIII. CONCLUSIONS

In summary, we have formulated a variational principle for a dissipative subsystem of an overall-conservative system by introducing constant Lagrange multipliers and Lagrangians nonlocal in time. We call it the variational principle for projected dynamics, or VPPD. We have also elaborated on applications of the VPPD to the special case of linear systems, particularly in the context of wave propagation in general linear media. The focus was on how the variational formulation helps in deriving reduced models, such as the quasistatic and GO models. In particular, we have proposed a variational formulation of dissipative GO in a linear medium that is inhomogeneous, nonstationary, nonisotropic, and exhibits both temporal and spatial dispersion simultaneously. The ‘‘spectral representation’’ of the dielectric tensor $\boldsymbol{\varepsilon}_{(\omega, \mathbf{k})}$ that enters GO equations is shown to be, strictly speaking, the Weyl symbol of the medium dielectric permittivity. This can be considered as an invariant definition of $\boldsymbol{\varepsilon}_{(\omega, \mathbf{k})}$, as opposed to *ad hoc* definitions that are commonly adopted in literature and have been a source of a continuing debate.

Our work is also intended as a stepping stone for extending the variational theory of modulational stability in general wave ensembles [15] to dissipative systems. Details will be described in a separate publication.

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APPENDIX A: COMPACT FORM OF THE PHASE-SPACE ACTION

As a side note, equations of Sec. IV B can be expressed in a more compact form in terms of the phase space coordinate z and the Poincaré two-form $\omega_{\alpha\beta}$,

$$z \doteq \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \omega_{\alpha\beta} \doteq \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad (\text{A1})$$

where \mathbb{I} is a $M \times M$ unit matrix. In particular, one can cast Eq. (32) as

$$\mathcal{A}[q, z] = \int_{t_1}^{t_2} \left[\Lambda^{(q)} + \frac{1}{2} \omega_{\alpha\beta} z^\alpha \dot{z}^\beta - H^{(z)} \right] dt \quad (\text{A2})$$

(Greek indexes span from 1 to $2M$) and Eq. (33) as

$$\begin{aligned} \delta \mathcal{A}[q, z] &= \left[p \cdot \delta q + \frac{1}{2} \omega_{\alpha\beta} z^\alpha \delta z^\beta \right] \Big|_{t_1}^{t_2} + \delta \mathcal{F}^{(q)} + \delta \mathcal{F}^{(z)}, \\ \delta \mathcal{F}^{(z)} &\doteq \int_{t_1}^{t_2} (\omega_{\alpha\beta} \dot{z}^\beta - \partial_\alpha H^{(z)}) \delta z^\alpha dt, \end{aligned} \quad (\text{A3})$$

where $\partial_\alpha \doteq \partial/\partial z^\alpha$ and $H^{(z)}(t, z) \doteq H^{(\xi)}(t, \xi, \eta)$. Accordingly, Hamilton's equations (26) are cast as follows:

$$\omega_{\alpha\beta} \dot{z}^\beta = \partial_\alpha H^{(z)}. \quad (\text{A4})$$

Also, Eq. (40) can be represented as follows:

$$\mathcal{A}^{(\text{eff})}[q] = \mathcal{A}[q, \hat{z}] + \frac{1}{2} \omega_{\alpha\beta} \hat{z}^\alpha(t_2) z_T^\beta, \quad (\text{A5})$$

where

$$z_T \doteq \begin{pmatrix} \xi_T \\ \eta_T \end{pmatrix}. \quad (\text{A6})$$

On physical trajectories one has $\hat{z}^\alpha(t_2) = z_T^\alpha$. Since $\omega_{\alpha\beta}$ is antisymmetric, this implies that the second term on the right side of Eq. (A5) vanishes on such trajectories.

