Princeton Plasma Physics Laboratory

PPPL-5291

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September 2016

Prepared for the U.S.Department of Energy under Contract DE-AC02-09CH11466.

Ponderomotive dynamics of waves in quasiperiodically modulated media

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(Dated: September 6, 2016)

Similarly to how charged particles experience time-averaged ponderomotive forces in high-frequency fields, linear waves also experience time-averaged refraction in modulated media. Here we propose a covariant variational theory of this "ponderomotive effect on waves" for a general nondissipative linear medium. Using the Weyl calculus, our formulation accommodates waves with temporal and spatial period comparable to that of the modulation (provided that parametric resonances are avoided). Our theory also shows that any wave is, in fact, a polarizable object that contributes to the linear dielectric tensor of the ambient medium. The dynamics of quantum particles is subsumed as a special case. As an illustration, ponderomotive potentials of quantum particles and photons are calculated within a number of models. We also explain a fundamental connection between these results and the commonly known expression for the electrostatic dielectric tensor of quantum plasmas.

I. INTRODUCTION

It is well known that a non-uniform high-frequency electromagnetic (EM) field can produce a time-averaged force, known as the ponderomotive force, on any particle that is charged or, more generally, has nonzero polarizability [1–6]. This effect has permitted a number of applications ranging from atomic cooling to particle acceleration [7, 8], but many other interesting opportunities remain. In particular, similar manipulations can be practiced on waves too, as shown recently in Ref. [9]. Specifically, any wave propagating through a temporally and (or) spatially modulated medium generally experiences time-averaged refraction determined by the modulation intensity [10]. It was also shown in Ref. [9] that this "ponderomotive effect on waves" subsumes the ponderomotive dynamics of particles as a special case because, quantum mechanically, particles can be represented as waves too. However, Ref. [9] assumes that the wave period (both temporal and spatial) is much smaller than the modulation period. This approximation limits the applicability of the theory. One may wonder then whether it can be relaxed (without specifying the type of waves being considered) and whether new interesting physics can be discovered then.

Here we answer these questions positively by proposing a more general theory of the ponderomotive effect on waves. For simplicity, we consider scalar dissipationless waves, but, in contrast with Ref. [9], our theory can describe waves with temporal and spatial period comparable to that of the modulation (provided that parametric resonances are avoided). Using the Weyl calculus, we explicitly derive the effective dispersion operator that governs the time-averaged dynamics of a wave in a quasiperiodically modulated medium. This result is later used to obtain the wave *ponderomotive Hamiltonian* (41). This formulation can be understood as a generalization of the oscillation-center (OC) theory, which is known from classical plasma physics [11–13], to any linear waves and quantum particles in particular. Our theory also shows that *any wave is, in fact, a polarizable object* that contributes to the linear dielectric tensor of the ambient medium. As an illustration, ponderomotive potentials of quantum particles and photons are calculated within a number of models and compared with simulations. In particular, we find that quantum effects can change the sign of the ponderomotive force. We also explain a fundamental connection between these results and the commonly known expression for the quantum-plasma electrostatic dielectric function. This work also serves as a stepping stone to improving the understanding of the modulational instabilities in general wave ensembles, as will be reported separately.

It is to be noted that effective Hamiltonians for temporally driven systems have been studied in condensed matter physics [14–21]. However, these studies are mainly focused on systems described by the Schrödinger equation and use the modulation period as the small parameter. In contrast, we study more general waves and expand in the modulation amplitude rather than period. This way, we can calculate the ponderomotive effect on waves using the Weyl calculus, which provides a direct connection with classical physics and the aforementioned OC theory in particular.

This work is organized as follows. In Sec. II the basic notation is defined. In Sec. III we present the variational formalism and the main assumptions used throughout the work. In Sec. IV we derive a general expression for the effective wave action. In Sec. V we present a theory of ponderomotive dynamics for eikonal waves. In Sec. VI we apply the theory to specific examples. In Sec. VII we show the fundamental connection between the ponderomotive potential that we derive in this paper and the commonly known dielectric tensor of quantum plasma. In Sec. VIII we summarize our main results. Some auxiliary calculations are presented in the Appendices. This includes an introduction to the Weyl calculus that we extensively use in the paper (Appendix A) and details of some of the calculations presented (Appendix B).

II. NOTATION

The following notation is used throughout the paper. The symbol "$\doteq$" denotes definitions. Unless otherwise specified, natural units are used in this work so that the speed of light equals one ($c = 1$), and so does the Planck constant ($\hbar = 1$). The Minkowski metric is adopted with signature $(+,-,-,-)$. Greek indices span from 0 to 3 and refer to spacetime coordinates, $x^\mu = (x^0, \mathbf{x})$, with $x^0 = t$. Also, $\partial_\mu \equiv \partial/\partial x^\mu = (\partial_t, \boldsymbol{\nabla})$, and $\mathrm{d}^4 x \equiv \mathrm{d}x^0\, \mathrm{d}^3\mathbf{x}$. Latin indices span from 1 to 3 and denote the spatial variables, i.e., $\mathbf{x} = (x^1, x^2, x^3)$, and $\partial_i \equiv \partial/\partial x^i$. Summation over repeated indexes is assumed. For arbitrary four-vectors a and b, we have: $a \cdot b \equiv a^\mu b_\mu = a^0 b^0 - \mathbf{a}\cdot\mathbf{b}$. The Dirac bra-ket notation is used to denote $|\Psi\rangle$ as a state of the Hilbert space defined over $\mathbb{R}^4$. In Euler-Lagrange equations (ELEs), the notation "δa :" denotes that the corresponding equation was obtained by extremizing the action integral with respect to a.

III. PHYSICAL MODEL

A. Wave action principle

We represent a wave field, either quantum or classical, as a scalar complex function $\Psi(x)$. The dynamics of any nondissipative linear wave can be described by the least action principle, $\delta\Lambda = 0$, where the real action Λ is bilinear in the wave field [22]. In the absence of parametric resonances [23], the action can be written in the form [24]

$$\Lambda \doteq \int \mathrm{d}^4x\, \mathrm{d}^4x'\; \Psi^*(x)\mathcal{D}(x,x')\Psi(x'), \tag{1}$$

where $\mathcal{D}$ is a Hermitian [$\mathcal{D}(x,x') = \mathcal{D}^*(x',x)$] scalar kernel that describes the underlying medium. Varying the action (1) leads to the following wave equations:

$$\delta\Psi^*(x): \quad 0 = \int \mathrm{d}^4x'\; \mathcal{D}(x,x')\Psi(x'), \tag{2a}$$

$$\delta\Psi(x): \quad 0 = \int \mathrm{d}^4x'\; \Psi^*(x')\mathcal{D}(x',x). \tag{2b}$$

For the rest of this work, it will be convenient to describe the wave $\Psi(x)$ also as an abstract vector $|\Psi\rangle$ in the Hilbert space of wave states, such that [22, 25]

$$\Psi(x) = \langle x|\Psi\rangle\,. \tag{3}$$

Here $|x\rangle$ are the eigenstates of the coordinate operator $\hat{x}$ such that $\langle x'|\hat{x}^\mu|x\rangle = x^\mu\langle x'|x\rangle = x^\mu\delta^4(x'-x)$. Let us introduce the momentum (wavevector) operator $\hat{p}$ such that $\langle x'|\hat{p}_\mu|x\rangle = i\partial\delta^4(x'-x)/\partial x^\mu$ in the coordinate representation. Thus, the action (1) can be rewritten as

$$\Lambda = \langle\Psi|\,\hat{\mathcal{D}}\,|\Psi\rangle\,, \tag{4}$$

where $\hat{\mathcal{D}}$ is the Hermitian *dispersion operator* defined such that $\mathcal{D}(x,x') = \langle x|\hat{\mathcal{D}}|x'\rangle$. Treating $\langle\Psi|$ and $|\Psi\rangle$ as independent [22], we obtain the following ELEs:

$$\delta\langle\Psi|: \quad \hat{\mathcal{D}}\,|\Psi\rangle = 0, \tag{5a}$$

$$\delta|\Psi\rangle: \quad \langle\Psi|\,\hat{\mathcal{D}} = 0, \tag{5b}$$

which can be understood as a generalized vector form of Eqs. (2). Specifically, Eqs. (2) are obtained by projecting Eqs. (5a) and (5b) by $\langle x|$ and $|x\rangle$, respectively.

B. Problem outline

Below, we consider the propagation of a wave $|\Psi\rangle$, called the *probe wave* (PW), in a medium whose parameters are modulated by some other wave, which we call the *modulating wave* (MW). Accordingly, $\mathcal{D}(x,x')$ is a rapidly oscillating function. Our goal is to derive a reduced version of Eqs. (5) that describes the time-averaged dynamics of the PW.

We assume that the dispersion operator can be decomposed as

$$\hat{\mathcal{D}} = \hat{\mathcal{D}}_0 + \hat{\mathcal{D}}_{\mathrm{osc}}, \tag{6}$$

where $\hat{\mathcal{D}}_0$ represents the effect of the unperturbed background medium and $\hat{\mathcal{D}}_{\mathrm{osc}}$ represents a weak perturbation caused by a MW. Additionally, we assume

$$\hat{\mathcal{D}}_{\mathrm{osc}} = \sum_{n=1}^{\infty} \sigma^n \hat{\mathcal{D}}_n, \tag{7}$$

where $\sigma \ll 1$ is some linear measure of the MW amplitude [26], and $\hat{\mathcal{D}}_n$ are Hermitian. Finally, we consider that the MW frequency Ω and wavevector $\mathbf{K}$ satisfy the geometrical optics (GO) assumption

$$\epsilon_{\mathrm{mw}} \doteq \max\left\{\frac{1}{\Omega\tau}, \frac{1}{|\mathbf{K}|\ell}\right\} \ll 1, \tag{8}$$

where τ and ℓ are the characteristic inhomogeneity scales of the background medium and of the MW envelope. (For clarity, we do not make a distinction between the scales of the background medium and of the MW envelope.) Note that the condition (8) applies to the MW only and not to the PW, unlike in Ref. [9].

