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Two corrections to the drift-wave kinetic equation in the context of zonal-flow physics

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The drift-wave (DW) kinetic equation, that is commonly used in studies of zonal flows (ZF), excludes the exchange of enstrophy between DW and ZF and also effects beyond the geometrical-optics limit. Using the quasilinear approximation of the generalized Hasegawa–Mima model, we propose a modified theory that accounts for these effects within a wave kinetic equation (WKE) of the Wigner–Moyal type, which is commonly known in quantum mechanics. In the geometrical-optics limit, this theory features additional terms beyond the traditional WKE that ensure exact conservation of the *total* enstrophy and energy in the DW-ZF system. Numerical simulations are presented to illustrate the importance of these additional terms. The proposed theory can be viewed as a reformulation of the second-order cumulant expansion (also known as the CE2) in a more intuitive manner, namely, in terms of canonical phase-space variables.

I. INTRODUCTION

The formation of zonal flows (ZF) is a problem of fundamental interest in many contexts, including physics of planetary atmospheres, astrophysics, and fusion science [1–7]. In particular, the interaction of ZF and drift-wave (DW) turbulence in laboratory plasmas significantly affects the transport of energy, momentum, and particles, so understanding it is critical to improving plasma confinement. But modeling the underlying physics remains a difficult problem. The workhorse approach to describing the DW-ZF coupling, which is the wave kinetic equation (WKE) [5, 8], is limited to the ray approximation and, in fact, is oversimplified even as a geometrical-optics (GO) model. That leads to missing essential physics, as was recently pointed out in Ref. [9] and will be elaborated below. These issues can be fixed by using the more accurate quasilinear approach known as the second-order cumulant expansion, or CE2 [10–14], whose applications to DW-ZF physics were pursued in Refs. [15–17]. However, the CE2 is nowhere near as intuitive as the WKE, and its robustness with respect to further approximations remains obscure. Having an approach as accurate as the CE2 and as intuitive as the WKE would be more advantageous.

Here, we propose such approach for a DW turbulence model based on the generalized Hasegawa–Mima equation (gHME) [18, 19]. The idea is as follows. We start by splitting the gHME into two coupled equations that describe ZF and fluctuations, respectively, and then linearize the equation for fluctuations, like in the CE2 approach. We notice then that this linearized equation is similar to that for a quantum particle governed by a generalized (non-Hermitian) Hamiltonian. By drawing on this analogy, we then formulate an *exact* (modulo quasilinear approximation) kinetic equation for such particle, which is akin to the so-called Wigner–Moyal equation in quantum mechanics [20–22].

Compared to the CE2, the Wigner–Moyal formulation is more intuitive, namely, for two reasons: (i) like the traditional WKE (tWKE), it permits viewing DW quanta ("driftons") as particles, except now driftons are *quantumlike* particles, i.e., have nonzero wavelengths; and (ii) the separation between Hamiltonian effects and dissipation remains transparent and unambiguous even beyond the GO approximation. Compared to the tWKE, the new approach is more precise, also for two reasons: (i) it captures effects beyond the GO limit; and (ii) *even in the GO limit*, it predicts corrections to the tWKE that emerge from the newly found corrections to the drifton dispersion. (In this aspect, our paper can be understood as an expansion of the GO approximation introduced in Ref. [9].) These corrections are essential as they allow DW-ZF enstrophy exchange, which is not included in the tWKE. By deriving the GO limit from first principles, we eliminate this discrepancy and obtain a theory that exactly conserves the total enstrophy (as opposed to the DW enstrophy conservation predicted by the tWKE) and the total energy, in precise agreement with the underlying gHME. We also illustrate the substantial difference between the GO limit of our theory and the tWKE using numerical simulations.