APPENDIX B: COUPLING TO ADIABATIC OSCILLATORS

Here, we discuss an instructive special case, namely, the situation when the Hamiltonian $H^{(\xi)}$ in Eq. (27) does not depend on ξ explicitly. In this case, η is conserved and can be treated as a constant parameter, which we denote J . Hence, one can write

$$A^{(\text{eff})}[q] = \int_{t_1}^{t_2} [\Lambda^{(q)}(t, q, \dot{q}) - H^{(\xi)}(t, q, \dot{q}; J)] dt. \quad (\text{B1})$$

An alternative derivation of the same result, also known as the Routh reduction can be found, e.g., in Ref. [73].

In particular, Eq. (B1) can be applied [74] when $\mathcal{S}^{(\xi)}$ is an *adiabatic oscillator* in which (ξ, η) are the angle-action variables. (Then, J is called an adiabatic invariant [61].) In the special case when $\mathcal{S}^{(\xi)}$ is a *linear oscillator*, one can further write $H^{(\xi)} = J \cdot \Omega$, where Ω is the corresponding canonical frequency (i.e., such that $\dot{\xi} = \Omega$), which is independent of J but may depend on (t, q, \dot{q}) . Then,

$$A^{(\text{eff})}[q] = \int_{t_1}^{t_2} [\Lambda^{(q)}(t, q, \dot{q}) - J \cdot \Omega(t, q, \dot{q})] dt. \quad (\text{B2})$$

For example, one can use this to describe a charged particle in a weakly inhomogeneous field B . In this case, Ω is the gyrofrequency, and J is proportional to the particle magnetic moment μ , such that $J \cdot \Omega = \mu B$ [75]. Then, the last term in Eq. (B2) is nothing but the effective potential responsible for the diamagnetic force.

APPENDIX C: PARAMETRIC AND QUANTUM INTERACTIONS

1. Basic equations

In addition to the linear coupling discussed in Sec. V B, it is instructive to consider parametric interactions, particularly because they subsume quantum interactions and thus can be understood as most general. Specifically, let us consider

$$H^{(\text{int})} = \Psi^\dagger \cdot h(t, q) \cdot \Psi, \quad (\text{C1})$$

where h is some Hermitian operator. (We assumed that $H^{(\text{int})}$ is independent of derivatives of q for brevity, but the corresponding generalization is straightforward.) The total Lagrangian in this case can be written as

$$L = \Lambda^{(q)}(t, q, \dot{q}) + \frac{i}{2} (\Psi^\dagger \cdot \dot{\Psi} - \dot{\Psi}^\dagger \cdot \Psi) - \Psi^\dagger \cdot h(t, q) \cdot \Psi, \quad (\text{C2})$$

and the corresponding ELE are

$$\delta q : \quad \dot{p} = \partial_q \Lambda^{(q)} - \Psi^\dagger \cdot \partial_q h \cdot \Psi, \quad (\text{C3})$$

$$\delta \Psi^\dagger : \quad i \dot{\Psi} = h \cdot \Psi. \quad (\text{C4})$$

Notably, Eq. (C4) conserves $\Psi^\dagger \cdot \Psi$ even in the presence of coupling, in contrast with Eq. (58).

For projected dynamics, the effective action $\mathcal{A}^{(\text{eff})}$ is introduced just like in Sec. V C. When the solution for Ψ and Ψ^\dagger are substituted into Eq. (C2), the second term and the third term cancel each other due to Eq. (C4), so one ends up with

$$\mathcal{A}^{(\text{eff})}[q] = \int_{t_1}^{t_2} \Lambda^{(q)}(t, q, \dot{q}) dt + \mathcal{B}, \quad (\text{C5})$$

$$\mathcal{B} \doteq \frac{i}{2} [\nu^\dagger \cdot \hat{\Psi}(t_2) - \hat{\Psi}^\dagger(t_2) \cdot \nu]. \quad (\text{C6})$$