IV. GENERAL THEORY

The oscillating terms in the dispersion operator will be eliminated by introducing an appropriate variable transformation on the PW. Specifically, let $|\Psi\rangle = \hat{\mathcal{U}}\,|\psi\rangle$. Then, Eq. (4) transforms to

$$\Lambda = \langle\psi|\,\hat{\mathcal{D}}_{\mathrm{eff}}\,|\psi\rangle\,, \tag{9}$$

where $\hat{\mathcal{D}}_{\text{eff}}$ is the effective dispersion operator

$$\hat{\mathcal{D}}_{\text{eff}} \doteq \hat{\mathcal{U}}^\dagger \hat{\mathcal{D}} \hat{\mathcal{U}}. \tag{10}$$

Below, we search for a transformation $\hat{\mathcal{U}}$ such that, unlike $\hat{\mathcal{D}}$, the operator $\hat{\mathcal{D}}_{\text{eff}}$ contains no dependence on the MW phase. The corresponding $|\psi\rangle$ is then understood as the OC state of the PW in a modulated medium.

A. Near-identity unitary transformation

For convenience, we require that $\hat{\mathcal{U}}$ be unitary so that $\langle\Psi|\Psi\rangle = \langle\psi|\psi\rangle$. Then, it can be represented as

$$\hat{\mathcal{U}} = \exp(i\hat{\mathcal{T}}), \tag{11}$$

where $\hat{\mathcal{T}}$ is a Hermitian operator called the *generator* of the unitary transformation $\hat{\mathcal{U}}$. In light of Eq. (7), we search for $\hat{\mathcal{T}}$ and $\hat{\mathcal{D}}_{\text{eff}}$ using the standard perturbation approach based on Lie transforms [12, 27]. Specifically, we consider the operators as power series in σ so that

$$\hat{\mathcal{T}} = \sum_{n=1}^{\infty} \sigma^n \hat{\mathcal{T}}_n, \quad \hat{\mathcal{D}}_{\text{eff}} = \sum_{n=0}^{\infty} \sigma^n \hat{\mathcal{D}}_{\text{eff},n}, \tag{12}$$

where $\hat{\mathcal{T}}_n$ and $\hat{\mathcal{D}}_{\text{eff},n}$ are Hermitian. We substitute Eqs. (6), (7), (11), and (12) into Eq. (10). Collecting terms by equal powers in the parameter σ, we obtain the following set of equations [28]:

$$\hat{\mathcal{D}}_{\text{eff},0} = \hat{\mathcal{D}}_0, \tag{13a}$$
$$\hat{\mathcal{D}}_{\text{eff},1} = \hat{\mathcal{D}}_1 + i[\hat{\mathcal{D}}_0, \hat{\mathcal{T}}_1], \tag{13b}$$
$$\hat{\mathcal{D}}_{\text{eff},2} = \hat{\mathcal{D}}_2 + i[\hat{\mathcal{D}}_0, \hat{\mathcal{T}}_2] + \hat{\mathcal{C}}_2, \tag{13c}$$

where $\hat{\mathcal{C}}_2 \doteq i[\hat{\mathcal{D}}_1, \hat{\mathcal{T}}_1] - (1/2)[[\hat{\mathcal{D}}_0, \hat{\mathcal{T}}_1], \hat{\mathcal{T}}_1]$ and so on. Here the brackets denote commutators. We require that $\hat{\mathcal{D}}_{\text{eff},n}$ contains no high-frequency modulations, so we let

$$\hat{\mathcal{D}}_{\text{eff},1} = \langle\langle \hat{\mathcal{D}}_1 \rangle\rangle, \tag{14a}$$
$$\hat{\mathcal{D}}_{\text{eff},2} = \langle\langle \hat{\mathcal{D}}_2 \rangle\rangle + \langle\langle \hat{\mathcal{C}}_2 \rangle\rangle, \tag{14b}$$

where '$\langle\langle ... \rangle\rangle$' is a time average over a modulation period. Then, subtracting Eqs. (14) from Eqs. (13), we obtain

$$-i[\hat{\mathcal{D}}_0, \hat{\mathcal{T}}_1] = \hat{\mathcal{D}}_1 - \langle\langle \hat{\mathcal{D}}_1 \rangle\rangle, \tag{15a}$$
$$-i[\hat{\mathcal{D}}_0, \hat{\mathcal{T}}_2] = \hat{\mathcal{D}}_2 - \langle\langle \hat{\mathcal{D}}_2 \rangle\rangle + \hat{\mathcal{C}}_2 - \langle\langle \hat{\mathcal{C}}_2 \rangle\rangle, \tag{15b}$$

As usual, this procedure can be iterated to higher orders in σ. However, for the sake of conciseness, we shall only calculate $\hat{\mathcal{D}}_{\text{eff}}$ upto $\mathcal{O}(\sigma^2)$. Below, we demonstrate how to solve Eqs. (15) for $\hat{\mathcal{T}}_1$ and $\hat{\mathcal{T}}_2$.

B. $\mathfrak{D}_{\text{eff}}$ within the leading-order approximation

In order to explicitly obtain $\hat{\mathcal{D}}_{\text{eff}}$ and $\hat{\mathcal{T}}_n$, let us consider Eqs. (13)-(15) in the Weyl representation. (Readers who are not familiar with the Weyl calculus are encouraged to read Appendix A before continuing further.) For $n = 0$, the Weyl transformation of Eq. (13a) leads to

$$D_{\text{eff},0}(x,p) = D_0(x,p), \tag{16}$$

where $D_n(x,p)$ and $D_{\text{eff},n}(x,p)$ are the Weyl symbols (A1) of the operators $\hat{\mathcal{D}}_n$ and $\hat{\mathcal{D}}_{\text{eff},n}$, respectively. For $n = 1$, the Weyl transformation of Eq. (13b) gives

$$D_{\text{eff},1} = D_1 - \{\{D_0, T_1\}\}, \tag{17}$$

where '$\{\{\cdot,\cdot\}\}$' is the Moyal sine bracket (A10) and $T_n(x,p)$ are the Weyl symbols of $\hat{\mathcal{T}}_n$. It is to be noted that D_n, $D_{\text{eff},n}$, and T_n are real functions of the eight-dimensional phase space because the corresponding operators are Hermitian.

Since D_1 is a linear measure of the MW field, we adopt

$$D_1(x,p) = \text{Re}[\mathcal{D}_1(x,p) e^{i\Theta(x)}], \tag{18}$$

where the real function $\Theta(x)$ is the MW phase and $\mathcal{D}_1(x,p)$ is the Weyl symbol characterizing the slowly-varying MW envelope [29, 30]. The gradients of the phase

$$\Omega(x) \doteq -\partial_t \Theta, \quad \mathbf{K}(x) \doteq \nabla\Theta, \tag{19}$$

determine the MW local frequency and wavevector, respectively. We introduce the MW four-wavevector $K_\mu(x) \doteq -\partial_\mu \Theta = (\Omega, -\mathbf{K})$, which is considered a slow function. [Accordingly, the contravariant representation of the MW four-wavevector is $K^\mu(x) = (\Omega, \mathbf{K})$.]

Since D_1 is quasi-periodic [31], $\langle\langle D_1 \rangle\rangle = 0$. Following Eq. (14a), then $D_{\text{eff},1} = 0$, which also gives

$$\{\{D_0, T_1\}\} = D_1. \tag{20}$$

Let us search for T_1 in the polar representation:

$$T_1 = \text{Re}[\mathcal{T}_1(x,p) e^{i\Theta(x)}], \tag{21}$$

where $\mathcal{T}_1(x,p)$ is to be determined. Substituting Eqs. (18) and (21) into Eq. (20) and equating terms with the same phase, we obtain (Appendix B)

$$\begin{aligned}
\mathcal{D}_1(x,p) e^{i\Theta(x)} &= \{\{D_0, \mathcal{T}_1 e^{i\Theta}\}\} \\
&= \mathcal{T}_1 \{\{D_0, e^{i\Theta}\}\} + \mathcal{O}(\epsilon_{\text{mw}}) \\
&= -i\mathcal{T}_1 (D_0 \star e^{i\Theta} - e^{i\Theta} \star D_0) + \mathcal{O}(\epsilon_{\text{mw}}) \\
&= -i\,[D_0(x, p + K/2) - D_0(x, p - K/2)] \\
&\quad \times \mathcal{T}_1(x,p) e^{i\Theta(x)} + \mathcal{O}(\epsilon_{\text{mw}}),
\end{aligned} \tag{22}$$

where '$\star$' is the Moyal product (A6) and $\mathcal{T}_1$ is pulled out of the sine bracket because it is a slowly-varying function. Solving for $\mathcal{T}_1$, we obtain

$$\mathcal{T}_1(x,p) = \frac{i\mathcal{D}_1(x,p)}{D_0(x, p + K/2) - D_0(x, p - K/2)} + \mathcal{O}(\epsilon_{\text{mw}}). \tag{23}$$

Now let us calculate $D_{\text{eff},2}$. From Eq. (13b), we have $[\hat{\mathcal{D}}_0, \hat{\mathcal{T}}_1] = i\hat{\mathcal{D}}_1$, so $\hat{\mathcal{C}}_2 = -(i/2)[\hat{\mathcal{T}}_1, \hat{\mathcal{D}}_1]$. Then, by applying the Weyl transform to Eq. (13c), we obtain

$$D_{\text{eff},2} = D_2 - \{\{D_0, T_2\}\} + C_2, \tag{24}$$

where $C_2(x,p) = (1/2)\{\{T_1, D_1\}\}$. After substituting D_1 and T_1, the Weyl symbol C_2 is found to be (Appendix B)

$$\begin{aligned} C_2(x,p) = &- \frac{1}{4} \sum_{n=\pm 1} \frac{|\mathcal{D}_1(x, p + nK/2)|^2}{D_0(x, p + nK) - D_0(x,p)} \\ &+ \text{Re}[\mathcal{C}_2(x,p) e^{i2\Theta(x)}] + \mathcal{O}(\epsilon_{\text{mw}}), \end{aligned} \tag{25}$$

where $\mathcal{C}_2(x,p)$ is a slowly-varying function whose explicit expression will not be needed for our purposes.