The paper is organized as follows. In Sec. II we introduce the gHME and its quasilinear approximation. In Sec. III we derive the Wigner–Moyal theory for DW and ZF. In Sec. IV we rederive the dispersion relation for the linear growth rate of ZF. In Sec. V we derive a corrected WKE that, in contrast to the tWKE, conserves the total enstrophy (and also energy). Numerical simulations are presented to compare the new WKE with the tWKE. In Sec. VI we summarize our main results. Some auxiliary calculations are also presented in Appendices. This includes a brief introduction to the Weyl calculus that we extensively use in our paper (Appendix A), a spectral representation of our theory (Appendix B), and proofs of the conservation properties of our models (Appendix C).

II. BASIC MODEL

Our theory is based on the gHME [18, 19],

$$\partial_t w + \mathbf{v}\cdot\nabla w + \beta\,\partial_x\psi = Q, \tag{1}$$

which is widely used to describe electrostatic two-dimensional (2-D) turbulent flows both in a magnetized plasma with a density gradient and in an atmospheric fluid on a rotating planet, where the role of DW is played by Rossby waves [1, 17]. Both contexts will be described on the same footing, so our results are applicable to DW and Rossby waves equally. We assume the usual geophysical coordinate system, where $\mathbf{x} = (x, y)$ is a 2-D coordinate, the x-axis is the ZF direction, and the y-axis is the direction of the local gradient of the plasma density or of the Coriolis parameter. (In the context of fusion plasmas, a different choice of coordinates is usually preferred in literature, where x and y are swapped.) The constant β is a measure of this gradient. The function $w(\mathbf{x}, t)$ is the generalized vorticity given by $w \doteq (\nabla^2 - L_\mathrm{D}^{-2}\hat{\alpha})\psi$, where $\hat{\alpha}$ is an operator such that $\hat{\alpha} = 1$ in parts of the spectrum corresponding to DW and $\hat{\alpha} = 0$ in those corresponding to ZF. (The symbol $\doteq$ denotes definitions.) Also, L_D is the plasma sound radius or the deformation radius. (For plasmas, one can take $L_\mathrm{D} = 1$ in normalized units [18]. Also, the barotropic model used in geophysics is recovered in the limit $L_\mathrm{D} \to \infty$ [10–13].) Also, $\psi(\mathbf{x}, t)$ is the electric potential or the stream function, $\mathbf{v} \doteq \mathbf{e}_z \times \nabla\psi$ is the fluid velocity on the $\mathbf{x}$ plane, and $\mathbf{e}_z$ is a unit vector normal to this plane. The term $Q(\mathbf{x}, t)$ describes external forces and dissipation. Systems with $Q = 0$ will be called isolated.

Let us decompose fields into their zonal-averaged and fluctuating components, denoted with bars and tildes, respectively. (For any g, its zonal average is $\bar{g} \doteq \int \mathrm{d}x\, g/L_x$, where L_x, henceforth assumed equal to one, is the system length along x axis.) In particular, $w = \bar{w} + \widetilde{w}$, where the two components of the generalized vorticity are related to ψ as [15]

$$\bar{w} = \nabla^2\bar{\psi}, \quad \widetilde{w} = \nabla_\mathrm{D}^2\widetilde{\psi}, \tag{2}$$

and $\nabla_\mathrm{D}^2 \doteq \nabla^2 - L_\mathrm{D}^{-2}$. Equations for $\widetilde{w}$ and $\bar{w}$ are obtained by taking the zonal-average and fluctuating parts of Eq. (1). This gives

$$\partial_t\widetilde{w} + \widetilde{\mathbf{v}}\cdot\nabla\bar{w} + \bar{\mathbf{v}}\cdot\nabla\widetilde{w} + \beta\,\partial_x\widetilde{\psi} + f_\mathrm{NL} = \widetilde{Q}, \tag{3a}$$

$$\partial_t\bar{w} + \overline{\widetilde{\mathbf{v}}\cdot\nabla\widetilde{w}} = \bar{Q}, \tag{3b}$$

where $\partial_x\bar{\psi} = 0$ and $f_\mathrm{NL} \doteq \widetilde{\mathbf{v}}\cdot\nabla\widetilde{w} - \overline{\widetilde{\mathbf{v}}\cdot\nabla\widetilde{w}}$ is a nonlinear term. Assuming f_NL is negligible in Eq. (3a), Eqs. (3) become

$$\partial_t\widetilde{w} + \widetilde{\mathbf{v}}\cdot\nabla\bar{w} + \bar{\mathbf{v}}\cdot\nabla\widetilde{w} + \beta\,\partial_x\widetilde{\psi} = \widetilde{Q}, \tag{4a}$$

$$\partial_t\bar{w} + \overline{\widetilde{\mathbf{v}}\cdot\nabla\widetilde{w}} = \bar{Q}. \tag{4b}$$