2. Derivation of the ELE for q

Let us show how a correct ELE for q is obtained from here by a straightforward calculation. First,

$$\delta\mathcal{A}^{(\text{eff})}[q] = \int_{t_1}^{t_2} [\partial_q \Lambda^{(q)} - \dot{p}] \delta q dt + \delta\mathcal{B}, \quad (\text{C7})$$

$$\delta\mathcal{B} = \frac{i}{2} [\nu^\dagger \cdot \delta\hat{\Psi}(t_2) - \delta\hat{\Psi}^\dagger(t_2) \cdot \nu]. \quad (\text{C8})$$

In order to calculate $\delta\hat{\Psi}(t_2)$, let us use the fact that $\hat{\Psi}(t_2) = U_h(t_2, t_1) \cdot \Psi_0$, where U_h is a unitary propagator given by the following time-ordered exponential:

$$U_h(t_2, t_1) = \mathbb{T} \exp \left[-i \int_{t_1}^{t_2} h(t', q(t')) dt' \right]. \quad (\text{C9})$$

If the variation δq is localized to an infinitesimal time interval Δt following some time τ , one can readily write

$$\begin{aligned} \delta\hat{\Psi}(t_2) &= U_h(t_2, \tau + \Delta t) \cdot \delta U_h(\tau + \Delta t, \tau) \cdot U_h(\tau, t_1) \cdot \Psi_0, \\ \delta U_h(\tau + \Delta t, \tau) &\approx -i \partial_q h(\tau, q(\tau)) \delta q(\tau) \Delta t, \end{aligned}$$

where one can further substitute $U_h(\tau, t_1) \cdot \Psi_0 = \hat{\Psi}(\tau)$. Then, for a general variation δq , one gets

$$\delta\hat{\Psi}(t_2) = -i \int_{t_1}^{t_2} [U_h(t_2, \tau) \cdot \partial_q h(\tau, q(\tau)) \cdot \hat{\Psi}(\tau)] \delta q(\tau) d\tau.$$

For $\delta\mathcal{B}/\delta q$ at any given time t , this gives

$$\frac{\delta\mathcal{B}}{\delta q} = \text{Re} [\nu^\dagger \cdot U_h(t_2, t) \cdot \partial_q h(t, q(t)) \cdot \hat{\Psi}(t)]. \quad (\text{C10})$$

Then, after substituting Eq. (60), one gets

$$\begin{aligned} \frac{\delta\mathcal{B}}{\delta q} &= -\text{Re} [\hat{\Psi}^\dagger(t_2) \cdot U_h(t_2, t) \cdot \partial_q h(t, q(t)) \cdot \hat{\Psi}(t)] \\ &= -\text{Re} [\hat{\Psi}^\dagger(t) \cdot \partial_q h(t, q(t)) \cdot \hat{\Psi}(t)]. \quad (\text{C11}) \end{aligned}$$

Here, we also used $\hat{\Psi}(t_2) = U_h(t_2, t) \cdot \Psi(t)$ and the fact that $[U_h(t_2, t)]^\dagger \cdot U_h(t_2, t)$ is a unit operator, because U_h is unitary. Since $\partial_q h(t, q(t))$ is Hermitian, one can omit ‘‘Re’’, so Eq. (C11) becomes

$$\frac{\delta\mathcal{B}}{\delta q} = -\hat{\Psi}^\dagger(t) \cdot \partial_q h(t, q(t)) \cdot \hat{\Psi}(t). \quad (\text{C12})$$

After substituting this into Eq. (C7) and requiring $\delta\mathcal{A}^{(\text{eff})}[q] = 0$, one obtains

$$\delta q : \quad \dot{p} = \partial_q \Lambda^{(q)} - \hat{\Psi}^\dagger \cdot \partial_q h \cdot \hat{\Psi}. \quad (\text{C13})$$

This is equivalent to Eq. (C3), as expected.