Following Eqs. (14b) and (24), we let $D_{\text{eff},2} = \langle\langle D_2 \rangle\rangle + \langle\langle C_2 \rangle\rangle$. Then, the symbol $T_2(x,p)$ satisfies

$$\{\{D_0, T_2\}\} = D_2 - \langle\langle D_2 \rangle\rangle + \text{Re}(\mathcal{C}_2 e^{i2\Theta}). \tag{26}$$

We then repeat the procedure shown in Eqs. (21)-(23) to obtain T_2 that satisfies Eq. (26). Finally, after collecting the previously obtained results of this section, the effective dispersion symbol is found to be [32]

$$\begin{aligned} D_{\text{eff}}(x,p) = &\, D_0(x,p) + \sigma^2 \langle\langle D_2(x,p) \rangle\rangle \\ &- \frac{\sigma^2}{4} \sum_{n=\pm 1} \frac{|\mathcal{D}_1(x, p + nK/2)|^2}{D_0(x, p + nK) - D_0(x,p)} \\ &+ \mathcal{O}(\epsilon_{\text{mw}}, \sigma^4). \end{aligned} \tag{27}$$

The Weyl symbol $D_{\text{eff}}(x,p)$ in Eq. (27) constitutes one of the main results of this work. The actual operator $\hat{\mathcal{D}}_{\text{eff}}$ can be obtained from the symbol (27) using the inverse Weyl transform (A2). Alternatively, one can find its coordinate representation $\mathcal{D}_{\text{eff}}(x,x')$ using Eq. (A3).

V. PONDEROMOTIVE DYNAMICS

With the effective dispersion operator obtained in Sec. IV, we can describe the time-averaged dynamics of the PW using

$$\hat{\mathcal{D}}_{\text{eff}} |\psi\rangle = 0. \tag{28}$$

Alternatively, we can apply the variational approach, and we express the action (9) in the *phase-space representation*. Following Refs. [24, 33], the action is written as

$$\Lambda = \int \mathrm{d}^4 x \, \mathrm{d}^4 p \, D_{\text{eff}}(x,p) W(x,p), \tag{29}$$

where $W(x,p)$ is the Wigner function [34] corresponding to the OC state $|\psi\rangle$; namely,

$$W(x,p) \doteq \int \frac{\mathrm{d}^4 s}{(2\pi)^4} \, e^{ip\cdot s} \langle x + s/2 | \psi \rangle \langle \psi | x - s/2 \rangle. \tag{30}$$

The variational approach is particularly convenient for deriving approximate models of wave dynamics [9, 35–42]. For illustration purposes, here we focus on the OC dynamics of PWs in the eikonal approximation, or the GO limit. Specifically, we proceed as follows.

A. Eikonal approximation

Let us consider the complex function $\psi \doteq \langle x|\psi\rangle$ in the following polar representation

$$\langle x|\psi\rangle = \psi(x) = \sqrt{\mathcal{I}_0(x)} e^{i\theta(x)}, \tag{31}$$

where $\mathcal{I}_0(x)$ and $\theta(x)$ are real functions. We assume that the phase θ is fast compared to the slowly-varying function $\mathcal{I}_0$. Specifically, we require

$$\epsilon_{\text{pw}} \doteq \max\left\{\frac{1}{\omega\tau}, \frac{1}{|\mathbf{k}|\ell}\right\} \ll 1, \tag{32}$$

where

$$\omega(x) \doteq -\partial_t \theta, \quad \mathbf{k}(x) \doteq \boldsymbol{\nabla}\theta \tag{33}$$

are the local PW frequency and the wavevector, respectively. As in Sec. III, τ and ℓ denote the characteristic homogeneity scales of the background medium and the MW envelope. For simplicity, we will denote the assumptions in Eqs. (8) and (32) by the same parameter

$$\epsilon \doteq \max\{\epsilon_{\text{mw}}, \epsilon_{\text{pw}}\} \ll 1. \tag{34}$$

[Note that, in conventional GO (such as that used in Ref. [9]), the PW needs to satisfy the harder constraint $\epsilon_{\text{GO}} \doteq \max\{\Omega/\omega, |\mathbf{K}|/|\mathbf{k}|\} \ll 1$ in the presence of a MW.]

Since ψ is assumed quasi-monochromatic, the Wigner function (30) is then, to the lowest order in ϵ [24],

$$W(x,p) = \mathcal{I}_0(x) \delta^4(p - k) + \mathcal{O}(\epsilon), \tag{35}$$

where $k_\mu(x) \doteq -\partial_\mu \theta = (\omega, -\mathbf{k})$. Substituting Eq. (35) into Eq. (29) leads to the following action:

$$\Lambda = \int \mathrm{d}^4 x \, \mathcal{I}_0(x) D_{\text{eff}}(x,k). \tag{36}$$

The action (36) has the form of Whitham's action, where

$$\mathcal{I} \doteq \mathcal{I}_0 \partial_\omega D_{\text{eff}}(x,k) \tag{37}$$

serves as the wave action density [38]. (From now on, $k = -\partial\theta$.) Treating $\mathcal{I}_0$ and θ as independent variables yields the following ELEs:

$$\delta\theta: \quad \partial_t \mathcal{I} + \boldsymbol{\nabla}\cdot(\mathcal{I}\mathbf{v}) = 0, \tag{38a}$$

$$\delta\mathcal{I}_0: \quad D_{\text{eff}}(x,k) = 0, \tag{38b}$$

where the flow velocity $\mathbf{v}$ is given by

$$\mathbf{v}(t,\mathbf{x}) \doteq -\frac{\partial_{\mathbf{k}} D_{\text{eff}}}{\partial_\omega D_{\text{eff}}}. \tag{39}$$

Here Eq. (38a) represents the action conservation theorem, and Eq. (38b) is the local wave dispersion relation.

B. Hayes's representation

Equation (38b) can be used to express the PW frequency ω as some function $H_{\text{eff}}(t, \mathbf{x}, \boldsymbol{\nabla}\theta)$:

$$\omega = H_{\text{eff}}(t, \mathbf{x}, \boldsymbol{\nabla}\theta). \tag{40}$$

This determines a *dispersion manifold* [33, 43]. The function H_{eff} can be represented as follows:

$$H_{\text{eff}}(t, \mathbf{x}, \mathbf{k}) \doteq H_0(t, \mathbf{x}, \mathbf{k}) + \sigma^2 \Phi(t, \mathbf{x}, \mathbf{k}), \tag{41}$$

where higher powers of σ are neglected, like in the previous section. (Henceforth, the small parameter σ will be omitted for clarity.) Here $H_0(t, \mathbf{x}, \mathbf{k})$ is the *unperturbed frequency* of the PW, so it satisfies $D_0(x, k_*) = 0$, where

$$k_*^\mu(t, \mathbf{x}, \mathbf{k}) \doteq (H_0(t, \mathbf{x}, \mathbf{k}), \mathbf{k}) \tag{42}$$

is the unperturbed PW four-wavevector. The function $\Phi(t, \mathbf{x}, \mathbf{k})$ can be understood as the PW *ponderomotive frequency shift.* (When multiplied by $\hbar$, Φ is also understood as the ponderomotive potential.) Using Eqs. (27) and (38b) together with the Taylor expansion

$$D_{\text{eff}}(x, k) \approx D_{\text{eff}}(x, k_*) + \partial_\omega D_{\text{eff}}(x, k_*)[\omega - H_0(t, \mathbf{x}, \mathbf{k})], \tag{43}$$

we obtain an explicit expression for Φ, which is

$$\Phi(t, \mathbf{x}, \mathbf{k}) = \Bigg[- \frac{\langle\langle D_2(x, k)\rangle\rangle}{\partial_\omega D_0(x, k)} + \frac{1}{4\partial_\omega D_0(x, k)} \times \sum_{n=\pm 1} \frac{|\mathcal{D}_1(x, k + nK/2)|^2}{D_0(x, k + nK) - D_0(x, k)} \Bigg]_{k=k_*}. \tag{44}$$

Hence, we can rewrite the action (36) in the Hayes's form [44]; namely,

$$\Lambda \simeq - \int \mathrm{d}^4x\, \mathcal{I}\, [\partial_t\theta + H_{\text{eff}}(t, \mathbf{x}, \boldsymbol{\nabla}\theta)]\,. \tag{45}$$

In this case, the corresponding ELE's are

$$\delta\theta: \quad \partial_t \mathcal{I} + \boldsymbol{\nabla} \cdot (\mathcal{I}\mathbf{u}) = 0, \tag{46a}$$

$$\delta\mathcal{I}: \quad \omega = H_{\text{eff}}(t, \mathbf{x}, \mathbf{k}), \tag{46b}$$

where $\mathbf{u}$ is the effective PW group velocity

$$\mathbf{u}(t, \mathbf{x}) \doteq \partial_{\mathbf{k}} H_{\text{eff}}(t, \mathbf{x}, \mathbf{k}). \tag{47}$$

Here Eq. (46b) is a Hamilton–Jacobi equation representing the local wave dispersion. Note that, on solutions of Eq. (46b), $\mathbf{u}(t, \mathbf{x})$ is the same as $\mathbf{v}(t, \mathbf{x})$ defined in Eq. (39), so Eqs. (46) are consistent with Eqs. (38).