Equations (4) compose the well-known quasilinear model [10]. In isolated systems, both sets of equations conserve the enstrophy and the energy (strictly speaking, free energy) defined as

$$\mathcal{Z} \doteq \frac{1}{2}\int \mathrm{d}^2x\, w^2, \quad \mathcal{E} \doteq -\frac{1}{2}\int \mathrm{d}^2x\, w\psi. \tag{5}$$

It is convenient to rewrite Eqs. (4) in terms of the ZF velocity $\bar{\mathbf{v}} = \mathbf{e}_x U$, whose only component $U(y, t)$ is $U = -\partial_y\bar{\psi}$. Specifically, one has $\widetilde{\mathbf{v}}\cdot\nabla\bar{w} = -(\partial_x\widetilde{\psi})(\partial_y^2 U)$, $\bar{\mathbf{v}}\cdot\nabla\widetilde{w} = U\partial_x\widetilde{w}$, and $\overline{\widetilde{\mathbf{v}}\cdot\nabla\widetilde{w}} = -\partial_y^2\,\overline{\widetilde{v}_x\widetilde{v}_y}$. We will also assume $\widetilde{Q} = \widetilde{\xi} - \mu_\mathrm{dw}\widetilde{w}$ and $\bar{Q} = -\mu_\mathrm{zf}\bar{w}$. Here, $\widetilde{\xi}$ is some external force with zero zonal average (eventually, we will assume it to be a white noise), and the constant coefficients μ_dw and μ_zf are intended to emulate the dissipation of DW and ZF caused by the external environment. Then, Eqs. (4) become

$$\partial_t\widetilde{w} + U\partial_x\widetilde{w} + [\beta - (\partial_y^2 U)]\partial_x\widetilde{\psi} = \widetilde{\xi} - \mu_\mathrm{dw}\widetilde{w}, \tag{6a}$$

$$\partial_t U + \mu_\mathrm{zf}U + \partial_y\overline{\widetilde{v}_x\widetilde{v}_y} = 0. \tag{6b}$$

Equations (6) are the same model as the one that underlies the CE2. Although not exact, this model is useful because it captures key aspects of ZF dynamics, such as formation and merging of zonal jets [13, 17, 23]. Below, we use it to derive a formulation of DW-ZF interactions alternative to the CE2, namely, as follows.

III. WIGNER–MOYAL FORMULATION

A. State vector

Consider a family of all reversible linear transformations of $\widetilde{w}(\mathbf{x}, t)$ of the form $\int \mathrm{d}^2x'\, K(\mathbf{x}, \mathbf{x}', t)\widetilde{w}(\mathbf{x}', t)$. These transformations map $\widetilde{w}(\mathbf{x}, t)$ into some family of image functions. Since these functions are mutually equivalent up to an isomorphism, the resulting family can be viewed as an single object, a time-dependent "state vector" $|\widetilde{w}\rangle$. (Analogous definitions will be assumed also for $|\widetilde{\psi}\rangle$ and $|\widetilde{\xi}\rangle$.) The original function $\widetilde{w}(\mathbf{x}, t)$ is then understood as a projection of $|\widetilde{w}\rangle$, namely, as its "coordinate representation" given by $\widetilde{w}(\mathbf{x}, t) = \langle\mathbf{x}|\widetilde{w}\rangle$. Here, $|\mathbf{x}\rangle$ are the eigenstates of the position operator $\hat{\mathbf{x}}$ normalized such that $\langle\mathbf{x}'|\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}\,\langle\mathbf{x}'|\mathbf{x}\rangle = \mathbf{x}\,\delta(\mathbf{x}' - \mathbf{x})$. This definition of a field is similar to that used in quantum mechanics for describing probability amplitudes [24]. Hence, it is convenient to describe the dynamics of $|\widetilde{w}\rangle$ using a quantumlike formalism. This is done as follows.