3. Approximate Lagrangian of weak coupling

Suppose that the coordinate q is chosen such that $q = O(a)$, where a is a small parameter. Assuming that the interaction is weak, a typical interaction Hamiltonian is then linear in q . Like H_0 , any zeroth-order term can be removed by a variable transformation, so we assume $h(t, q) = w \cdot q$, where $w = O(1)$ may depend only on t , if at all. We now seek to derive an approximation of L accurate up to $O(a^2)$; i.e., terms $o(a^2)$ will be neglected.

To construct such an approximation, we adopt an asymptotic representation of Ψ as a power series in a ; i.e., $\Psi = \sum_n \Psi_n$, where $\Psi_n = O(a^n)$. According to Eq. (C4), $\dot{\Psi}_n = o(a^n)$, so truncating this series as $\Psi = \Psi_0 + \Psi_1$ is sufficient. Since Ψ_0 is constant, $\mathcal{S}^{(\xi)}$ is then fully described by Ψ_1 , which serves as the new independent variable instead of the original Ψ . Hence, after omitting complete time derivatives (which do not affect ELE), Eq. (C2) becomes similar to Eq. (54), namely,

$$\begin{aligned} L &= \Lambda_0^{(q)}(t, q, \dot{q}) + \frac{i}{2} (\Psi_1^\dagger \cdot \dot{\Psi}_1 - \dot{\Psi}_1^\dagger \cdot \Psi_1) \\ &\quad + \Gamma^\dagger(t, q) \cdot \Psi_1 + \Psi_1^\dagger \cdot \Gamma(t, q), \quad (\text{C14}) \end{aligned}$$

where the new $\Lambda_0^{(q)}$ and Γ are given by

$$\Lambda_0^{(q)}(t, q, \dot{q}) \doteq \Lambda^{(q)}(t, q, \dot{q}) - \Psi_0^\dagger \cdot h(t, q) \cdot \Psi_0, \quad (\text{C15})$$

$$\Gamma(t, q) \doteq -h(t, q) \cdot \Psi_0. \quad (\text{C16})$$

(One may also recognize this as the leading-order Born approximation of the original problem.) This result shows that theory of linear dispersion for interaction Hamiltonians of the form (C1) can be constructed identically to that for interaction Hamiltonians of the form (51). In particular, this means that general predictions of linear dispersion theory are independent of whether oscillators comprising a medium are classical or quantum.

APPENDIX D: PROPERTIES OF $\tilde{\alpha}_\omega$ AND α_ω

Here, we discuss some properties of the functions $\tilde{\alpha}_\omega$ and α_ω that are introduced in Sec. VI A 3.

a. Function $\tilde{\alpha}_\omega$

First, Eq. (103) is proved as follows:

$$\begin{aligned}
& \int_{-\tau_m}^{\tau_m} d\tau \bar{\alpha}(\bar{t}, \tau) \operatorname{sgn}(\tau) e^{i\omega\tau} \\
&= \int_0^{\tau_m} d\tau \bar{\alpha}(\bar{t}, \tau) e^{i\omega\tau} - \int_{-\tau_m}^0 d\tau \bar{\alpha}(\bar{t}, \tau) e^{i\omega\tau} \\
&= \int_0^{\tau_m} d\tau \bar{\alpha}(\bar{t}, \tau) \operatorname{sgn}(\tau) e^{i\omega\tau} + \int_{\tau_m}^0 d\tau \bar{\alpha}(\bar{t}, -\tau) e^{-i\omega\tau} \\
&= \int_0^{\tau_m} d\tau \bar{\alpha}(\bar{t}, \tau) e^{i\omega\tau} - \int_0^{\tau_m} d\tau \bar{\alpha}(\bar{t}, -\tau) e^{-i\omega\tau} \\
&= \int_0^{\tau_m} d\tau \bar{\alpha}(\bar{t}, \tau) e^{i\omega\tau} + \int_0^{\tau_m} d\tau \bar{\alpha}^T(\bar{t}, \tau) e^{-i\omega\tau} \\
&= \tilde{\alpha}_\omega(\bar{t}) + \tilde{\alpha}_\omega^\dagger(\bar{t}) = 2\tilde{\alpha}_{\omega,H}(\bar{t}), \tag{D1}
\end{aligned}$$

where we used that $\bar{\alpha}$ is real by definition. Equation (104) is proved similarly.