Another comment is the following. When $|K| \ll |k_*|$, the effective Hamiltonian can be approximated to

$$H_{\text{eff}}(t, \mathbf{x}, \mathbf{k}) \simeq H_0(t, \mathbf{x}, \mathbf{k}) + \Bigg[- \frac{\langle\langle D_2(x, k)\rangle\rangle}{\partial_\omega D_0(x, k)} + \frac{\sigma^2}{4\partial_\omega D_0(x, k)} K^\mu \frac{\partial}{\partial k^\mu} \left(\frac{|\mathcal{D}_1(x, k)|^2}{K^\nu \partial_{k^\nu} D_0(x, k)} \right) \Bigg]_{k=k_*}, \tag{48}$$

where $K^\mu \partial_{k^\mu} \equiv \Omega \partial_\omega + \mathbf{K} \cdot \partial_{\mathbf{k}}$. When $D(x, p)$ in Eq. (6) is of the Hayes's form $[D(x, p) = \omega - H(t, \mathbf{x}, \mathbf{k})]$, Eq. (48) recovers the same expression for H_{eff} that was previously reported in Ref. [9].

C. Point-particle model and ray equations

The ray equations corresponding to Eqs. (46) can be obtained by assuming the point-particle limit. Specifically, let us adopt the ansatz

$$\mathcal{I}(t, \mathbf{x}) = \delta^3(\mathbf{x} - \mathbf{X}(t)), \tag{49}$$

where $\mathbf{X}(t)$ is the location of the wave packet. As in Ref. [45], integrating the action (45) in space yields the canonical phase-space action of a point-particle; namely,

$$\Lambda = \int \mathrm{d}t\, [\mathbf{P} \cdot \dot{\mathbf{X}} - H_{\text{eff}}(t, \mathbf{X}, \mathbf{P})], \tag{50}$$

where $\mathbf{P}(t) \doteq \boldsymbol{\nabla}\theta(t, \mathbf{X}(t))$. Here, $\mathbf{X}(t)$ and $\mathbf{P}(t)$ serve as canonical variables. The corresponding ELEs are

$$\delta\mathbf{P}: \quad \dot{\mathbf{X}} = \partial_{\mathbf{P}} H_{\text{eff}}(t, \mathbf{X}, \mathbf{P}), \tag{51a}$$

$$\delta\mathbf{X}: \quad \dot{\mathbf{P}} = -\partial_{\mathbf{X}} H_{\text{eff}}(t, \mathbf{X}, \mathbf{P}). \tag{51b}$$

Equations (51) describe the ponderomotive dynamics of PW rays. These equations include the time-averaged refraction of a PW under the action of a MW. The ponderomotive dynamics of charged particles is subsumed here as a special case. Below, we discuss this in detail.

VI. DISCUSSION AND EXAMPLES

A. Example 1: Schrödinger particle in an electrostatic field

We consider a nonrelativistic particle interacting in a modulated electrostatic potential. The particle dynamics can be described using the Schrödinger equation

$$i\partial_t \Psi = \left[-\boldsymbol{\nabla}^2/2m + qV(x) \right] \Psi, \tag{52}$$

where m and q are the particle mass and charge, the electrostatic potential $V(x) = \operatorname{Re}\left[V_c(x)e^{i\Theta(x)}\right]$ is assumed small, $\Theta(x)$ is a real fast phase, and $V_c(x)$ is a complex function describing the slowly-varying potential envelope. In this case, the dispersion operator is

$$\hat{\mathfrak{D}} \doteq \hat{p}_0 - \hat{\mathbf{p}}^2/2m - qV(\hat{x}). \tag{53}$$

The corresponding Weyl symbols are (Appendix A)

$$D_0(p) = p_0 - \mathbf{p}^2/2m, \quad D_{\text{osc}}(x) = -qV(x). \tag{54}$$

The symbol D_{eff} is calculated using Eq. (27). Note that $D_1 = -\operatorname{Re}(qV_c e^{i\Theta})$ so $\mathcal{D}_1 = -qV_c$ and $D_n = 0$ for

$n \geq 2$. Substituting into Eq. (27), we obtain

$$\begin{aligned}
&D_{\text{eff}}(x,p) \\
&= D_0(p) - \frac{1}{4}\sum_{n=\pm 1} \frac{|\mathcal{D}_1(x)|^2}{D_0(p+nK) - D_0(p)} \\
&= p_0 - \frac{\mathbf{p}^2}{2m} - \sum_{n=\pm 1} \frac{q^2|V_c(x)|^2/4}{\left(p_0 + n\Omega - \frac{(\mathbf{p}+n\mathbf{K})^2}{2m}\right) - \left(p_0 - \frac{\mathbf{p}^2}{2m}\right)} \\
&= p_0 - \frac{\mathbf{p}^2}{2m} - \frac{q^2|\mathbf{K}V_c|^2/m}{4(\Omega - \mathbf{p}\cdot\mathbf{K}/m)^2 - (\mathbf{K}^2/m)^2}. \qquad (55)
\end{aligned}$$

Inserting Eq. (55) into Eq. (36) leads to the action in the Hayes form (45), where the effective Hamiltonian is

$$H_{\text{eff}}(t,\mathbf{x},\mathbf{k}) = \frac{\mathbf{k}^2}{2m} + \frac{q^2|\mathbf{K}V_c|^2/m}{4(\Omega - \mathbf{k}\cdot\mathbf{K}/m)^2 - (\mathbf{K}^2/m)^2}. \quad (56)$$

In the fluid description of the particle wave packet, the corresponding ELEs are given by Eqs. (46). The corresponding ray equations are obtained from the point-particle Lagrangian (50). When introducing the missing $\hbar$ factors, the effective point-particle Hamiltonian is

$$H_{\text{eff}}(t,\mathbf{X},\mathbf{P}) = \frac{\mathbf{P}^2}{2m} + \frac{q^2|\mathbf{K}V_c|^2/m}{4(\Omega - \mathbf{P}\cdot\mathbf{K}/m)^2 - (\hbar\mathbf{K}^2/m)^2}. \quad (57)$$

The classical ponderomotive Hamiltonian [3, 40]

$$H_{\text{eff,cl}}(t,\mathbf{X},\mathbf{P}) = \frac{\mathbf{P}^2}{2m} + \frac{q^2|\mathbf{K}V_c|^2}{4m(\Omega - \mathbf{P}\cdot\mathbf{K}/m)^2} \quad (58)$$

is recovered from Eq. (57) in the limit of small $\mathbf{K}$. In constrast with $H_{\text{eff,cl}}$, Eq. (57) predicts that the ponderomotive potential can be attractive, as confirmed numerically in Fig. 1. This effect is similar to that reported in Refs. [15, 46]. Note that the ray trajectories generated by H_{eff} accurately match the motion of the wave packet's center. Also note that, at $\Omega = 0$, the ponderomotive potential is resonant at $2\mathbf{K}\cdot\mathbf{P} = \pm\hbar\mathbf{K}^2$. This relation can be written as $\lambda_{\text{dB}} = 2d\cos\zeta$, where λ_{dB} is the particle de Broiglie wavelength, $d \doteq 2\pi/K$ is the characteristic length of the lattice, and ζ is the angle between the $\mathbf{K}$ and $\mathbf{P}$ vectors. One may recognize this as the Bragg resonance. In other words, our wave theory presents Bragg scattering as a variation of the ponderomotive effect. One can also identify a parallel between Eq. (57) and the linear susceptibility of quantum plasma [47, 48]. This will be further discussed in Sec. VII.

B. Example 2: Ponderomotive dynamics of a relativistic spinless particle

In this section we calculate the ponderomotive Hamiltonian of a relativistic spinless particle interacting with a slowly-varying background EM field and a high-frequency EM modulation [49]. The particle dynamics can be described using the Klein-Gordon equation

$$\left[(i\partial_t - qV)^2 - (-i\boldsymbol{\nabla} - q\mathbf{A})^2 - m^2\right]\Psi = 0, \quad (59)$$

where $V(x)$ and $\mathbf{A}(x)$ are the scalar and vector potentials, respectively. The four-potential is given by

$$A^\mu(x) \doteq A^\mu_{\text{bg}}(x) + A^\mu_{\text{osc}}(x), \quad (60)$$

where $A^\mu_{\text{bg}}(x)$ is the four-potential describing the background EM field and $A^\mu_{\text{osc}}(x) \doteq \text{Re}[A^\mu_c(x)e^{i\Theta(x)}]$ is the modulated EM wave four-potential and is considered small. As before, $\Theta(x)$ is a real fast phase, and $A^\mu_c(x)$ is the slowly-varying complex amplitude of the MW. In terms of operators, the dispersion operator is given by

$$\hat{\mathfrak{D}}_0 = [\hat{p}^\mu - qA^\mu_{\text{bg}}(\hat{x})][\hat{p}_\mu - qA_{\text{bg},\mu}(\hat{x})] - m^2, \quad (61a)$$

$$\begin{aligned}
\hat{\mathfrak{D}}_{\text{osc}} = &-\{qA^\mu_{\text{osc}}(\hat{x})[\hat{p}_\mu - qA_{\text{bg},\mu}(\hat{x})] + \text{h.c.}\} \\
&+ q^2 A^\mu_{\text{osc}}(\hat{x})A_{\text{osc},\mu}(\hat{x}). \qquad (61b)
\end{aligned}$$