Second, let us explicitly derive an approximation for $\tilde{\alpha}_\omega$ in the limit (99). Using Eq. (92) for $\bar{\alpha}$ together with Eq. (100) for \mathcal{U} , we obtain

$$\begin{aligned}
\bar{\alpha}(\bar{t}, \tau) e^{i\omega\tau} &= \frac{i}{2} \left[v^\dagger(\bar{t} + \tau/2) \cdot e^{i(\omega - H_0)\tau} \cdot v(\bar{t} - \tau/2) \right. \\
&\quad \left. - v^T(\bar{t} + \tau/2) \cdot e^{i(\omega + H_0)\tau} \cdot v^*(\bar{t} - \tau/2) \right].
\end{aligned}$$

Then, using $v(\bar{t} \pm \tau/2) \approx v(\bar{t})$, one gets

$$\begin{aligned}
\tilde{\alpha}_{\omega,H} &= \frac{1}{2} v^\dagger \cdot \left\{ \frac{1 - \cos[(H_0 - \omega)\tau_m]}{H_0 - \omega} \right. \\
&\quad \left. + \frac{1 - \cos[(H_0 + \omega)\tau_m]}{H_0 + \omega} \right\} \cdot v,
\end{aligned}$$

$$\tilde{\alpha}_{\omega,A} = \frac{1}{2} v^\dagger \cdot \left\{ \frac{\sin[(H_0 - \omega)\tau_m]}{H_0 - \omega} - \frac{\sin[(H_0 + \omega)\tau_m]}{H_0 + \omega} \right\} \cdot v,$$

or, in the continuous-spectrum limit (Sec. VI A 2),

$$\tilde{\alpha}_{\omega,H} = \int \frac{1 - \cos[(\Omega - \omega)\tau_m]}{2(\Omega - \omega)} [f(\bar{t}, \Omega) - f(\bar{t}, -\Omega)] d\Omega, \tag{D2}$$

$$\tilde{\alpha}_{\omega,A} = \int \frac{\sin[(\Omega - \omega)\tau_m]}{2(\Omega - \omega)} [f(\bar{t}, \Omega) - f(\bar{t}, -\Omega)] d\Omega. \tag{D3}$$

Notably, the anti-Hermitian part of $\tilde{\alpha}_\omega$ vanishes in the zero- ω limit; i.e., $\tilde{\alpha}_{0,A} = 0$, so $\tilde{\alpha}_0 \approx \tilde{\alpha}_{0,H}$, and

$$\begin{aligned}
\tilde{\alpha}_{0,H} &\approx v^\dagger \cdot H_0^{-1} \cdot [1 - \cos(H_0\tau_m)] \cdot v \\
&\rightarrow \int \Omega^{-1} [1 - \cos(\Omega\tau_m)] f(\bar{t}, \Omega) d\Omega. \tag{D4}
\end{aligned}$$

b. Function α_ω

Asymptotic expressions for α_ω under the condition (99) can be obtained by taking the large- τ_m limit of Eqs. (D2) and (D3). The oscillating terms vanish for $\Omega \neq \omega$. For $\Omega = \omega$, one has $1 - \cos[(\Omega - \omega)\tau_m] \equiv 0$, so, instead of averaging the corresponding cosine to zero, we entirely exclude the point $\Omega = \omega$ from the integration domain. Also, $\sin[(\Omega - \omega)\tau_m]/(\Omega - \omega) \rightarrow \pi\delta(\Omega - \omega)$. Hence, one gets

$$\alpha_{\omega,H} = \text{PV} \int \frac{f(\bar{t}, \Omega) - f(\bar{t}, -\Omega)}{2(\Omega - \omega)} d\Omega, \tag{D5}$$