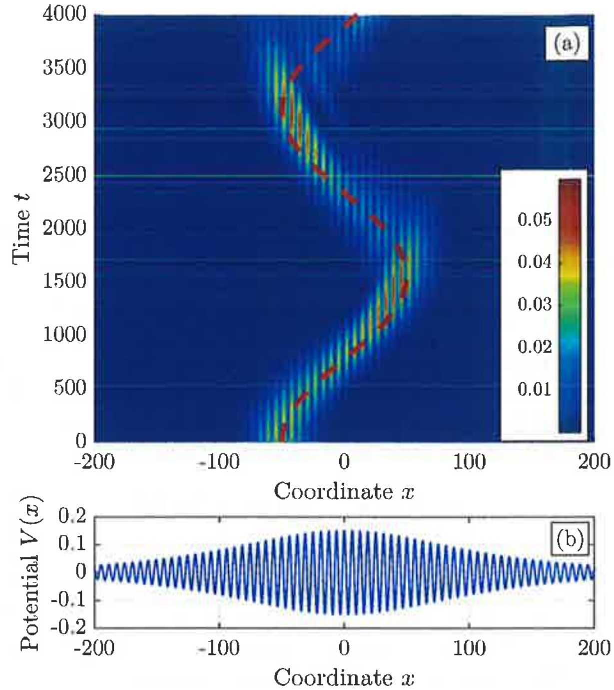

FIG. 1: (a) Comparison of the simulation results obtained by numerically integrating the full-wave Eq. (52) (solid fill) and the ray-tracing Eqs. (51) with H_{eff} taken from Eq. (57) (dashed) for a stationary MW. The initial wave packet is $\Psi_0(x) = (2\pi\eta^2)^{-1/4}\exp\left[-(x-\mu)^2/(4\eta^2)\right]$, where $\mu = -50$ and $\eta = 15$, and it is normalized such that $\int dx\, \Psi_0^2(x) = 1$. (The simulation is one-dimensional, and x denotes the spatial coordinate, unlike in the main text, where x denotes the spacetime coordinate.) The initial conditions for the ray trajectory are $X(0) = -50$ and $P(0) = 0$. (b) MW profile of the form $V(x) = 0.15\,\text{sech}(x/80)\cos(x)$. Natural units are used such that $m = 1$, $q = 1$, $\hbar = 1$, and $K = 1$. At later times (not shown), diffraction effects become important, so the eikonal theory becomes inapplicable.

The corresponding Weyl symbols are (Appendix A)

$$D_0(x,p) = [p^\mu - qA^\mu_{\rm bg}(x)][p_\mu - qA_{{\rm bg},\mu}(x)] - m^2, \quad (62a)$$
$$D_1(x,p) = -2qA^\mu_{\rm osc}(x)[p_\mu - qA_{{\rm bg},\mu}(x)], \quad (62b)$$
$$D_2(x,p) = q^2 A^\mu_{\rm osc}(x) A_{{\rm osc},\mu}(x). \quad (62c)$$

Substituting into Eq. (27), we obtain

$$D_{\rm eff}(x,p) = \pi^\mu \pi_\mu - m^2 + \frac{q^2|A_c|^2}{2} - \sum_{n=\pm 1} \frac{q^2|A_c\cdot(\pi + nK/2)|^2}{2n\pi\cdot K + n^2 K\cdot K}, \quad (63)$$

where $\pi^\mu \doteq p^\mu - qA^\mu_{\rm bg}$, $|A_c|^2 = A_c\cdot A_c^* = |V_c|^2 - |\mathbf{A}_c|^2$, and $|A_c\cdot(\pi+nK/2)|^2 = [A_c\cdot(\pi+nK/2)][A_c^*\cdot(\pi+nK/2)]$.

Following the procedure in Sec. V, we determine the effective Hamiltonian for a point particle. Introducing the missing c and $\hbar$ factors, we obtain

$$H_{\rm eff}(t,\mathbf{X},\mathbf{P}) = \gamma mc^2 + qV_{\rm bg} - \frac{q^2(A_c\cdot A_c^*)}{4\gamma mc^2} + \frac{1}{2\gamma mc^2}\sum_{n=\pm 1}\frac{q^2|A_c\cdot(\Pi_* + n\hbar K/2)|^2}{2n\Pi_*\cdot\hbar K + n^2\hbar^2 K\cdot K}, \quad (64)$$

where

$$\gamma(t,\mathbf{X},\mathbf{P}) \doteq \sqrt{1 + \left(\frac{\mathbf{P}}{mc} - \frac{q\mathbf{A}_{\rm bg}}{mc^2}\right)^2}, \quad (65)$$

is the unperturbed Lorentz factor, $\Pi^\mu_* \doteq (\gamma mc, \mathbf{P} - q\mathbf{A}_{\rm bg}/c)$ is the unperturbed kinetic four-momentum, $A^\mu_c(t,\mathbf{X}) = (V_c, \mathbf{A}_c)$ is the modulated four-potential, and $K^\mu(t,\mathbf{X}) = (\Omega/c, \mathbf{K})$ is the MW four-wavevector. All quantities are evaluated at the particle position $\mathbf{X}(t)$.

Several interesting limits can be studied with the effective Hamiltonian (64). In the Lorentz gauge, where $\partial_\mu A^\mu_{\rm osc} = 0$, we have $K\cdot A_c = \mathcal{O}(\epsilon)$. Then, $H_{\rm eff}$ becomes

$$H_{\rm eff}(t,\mathbf{X},\mathbf{P}) = \gamma mc^2 + qV_{\rm bg} - \frac{q^2(A_c\cdot A_c^*)}{4\gamma mc^2} - \left(\frac{q^2|A_c\cdot\Pi_*|^2}{\gamma mc^2}\right)\frac{K\cdot K}{4(\Pi_*\cdot K)^2 - (\hbar K\cdot K)^2}. \quad (66)$$

Let us analyze the terms appearing in Eq. (66). For example, $K\cdot K = (\Omega/c)^2 - \mathbf{K} = 0$ for a vacuum wave, so the second line vanishes. The remaining terms can be understood as the lowest-order expansion (in $|A_c|^2$) of the effective ponderomotive Hamiltonian $H_{\rm eff} = mc^2[1 + (\mathbf{P}/mc - q\mathbf{A}_{\rm bg}/mc^2)^2 - q^2(A_c\cdot A_c^*)/2m^2c^4]^{1/2} + qV_{\rm bg}$ that a relativistic spinless particle experiences in an oscillating EM pulse [50–53]. In the case where $K\cdot K \neq 0$, the term in the second line of Eq. (66) persists and accounts for Compton scattering, much like the Bragg scattering discussed in Sec. VI A.

Also, let us consider a particle that interacts with an oscillating electrostatic field so that $A^\mu_c = (V_c, 0)$. In this case, Eq. (64) gives (Appendix B)

$$H_{\rm eff}(t,\mathbf{X},\mathbf{P}) = \gamma mc^2 + qV_{\rm bg} + \frac{q^2|V_c|^2}{4\gamma m} \times \frac{\mathbf{K}^2 - (\mathbf{v}_*\cdot\mathbf{K}/c)^2 - (\hbar\mathbf{K}/2\gamma mc)^2(K\cdot K)}{(\Omega - \mathbf{v}_*\cdot\mathbf{K})^2 - (\hbar K\cdot K/2\gamma m)^2}. \quad (67)$$

where $\mathbf{v}_* \doteq \mathbf{\Pi}/(\gamma m)$ is the unperturbed particle velocity. The last term in Eq. (67) is the relativistic ponderomotive energy. [In the nonrelativistic limit, when $\gamma \simeq 1$ and $\hbar|\mathbf{K}| \ll mc$, Eq. (67) reduces to Eq. (57), as expected.] When quantum corrections are negligible, we obtain

$$H_{\rm eff}(t,\mathbf{X},\mathbf{P}) = \gamma mc^2 + qV_{\rm bg} + \frac{q^2|\mathbf{K}V_c|^2}{4M(\Omega - \mathbf{v}_*\cdot\mathbf{K})^2}, \quad (68)$$

where $M \doteq m\gamma|\mathbf{K}|^2 / [\,|\mathbf{K}|^2 - (\mathbf{v}_*\cdot\mathbf{K}/c)^2]$. When $\mathbf{v}_*$ is pointed along $\mathbf{K}$, one has $M = m\gamma^3$, which is understood as the longitudinal mass. In contrast, when $\mathbf{v}$ is transverse to $\mathbf{K}$, one has $M = m\gamma$, which is understood as the transverse mass [54].

C. Example 3: Electrostatic wave in a density modulated plasma

As another example, let us consider an EM wave $\Psi(x)$ propagating in a density-modulated plasma. The PW dynamics is described by

$$\partial_t^2\Psi = \nabla^2\Psi - \omega_p^2\Psi, \quad (69)$$

where $\omega_p^2(x) \doteq 4\pi q^2 n(x)/m$ is the plasma frequency squared [55]. The plasma density is modulated such that $n(x) = n_{\rm bg}(x) + n_{\rm osc}(x)$, where $n_{\rm bg}(x)$ is the slowly-varying background plasma density and $n_{\rm osc}(x) \doteq {\rm Re}[n_c(x)e^{i\Theta(x)}]$ is a fast modulation of small amplitude. The dispersion operator is given by

$$\hat{\mathcal{D}} = \hat{p}\cdot\hat{p} - \omega_p^2(\hat{x}), \quad (70)$$

so the corresponding Weyl symbols are

$$D_0(x,p) = p\cdot p - \omega^2_{p,\rm bg}(x), \quad (71a)$$
$$D_{\rm osc}(x) = -{\rm Re}[\omega^2_{p,c}(x)e^{i\Theta(x)}], \quad (71b)$$

where $\omega^2_{p,\rm bg}(x) \doteq 4\pi q^2 n_{\rm bg}/m$ and $\omega^2_{p,c}(x) \doteq 4\pi q^2 n_c/m$.