$$\alpha_{\omega,A} = \frac{\pi}{2} [f(\bar{t}, \omega) - f(\bar{t}, -\omega)], \tag{D6}$$

where PV denotes the Cauchy principal value. Equation (D5) can be understood also as $\alpha_{\omega,H} = -\text{H}\alpha_{\omega,A}$, where H is the Hilbert transform. Using that $\text{H}^2 = -1$, we then obtain $\alpha_{\omega,A} = \text{H}\alpha_{\omega,H}$. Thus, in summary,

$$\alpha_{\omega,H} = \frac{1}{\pi} \text{PV} \int \frac{\alpha_{\omega,A}}{\Omega - \omega} d\Omega, \tag{D7}$$

$$\alpha_{\omega,A} = -\frac{1}{\pi} \text{PV} \int \frac{\alpha_{\omega,H}}{\Omega - \omega} d\Omega, \tag{D8}$$

which can be recognized as the Kramers-Kronig relations.

APPENDIX E: CALCULATION OF $\delta\langle\mathcal{B}\rangle$

To calculate $\delta\langle\mathcal{B}\rangle$ used in Sec. VI C, we proceed as follows. First, using Eqs. (122) and (123), we get

$$\delta\mathcal{B}_+ = \text{Re} \left\{ -\frac{i}{8} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' e^{-i\theta(t)} Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot \delta[Q(t') e^{i\theta(t')}] \right\}. \tag{E1}$$

Using the notation $E(\bar{t}, \tau) \doteq e^{i\theta(t') - i\theta(t)}$, we further rewrite this as follows:

$$\begin{aligned}
\delta\mathcal{B}_+ &= \frac{1}{8} \operatorname{Re} \left\{ \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot [Q(t') \delta\theta(t') - i\delta Q(t')] \right\} \\
&= \frac{1}{16} \left\{ \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot Q(t') \delta\theta(t') + \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' [E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot Q(t')]^\dagger \delta\theta(t') \right. \\
&\quad \left. - i \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot \delta Q(t') + i \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' [E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot \delta Q(t')] \right\} \\
&= \frac{1}{16} \left\{ \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot Q(t') \delta\theta(t') + \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, -\tau) Q^\dagger(t') \cdot V^\dagger(t') \cdot V(t) \cdot Q(t) \delta\theta(t') \right. \\
&\quad \left. - i \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot \delta Q(t') + i \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, -\tau) \delta Q^\dagger(t') \cdot V^\dagger(t') \cdot V(t) \cdot Q(t) \right\} \\
&= \frac{1}{16} \left\{ \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot Q(t') \delta\theta(t') + \int_{t_1}^{t_2} dt' \int_{t_1}^{t_2} dt E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot Q(t') \delta\theta(t) \right. \\
&\quad \left. - i \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot \delta Q(t') + i \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) \delta Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot Q(t') \right\} \\
&= \frac{1}{8} \left\{ \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot Q(t') \frac{\delta\theta(t') + \delta\theta(t)}{2} \right. \\
&\quad \left. - \frac{i}{2} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot \delta Q(t') + \frac{i}{2} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) \delta Q^\dagger(t) \cdot V^\dagger(t) \cdot V(t') \cdot Q(t') \right\} \\
&= \frac{1}{8} \operatorname{Tr} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) \left\{ W(\bar{t}, \tau) \frac{\delta\theta(t') + \delta\theta(t)}{2} + \frac{i}{2} [Q(t') \cdot \delta Q^\dagger(t) - \delta Q(t') \cdot Q^\dagger(t)] \right\} \cdot V^\dagger(t) \cdot V(t').
\end{aligned}$$