Substituting Eqs. (71) into Eq. (27), we obtain

$$D_{\rm eff}(x,p) = p\cdot p - \omega^2_{p,\rm bg} + \frac{|\omega^2_{p,c}|^2(K\cdot K)/8}{(p\cdot K)^2 - (K\cdot K)^2/4}. \quad (72)$$

Then, the effective Hamiltonian is given by

$$H_{\rm eff}(t,\mathbf{x},\mathbf{k}) = \omega_0(t,\mathbf{x},\mathbf{k}) - \frac{|\omega^2_{p,c}|^2}{16\omega_0^3}\Xi, \quad (73)$$

where $\omega_0(t,\mathbf{x},\mathbf{k}) \doteq (c^2\mathbf{k}^2 + \omega_{p,\mathrm{bg}}^2)^{1/2}$ is the unperturbed EM wave frequency,

$$\Xi \doteq \frac{\Omega^2 - c^2\mathbf{K}^2}{(\Omega - \mathbf{v}_* \cdot \mathbf{K})^2 - (\Omega^2 - c^2\mathbf{K}^2)^2/4\omega_0^2}, \tag{74}$$

and $\mathbf{v}_* = c^2\mathbf{k}/\omega_0$ is the unperturbed EM wave group velocity. (For clarity, we inserted the missing c factors.) The second term in Eq. (73) represents the ponderomotive frequency shift. Hence, even classical EM waves in a modulated plasma experience ponderomotive effects.

Similarly to the previous examples, the denominator in Eq. (73) also contains photon recoil effects. The Bragg resonance condition is also included; i.e., for EM waves propagating in static modulated media ($\Omega = 0$), the Bragg resonance occurs at $2\mathbf{k}\cdot\mathbf{K} = \pm\mathbf{K}^2$. In the opposite limit where $\Omega \gg \mathbf{v}_* \cdot \mathbf{K}$, Eq. (73) becomes

$$H_{\mathrm{eff}}(t,\mathbf{x},\mathbf{k}) = \omega_0 + \left(\frac{|\omega_{p,c}^2|^2}{16\omega_0^3}\right) \frac{N^2-1}{1-\mu^2(N^2-1)^2}, \tag{75}$$

where $\mu \doteq \Omega/2\omega_0$ and $N \doteq c|\mathbf{K}|/\Omega$ is the MW refraction index. Note that the GO result reported in Ref. [9] is recovered in the limit $\mu \ll 1$.

VII. MODULATIONAL DYNAMICS AND POLARIZABILITY OF WAVE QUANTA

A. Basic equations

Knowing the effective Hamiltonian H_{eff} of PWs, we can easily derive the self-consistent dynamics of a MW when it interacts with an ensemble of PWs. (As a special case, when PWs are free charged particles, such ensemble is a plasma.) To do this, let us consider the Lagrangian of the whole system in the form $\Lambda_\Sigma = \Lambda_{\mathrm{mw}} + \Lambda_{\mathrm{pw}}$, where Λ_{mw} is the MW action, and Λ_{pw} is the cumulative action of all PWs. We attribute the interaction action to Λ_{pw}, so, by definition, Λ_{mw} is the system action absent PWs. Then, Λ_{mw} equals the action of the MW EM field in vacuum, $\Lambda_{\mathrm{mw}} \approx \int \mathrm{d}^4x\,(\mathbf{E}_{\mathrm{mw}}^2 - \mathbf{B}_{\mathrm{mw}}^2)/(8\pi)$ [56]. Since the MW is assumed to satisfy the GO approximation, its electric and magnetic fields can be expressed as

$$\mathbf{E}_{\mathrm{mw}} = \mathrm{Re}\,(\mathbf{E}_c e^{i\Theta}), \quad \mathbf{B}_{\mathrm{mw}} = \mathrm{Re}\,(\mathbf{B}_c e^{i\Theta}), \tag{76}$$

where the envelopes $\mathbf{E}_c$ and $\mathbf{B}_c$ are slow compared to Θ. Then, we can approximate Λ_{mw} as

$$\Lambda_{\mathrm{mw}} = \int \mathrm{d}^4x \left(\frac{|\mathbf{E}_c|^2}{16\pi} - \frac{c^2|\mathbf{K}\times\mathbf{E}_c|^2}{16\pi\Omega^2}\right), \tag{77}$$

where we substituted Faraday's law $\mathbf{B}_c \approx (c\mathbf{K}/\Omega)\times\mathbf{E}_c$.

To calculate Λ_{pw}, we assume that PWs are mutually incoherent and do not interact other than via the MW. (When PWs are charged particles, this is known as the collisionless-plasma approximation. In a broader context, this can be recognized as the quasilinear approximation.) Then, $\Lambda_{\mathrm{pw}} = \sum_i \Lambda_i$, where Λ_i are the actions of individual PWs. Let us adopt Λ_i in the form (45). Hence,

$$\Lambda_{\mathrm{pw}} = \Lambda_{\mathrm{PW},0} - \sum_i \int \mathrm{d}^4x\,\mathcal{I}_i(t,\mathbf{x})\,\Phi_i(t,\mathbf{x},\nabla\theta_i), \tag{78}$$

where $\Lambda_{\mathrm{PW},0} = -\sum_i \int \mathrm{d}^4x\,\mathcal{I}_i\,[\partial_t\theta_i + H_{0,i}(t,\mathbf{x},\nabla\theta_i)]$ is independent of MW variables, so it can be dropped. (In this section, we are only interested in ELEs for the MW, and $\Lambda_{\mathrm{PW},0}$ does not contribute to those.) Let us consider PWs in groups s such that, within each group, PWs have the same ponderomotive frequency shift Φ_s. Then,

$$\Lambda_{\mathrm{pw}} = -\sum_s \int \mathrm{d}^4x\,\mathrm{d}^3\mathbf{p}\,f_s(t,\mathbf{x},\mathbf{p})\,\Phi_s(t,\mathbf{x},\mathbf{p}), \tag{79}$$

where $f_s \doteq \sum_{i\in s} \mathcal{I}_i(t,\mathbf{x})\,\delta[\mathbf{p} - \nabla\theta_i(t,\mathbf{x})]$ represents the total phase-space density of species s. This gives $\Lambda_\Sigma = \int \mathrm{d}^4x\,\mathfrak{L}$, where the Lagrangian density $\mathfrak{L}$ is

$$\mathfrak{L} = \frac{|\mathbf{E}_c|^2}{16\pi} - \frac{c^2|\mathbf{K}\times\mathbf{E}_c|^2}{16\pi\Omega^2} - \sum_s \int \mathrm{d}^3\mathbf{p}\,f_s(t,\mathbf{x},\mathbf{p})\,\Phi_s(t,\mathbf{x},\mathbf{p}).$$

Since Φ_s is bilinear in the MW field and independent on the MW phase, it can be expressed as

$$\Phi_s = -\frac{1}{4}\,\mathbf{E}_c^* \cdot \boldsymbol{\alpha}_s \cdot \mathbf{E}_c, \tag{80}$$

where $\boldsymbol{\alpha}_s$ is some complex tensor that can depend on Ω and $\mathbf{K}$ but not on $\mathbf{E}_c$ or $\mathbf{E}_c^*$. Explicitly, it is defined as

$$\boldsymbol{\alpha}_s \doteq -4\frac{\partial^2}{\partial\mathbf{E}_c\,\partial\mathbf{E}_c^*}\,\Phi_s(\mathbf{E}_c,\mathbf{E}_c^*,\Omega,\mathbf{K};\,t,\mathbf{x},\mathbf{p}), \tag{81}$$

or, equivalently, $\boldsymbol{\alpha}_s \doteq -4\partial^2 H_{\mathrm{eff},s}/(\partial\mathbf{E}_c\,\partial\mathbf{E}_c^*)$. Then,

$$\mathfrak{L} = \frac{1}{16\pi}\,\mathbf{E}_c^* \cdot \boldsymbol{\varepsilon}(t,\mathbf{x},\Omega,\mathbf{K}) \cdot \mathbf{E}_c - \frac{c^2|\mathbf{K}\times\mathbf{E}_c|^2}{16\pi\Omega^2}, \tag{82}$$

where we introduced $\boldsymbol{\varepsilon} \doteq 1 + \boldsymbol{\chi}$ and

$$\boldsymbol{\chi} \doteq 4\pi \sum_s \int \mathrm{d}^3\mathbf{p}\,f_s(t,\mathbf{x},\mathbf{p})\,\boldsymbol{\alpha}_s(t,\mathbf{x},\mathbf{p},\Omega,\mathbf{K}). \tag{83}$$

By treating $(\Theta,\mathbf{E}_c,\mathbf{E}_c^*)$ as independent variables, we then obtain the following ELEs:

$$\delta\Theta: \quad \partial_t(\partial_\Omega\mathfrak{L}) - \nabla\cdot(\partial_{\mathbf{K}}\mathfrak{L}) = 0, \tag{84}$$

$$\delta\mathbf{E}_c^*: \quad (\Omega/c)^2\,\boldsymbol{\varepsilon}\cdot\mathbf{E}_c + \mathbf{K}\times(\mathbf{K}\times\mathbf{E}_c) = 0, \tag{85}$$

plus a conjugate equation for $\mathbf{E}_c^*$. [Remember that Ω and $\mathbf{K}$ are related to Θ via Eq. (19).] We then recognize these ELEs as the GO equations describing EM waves in a dispersive medium with dielectric tensor $\boldsymbol{\varepsilon}$ [55, 57]. Thus, $\boldsymbol{\chi}$ is the susceptibility of the medium, and $\boldsymbol{\alpha}_s$ serves as the *linear polarizability* of PWs of type s [58].

Hence, Eq. (80) can be interpreted as a fundamental relation between the ponderomotive energy and the linear polarizability. This relation represents a generalization of the well-known "K-χ theorem" [3, 59], which establishes the same equalities for classical particles, to general waves. Those include quantum particles as a special case and also photons, plasmons, phonons, etc. According to the theory presented here, any such object can be assigned a ponderomotive energy and thus has a polarizability (81). Some examples are discussed below.

B. Examples

As a first example, let us consider a nonrelativistic quantum electron with charge q, mass m, and OC momentum $\mathbf{P} \equiv m\mathbf{v}$. Suppose the electron interacts with an electrostatic MW (so $\mathbf{B}_c = 0$). Then, H_{eff} can be taken from Eq. (57), and Eq. (81) readily yields that the electron polarizability is a diagonal matrix given by

$$\alpha_e = -\mathbb{I}_3 \frac{q^2}{m} \left[(\Omega - \mathbf{K}\cdot\mathbf{v})^2 - (\hbar\mathbf{K}^2/2m)^2\right]^{-1}. \qquad (86)$$

(Here, $\mathbb{I}_3$ is a 3×3 unit matrix.) The susceptibility of nondegenerate plasma formed by such electrons is [60]

$$\chi = -\mathbb{I}_3 \frac{4\pi q^2}{m} \int \mathrm{d}^3\mathbf{v}\, \frac{f(t, \mathbf{x}, \mathbf{v})}{(\Omega - \mathbf{K}\cdot\mathbf{v})^2 - (\hbar\mathbf{K}^2/2m)^2},$$

which is precisely the textbook result [47]. This shows that the commonly known expression for the dielectric tensor of quantum plasma is actually a reflection of the less-known quantum ponderomotive energy.

Second, consider an EM wave in a nonmagnetized density-modulated cold electron plasma. Using the continuity equation for the electron fluid and also the fluid momentum equation, one readily finds that

$$\omega_{p,c}^2 = \left(\frac{iq\omega_{p,\text{bg}}^2}{m\Omega^2}\right) \mathbf{K}\cdot\mathbf{E}_c, \qquad (87)$$

where we assume the same notation as in Sec. VI C. Then, using Eq. (73), one gets

$$\Phi = -\hbar\omega_0 \left(\frac{\omega_{p,\text{bg}}^4 \Xi}{16\omega_0^4}\right) \frac{q^2(\mathbf{E}_c^*\cdot\mathbf{K}\mathbf{K}\cdot\mathbf{E}_c)}{m^2\Omega^4}. \qquad (88)$$

where $\mathbf{KK}$ is a dyadic tensor. (The factor $\hbar$ is introduced in order to treat Φ as a per-photon energy rather than as a classical frequency.) Hence, according to Eq. (81), a photon has a linear polarizability

$$\alpha_{\text{ph}} = \hbar\omega_0 \left(\frac{\omega_{p,\text{bg}}^4 \Xi}{4\omega_0^4}\right) \frac{q^2\mathbf{KK}}{m^2\Omega^4}. \qquad (89)$$

In principle, one must account for this polarizability when calculating $\boldsymbol{\varepsilon}$; i.e., photons contribute to the linear dielectric tensor just like electrons and ions. That said, the photon polarizability is typically small compared to the electron polarizability because, loosely,

$$\frac{\alpha_{\text{ph}}}{\alpha_e} \sim \left(\frac{\hbar\omega_0}{mc^2}\right) \frac{\omega_{p,\text{bg}}^4}{\omega_0^4}, \qquad (90)$$

where $\hbar\omega_0/mc^2 \ll 1$. Hence, ignoring the photon contribution to the plasma dielectric tensor is justified except at large enough photon densities.

Similar calculations are also possible for dissipative dynamics and vector waves and also help understand the modulational dynamics of wave ensembles in a general context. However, elaborating on these topics is outside the scope of this paper and is left to future publications.

VIII. CONCLUSIONS

In this work, we show that scalar waves, both quantum and classical, can experience time-averaged refraction when propagating in modulated media. This phenomenon is analogous to the ponderomotive effect encountered by charged particles in high-frequency EM fields. We propose a covariant variational theory of this "ponderomotive effect on waves" for a general nondissipative linear medium. Using the Weyl calculus, our formulation is able to describe waves with temporal and spatial period comparable to that of the modulation (provided that parametric resonances are avoided). This theory can be understood as a generalization of the oscillation-center theory, which is known from classical plasma physics, to any linear waves or quantum particles in particular. This work also shows that any wave is, in fact, a polarizable object that contributes to the linear dielectric tensor of the ambient medium. Three examples of applications of the theory are given: a Schrödinger particle propagating in an oscillating electrostatic field, a Klein-Gordon particle interacting with modulated EM fields, and an EM wave propagating in a density-modulated plasma.

This work can be expanded in several directions. First, one can extend the theory to dissipative waves [61] and vector waves with polarization effects [39, 62], which could be important at Bragg resonances. Second, the theory presented here can be used as a stepping stone to improving the understanding of the modulational instabilities in general wave ensembles. This requires a generalization of the analysis presented in Sec. VII and will be reported in a separate paper.

The authors thank N. J. Fisch for valuable discussions. This work was supported by the U.S. DOE through Contract No. DE-AC02-09CH11466, by the NNSA SSAA Program through DOE Research Grant No. DE-NA0002948, and by the U.S. DOD NDSEG Fellowship through Contract No. 32-CFR-168a.

Appendix A: The Weyl transform

This appendix summarizes our conventions for the Weyl transform. (For more information, see the excellent reviews in Refs. [33, 63–65].) The Weyl symbol $A(x,p)$ of any given operator $\hat{A}$ is defined as

$$A(x,p) \doteq \int \mathrm{d}^4 s\, e^{ip\cdot s} \langle x + s/2|\hat{A}|x - s/2\rangle\,, \tag{A1}$$

where $p\cdot s = p^0 s^0 - \mathbf{p}\cdot\mathbf{s}$ and the integrals span over $\mathbb{R}^4$. We shall refer to this description of the operators as a *phase-space representation* since the symbols (A1) are functions of the eight-dimensional phase space. Conversely, the inverse Weyl transformation is given by

$$\hat{A} = \int \frac{\mathrm{d}^4x\,\mathrm{d}^4p\,\mathrm{d}^4s}{(2\pi\epsilon)^4}\, e^{ip\cdot s} A(x,p)\, |x - s/2\rangle \langle x + s/2|\,. \tag{A2}$$

Hence, $\mathcal{A}(x,x') = \langle x|\hat{A}|x'\rangle$ can be expressed as

$$\mathcal{A}(x,x') = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, e^{ip\cdot(x'-x)} A\left(\frac{x+x'}{2}, p\right). \tag{A3}$$

In the following, we will outline a number of useful properties of the Weyl transform.

- For any operator $\hat{A}$, the trace $\mathrm{Tr}[\hat{A}] \doteq \int \mathrm{d}^4x\, \langle x|\hat{A}|x\rangle$ can be expressed as

$$\mathrm{Tr}[\hat{A}] = \int \frac{\mathrm{d}^4x\,\mathrm{d}^4p}{(2\pi)^4} A(x,p). \tag{A4}$$

- If $A(x,p)$ is the Weyl symbol of $\hat{A}$, then $A^*(x,p)$ is the Weyl symbol of $\hat{A}^\dagger$. As a corollary, the Weyl symbol of a Hermitian operator is real.

- For any $\hat{\mathcal{C}} = \hat{A}\hat{\mathcal{B}}$, the corresponding Weyl symbols satisfy [66, 67]

$$C(x,p) = A(x,p) \star B(x,p). \tag{A5}$$

  Here '$\star$' refers to the *Moyal product*, which is given by

$$A(x,p) \star B(x,p) \doteq A(x,p) e^{i\hat{\mathcal{L}}/2} B(x,p), \tag{A6}$$

  and $\hat{\mathcal{L}}$ is the *Janus operator*, which is given by

$$\hat{\mathcal{L}} \doteq \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_x} - \overleftarrow{\partial_x} \cdot \overrightarrow{\partial_p} \equiv \{\overleftarrow{\cdot}, \overrightarrow{\cdot}\}. \tag{A7}$$

  The arrows indicate the direction in which the derivatives act, and $A\hat{\mathcal{L}}B = \{A, B\}$ is the canonical Poisson bracket in the eight-dimensional phase space, namely,

$$\hat{\mathcal{L}} = \frac{\overleftarrow{\partial}}{\partial p^0}\frac{\overrightarrow{\partial}}{\partial x^0} - \frac{\overleftarrow{\partial}}{\partial x^0}\frac{\overrightarrow{\partial}}{\partial p^0} + \frac{\overleftarrow{\partial}}{\partial \mathbf{x}}\cdot\frac{\overrightarrow{\partial}}{\partial \mathbf{p}} - \frac{\overleftarrow{\partial}}{\partial \mathbf{p}}\cdot\frac{\overrightarrow{\partial}}{\partial \mathbf{x}}. \tag{A8}$$

- The Moyal product is associative; i.e.,

$$A \star B \star C = (A \star B) \star C = A \star (B \star C). \tag{A9}$$

- The anti-symmetrized Moyal product defines the so-called *Moyal bracket*

$$\{\{A, B\}\} \doteq \frac{1}{i}(A \star B - B \star A) = 2A \sin\left(\frac{\epsilon\hat{\mathcal{L}}}{2}\right) B. \tag{A10}$$

  Because of the latter equality, the Moyal bracket is also often called the *sine bracket*. To the lowest order in ϵ,

$$\{\{A, B\}\} \simeq \{A, B\}. \tag{A11}$$

- Now we tabulate some Weyl transforms of various operators. (We use a two-sided arrow to show the correspondence with the Weyl transform.) First of all, the Weyl transforms of the identity, position, and momentum operators are given by

$$\hat{1} \Leftrightarrow 1, \quad \hat{x}^\mu \Leftrightarrow x^\mu, \quad \hat{p}_\mu \Leftrightarrow p_\mu. \tag{A12}$$

  If f and g are any two functions, then

$$f(\hat{x}) \Leftrightarrow f(x), \quad g(\hat{p}) \Leftrightarrow g(p). \tag{A13}$$

  Similarly, using Eq. (A6), we have

$$\hat{p}_\mu f(\hat{x}) \Leftrightarrow p_\mu f(x) + (i/2)\partial_\mu f(x), \tag{A14}$$

$$f(\hat{x})\hat{p}_\mu \Leftrightarrow p_\mu f(x) - (i/2)\partial_\mu f(x). \tag{A15}$$