Next, we notice that $\delta\mathcal{B}_-[Q, \theta] = \delta\mathcal{B}_+[Q^*, -\theta] = (\delta\mathcal{B}_+[Q^*, -\theta])^*$. Hence,

$$\begin{aligned}
\delta\mathcal{B}_- &= \frac{1}{8} \operatorname{Tr} \left[\int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, -\tau) \left\{ W^*(\bar{t}, \tau) \frac{-\delta\theta(t') - \delta\theta(t)}{2} + \frac{i}{2} [Q^*(t') \cdot \delta Q^\dagger(t) - \delta Q^*(t') \cdot Q^\dagger(t)] \right\} \cdot V^\dagger(t) \cdot V(t') \right]^* \\
&= -\frac{1}{8} \operatorname{Tr} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) \left[W(\bar{t}, \tau) \frac{\delta\theta(t') + \delta\theta(t)}{2} + \delta Y(\bar{t}, \tau) \right] \cdot V^\dagger(t) \cdot V^*(t'),
\end{aligned}$$

where W is given by Eq. (132) and δY is given by Eq. (144). For $\delta\langle\mathcal{B}\rangle = \delta\mathcal{B}_+ + \delta\mathcal{B}_-$, this leads to

$$\begin{aligned}
\delta\langle\mathcal{B}\rangle &= \frac{1}{8} \operatorname{Tr} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) \left[W(\bar{t}, \tau) \frac{\delta\theta(t') + \delta\theta(t)}{2} + \delta Y(\bar{t}, \tau) \right] \cdot [V^\dagger(t) \cdot V(t') - V^\dagger(t) \cdot V^*(t')] \\
&= \frac{1}{8} \operatorname{Tr} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) \left[W(\bar{t}, \tau) \frac{\delta\theta(t') + \delta\theta(t)}{2} + \delta Y(\bar{t}, \tau) \right] \cdot [-2i\alpha(t, t')] \\
&= -\frac{i}{4} \operatorname{Tr} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' E(\bar{t}, \tau) \left[W(\bar{t}, \tau) \frac{\delta\theta(t') + \delta\theta(t)}{2} + \delta Y(\bar{t}, \tau) \right] \cdot \alpha(t, t') \\
&= -\frac{i}{4} \operatorname{Tr} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' e^{i\omega(\bar{t})\tau} [W(\bar{t}, \tau) \delta\theta(\bar{t}) + \delta Y(\bar{t}, \tau)] \cdot \bar{\alpha}(\bar{t}, \tau) + O(\epsilon_\omega^2) \\
&\approx \delta\mathcal{B}_\theta + \delta\mathcal{B}_Q,
\end{aligned} \tag{E2}$$

where $\delta\mathcal{B}_\theta$ and $\delta\mathcal{B}_Q$ are given by Eqs. (142) and (143), respectively.

[1] Practical applications of this fact include the development of advanced algorithms for numerical simulations

that enjoy exceptional long-term accuracy [76–78].

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- [65] This implies M real BC per each end point, as in the original problem (Sec. III B). This is in contrast with Ref. [5], where M complex BC were assumed per end point in the quantumlike representation [namely, $\delta\psi(t_{1,2}) = 0$], so the problem was overdefined.
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- [68] As asymptotic approximation of $\mathcal{J}[f] \doteq \int_a^b g(t) e^{i\theta(t)} dt$, where g evolves at some rate $\mathcal{T}^{-1} \ll \omega \doteq -\dot{\theta}$, can be found via integration by parts. This gives $\mathcal{J}[g] = \int_a^b [(d/dt)(ig e^{i\theta}/\omega) - ig'(t)/\omega] dt$ or, equivalently, $\mathcal{J}[g] = (ig e^{i\theta}/\omega)|_a^b - \mathcal{J}[ig'(t)/\omega]$, where the functional on the right is smaller than the functional on the left by a factor $O(\omega\mathcal{T}) \gg 1$. The calculation can also be iterated. Then, $\mathcal{J}[g]$ can be represented as an asymptotic series of terms that are determined by f, θ and their derivatives evaluated locally at the ends of the integration domain.
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