Appendix B: Auxiliary calculations

Here we include additional details on some of the derived results presented in this work. Specifically, to obtain Eq. (22), first note that $\Theta(x)$ is fast while $K_\mu(x) \doteq -\partial_\mu \Theta$ is slowly varying. Then,

$$\begin{aligned}
&A(x,p) \star e^{i\Theta} \\
&\quad = A(x,p) e^{i\hat{\mathcal{L}}/2} e^{i\Theta} \\
&\quad = A(x,p) \left(\sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \frac{\overleftarrow{\partial}^n}{\partial p^n} \cdot \frac{\overrightarrow{\partial}^n}{\partial x^n} \right) e^{i\Theta} \\
&\quad = A(x,p) \left[\sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \frac{\overleftarrow{\partial}^n}{\partial p^n} \cdot \left(i \frac{\partial \Theta}{\partial x} \right)^n \right] e^{i\Theta} + \mathcal{O}(\epsilon_{\mathrm{mw}}) \\
&\quad = A(x,p) \left[\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\overleftarrow{\partial}^n}{\partial p^n} \cdot \left(\frac{K}{2} \right)^n \right] e^{i\Theta} + \mathcal{O}(\epsilon_{\mathrm{mw}}) \\
&\quad = A(x, p + K/2) e^{i\Theta} + \mathcal{O}(\epsilon_{\mathrm{mw}}),
\end{aligned} \tag{B1}$$

where we used a single dot product to denote the contraction of the tensors involved. Similarly,

$$e^{i\Theta} \star A(x,p) = A(x, p - K/2) e^{i\Theta} + \mathcal{O}(\epsilon_{\mathrm{mw}}). \tag{B2}$$

For the calculation shown in Eq. (25), we need the following result:

$$
\begin{aligned}
&A(x,p)e^{i\Theta_1} \star B(x,p)e^{i\Theta_2}\\
&\quad = A(x,p)e^{i\Theta_1} e^{i\hat{\mathcal{L}}/2} B(x,p)e^{i\Theta_2}\\
&\quad = A(x,p)e^{i\Theta_1} e^{i(\overleftarrow{\partial_p}\cdot\overrightarrow{\partial_x} - \overleftarrow{\partial_x}\cdot\overrightarrow{\partial_p})/2} e^{i\Theta_2} B(x,p)\\
&\quad = A(x,p)e^{i\Theta_1} e^{-(\overleftarrow{\partial_p}\cdot\partial_x\Theta_2 - \partial_x\Theta_1\cdot\overrightarrow{\partial_p})/2} e^{i\Theta_2} B(x,p)\\
&\qquad + \mathcal{O}(\epsilon_{\rm mw})\\
&\quad = A(x,p)e^{i\Theta_1} e^{\overleftarrow{\partial_p}\cdot(K_2/2)} e^{-(K_1/2)\cdot\overrightarrow{\partial_p}} e^{i\Theta_2} B(x,p)\\
&\qquad + \mathcal{O}(\epsilon_{\rm mw})\\
&\quad = A(x,p+K_2/2)B(x,p-K_1/2)e^{i(\Theta_1+\Theta_2)} + \mathcal{O}(\epsilon_{\rm mw}).
\end{aligned}
\tag{B3}
$$

Substituting this result, we then obtain

$$
\begin{aligned}
C_2 &= \{\{T_1, D_1\}\}/2\\
&= \{\{\mathcal{T}_1(x,p)e^{i\Theta}, \mathcal{D}_1^*(x,p)e^{-i\Theta}\}\}/8\\
&\quad + \{\{\mathcal{T}_1^*(x,p)e^{-i\Theta}, \mathcal{D}_1(x,p)e^{i\Theta}\}\}/8\\
&\quad + \{\{\mathcal{T}_1(x,p)e^{i\Theta}, \mathcal{D}_1(x,p)e^{i\Theta}\}\}/8\\
&\quad + \{\{\mathcal{T}_1^*(x,p)e^{-i\Theta}, \mathcal{D}_1^*(x,p)e^{-i\Theta}\}\}/8\\
&= \mathcal{T}_1(x,p-K/2)\mathcal{D}_1^*(x,p-K/2)/(8i)\\
&\quad - \mathcal{T}_1(x,p+K/2)\mathcal{D}_1^*(x,p+K/2)/(8i)\\
&\quad + \mathcal{T}_1^*(x,p+K/2)\mathcal{D}_1(x,p+K/2)/(8i)\\
&\quad - \mathcal{T}_1^*(x,p-K/2)\mathcal{D}_1(x,p-K/2)/(8i)\\
&\quad + \mathrm{Re}[\mathcal{C}_2(x,p)e^{2i\Theta(x)}] + \mathcal{O}(\epsilon_{\rm mw}),
\end{aligned}
\tag{B4}
$$

where $\mathcal{C}_2(x,p)$ is some function, whose explicit expression is not important for our purposes. Substituting Eq. (23) into Eq. (B4), we obtain

$$
\begin{aligned}
C_2 &= -\frac{1}{4}\left[\frac{|\mathcal{D}_1(x,p+K/2)|^2}{D_0(x,p+K) - D_0(x,p)|^2}\right.\\
&\qquad \left. + \frac{|\mathcal{D}_1(x,p-K/2)|^2}{D_0(x,p-K) - D_0(x,p)}\right]\\
&\quad + \mathrm{Re}[\mathcal{C}_2(x,p)e^{2i\Theta(x)}] + \mathcal{O}(\epsilon_{\rm mw})\\
&= -\frac{1}{4}\sum_{n=\pm1}\frac{|\mathcal{D}_1(x,p+nK/2)|^2}{D_0(x,p+nK) - D_0(x,p)|^2}\\
&\quad + \mathrm{Re}[\mathcal{C}_2(x,p)e^{2i\Theta(x)}] + \mathcal{O}(\epsilon_{\rm mw}).
\end{aligned}
\tag{B5}
$$

The calculation of Eq. (66) is presented below. Starting from Eq. (64) and letting $A_c^\mu = (V_c, 0)$, we have

$$
\begin{aligned}
&H_{\rm eff}(t,\mathbf{X},\mathbf{P}) - \gamma mc^2 - qV_{\rm bg}\\
&\quad = -\frac{q^2(A_c\cdot A_c^*)}{4\gamma mc^2} + \frac{1}{2\gamma mc^2}\sum_{n=\pm1}\frac{q^2|A_c\cdot(\Pi_* + n\hbar K/2)|^2}{2n\Pi_*\cdot\hbar K + n^2\hbar^2 K\cdot K},\\
&\quad = -\frac{q^2|V_c|^2}{4\gamma mc^2} + \frac{q^2|V_c|^2}{2\gamma mc^2}\left[\frac{(\gamma mc + \hbar\Omega/2c)^2}{2\Pi_*\cdot\hbar K + \hbar^2 K\cdot K} - \frac{(\gamma mc - \hbar\Omega/2c)^2}{2\Pi_*\cdot\hbar K - \hbar^2 K\cdot K}\right]\\
&\quad = -\frac{q^2|V_c|^2}{4\gamma mc^2} + \left(\frac{q^2|V_c|^2}{2\gamma mc^2}\right)\frac{(\gamma mc + \hbar\Omega/2c)^2(2\Pi_*\cdot\hbar K - \hbar^2 K\cdot K) - (\gamma mc - \hbar\Omega/2c)^2(2\Pi_*\cdot\hbar K + \hbar^2 K\cdot K)}{4(\Pi_*\cdot\hbar K)^2 - (\hbar^2 K\cdot K)^2}\\
&\quad = -\frac{q^2|V_c|^2}{4\gamma mc^2} + \left(\frac{q^2|V_c|^2}{8\gamma mc^2}\right)\frac{4\gamma m\Omega(\Pi_*\cdot K) - 2\gamma^2 m^2c^2(K\cdot K) - \hbar^2\Omega^2(K\cdot K)/2c^2}{(\Pi_*\cdot K)^2 - (\hbar K\cdot K/2)^2}\\
&\quad = \left(\frac{q^2|V_c|^2}{8\gamma mc^2}\right)\frac{4\gamma m\Omega(\Pi_*\cdot K) - 2\gamma^2 m^2c^2(K\cdot K) - 2(\Pi_*\cdot K)^2 - \hbar^2\Omega^2(K\cdot K)/2c^2 + \hbar^2(K\cdot K)^2/2}{(\Pi_*\cdot K)^2 - (\hbar K\cdot K/2)^2}\\
&\quad = \left(\frac{q^2|V_c|^2}{8\gamma mc^2}\right)\frac{4\gamma m\Omega(\gamma m\Omega - \mathbf{\Pi}\cdot\mathbf{K}) - 2\gamma^2 m^2c^2(\Omega^2/c^2 - \mathbf{K}^2) - 2(\gamma m\Omega - \mathbf{\Pi}\cdot\mathbf{K})^2 - \hbar^2\mathbf{K}^2(K\cdot K)/2}{(\Pi_*\cdot K)^2 - (\hbar K\cdot K/2)^2}\\
&\quad = \left(\frac{q^2|V_c|^2}{4\gamma m}\right)\frac{\mathbf{K}^2 - (\mathbf{\Pi}\cdot\mathbf{K}/\gamma mc)^2 - (\hbar\mathbf{K}/2\gamma mc)^2(K\cdot K)}{(\Omega - \mathbf{\Pi}\cdot\mathbf{K}/\gamma m)^2 - (\hbar K\cdot K/2\gamma m)^2}\\
&\quad = \left(\frac{q^2|V_c|^2}{4\gamma m}\right)\frac{\mathbf{K}^2 - (\mathbf{\Pi}\cdot\mathbf{K}/\gamma mc)^2}{(\Omega - \mathbf{\Pi}\cdot\mathbf{K}/\gamma m)^2} + \mathcal{O}\left(\hbar^2\right).
\end{aligned}
\tag{B6}
$$

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