In addition to the coordinate operator $\hat{\mathbf{x}}$, let us introduce a momentum (wave-vector) operator $\hat{\mathbf{p}}$ such that, in the coordinate representation, $\hat{\mathbf{p}} \doteq -i\nabla$. Accordingly, $|\widetilde{w}\rangle = -\hat{p}_\mathrm{D}^2\,|\widetilde{\psi}\rangle$, where

$$\hat{p}_\mathrm{D}^2 \doteq \hat{p}^2 + L_\mathrm{D}^{-2}, \quad \hat{p}^2 \doteq \hat{\mathbf{p}}\cdot\hat{\mathbf{p}}. \tag{7}$$

Hence, Eq. (6a) can be represented in the following form:

$$i\partial_t\,|\widetilde{w}\rangle = \hat{H}\,|\widetilde{w}\rangle + i\,|\widetilde{\xi}\rangle. \tag{8}$$

The operator $\hat{H}$ is given by

$$\hat{H} \doteq -\beta\hat{p}_x\hat{p}_{\rm D}^{-2} + \hat{U}\hat{p}_x + \hat{U}''\hat{p}_x\hat{p}_{\rm D}^{-2} - i\mu_{\rm dw}. \qquad (9)$$

Also, $\hat{U} \doteq U(\hat{y}, t)$, and the prime above U henceforth denotes ∂_y; in particular, $\hat{U}'' \doteq \partial_y^2 U(\hat{y}, t)$.

B. Generalized von Neumann equation

Let us express Eq. (9) as $\hat{H} = \hat{H}_H + i\hat{H}_A$, where $\hat{H}_H \doteq (\hat{H} + \hat{H}^\dagger)/2$ and $\hat{H}_A \doteq (\hat{H} - \hat{H}^\dagger)/(2i)$ are the Hermitian and anti-Hermitian parts of $\hat{H}$, correspondingly. Explicitly, these operators can be written as

$$\hat{H}_H = -\beta\hat{p}_x\hat{p}_{\rm D}^{-2} + \hat{U}\hat{p}_x + [\hat{U}'', \hat{p}_x\hat{p}_{\rm D}^{-2}]_+/2, \qquad (10a)$$

$$\hat{H}_A = [\hat{U}'', \hat{p}_x\hat{p}_{\rm D}^{-2}]_-/(2i) - \mu_{\rm dw}, \qquad (10b)$$

where $[\cdot,\cdot]_-$ denotes the commutator given by $[\hat{A}, \hat{B}]_- = \hat{A}\hat{B} - \hat{B}\hat{A}$, and $[\cdot,\cdot]_+$ denotes the anti-commutator given by $[\hat{A}, \hat{B}]_+ = \hat{A}\hat{B} + \hat{B}\hat{A}$. Let us also introduce a Hermitian operator $\hat{W} \doteq |\widetilde{w}\rangle\langle\widetilde{w}|$, which, by analogy with quantum mechanics, is interpreted as the "fluctuating-vorticity density" operator. It is seen from Eq. (8) that $\hat{W}$ satisfies

$$i\partial_t\hat{W} = [\hat{H}_H, \hat{W}]_- + i[\hat{H}_A, \hat{W}]_+ + i\hat{F}, \qquad (11)$$

where $\hat{F} \doteq |\widetilde{\xi}\rangle\langle\widetilde{w}| + |\widetilde{w}\rangle\langle\widetilde{\xi}|$. In particular, taking the trace of this equation also gives an equation for the "total number of DW quanta," $N \doteq {\rm Tr}\,\hat{W} = \int {\rm d}^2x\,\langle\mathbf{x}|\hat{W}|\mathbf{x}\rangle = \int {\rm d}^2x\,\widetilde{w}^2 = \langle\widetilde{w}|\widetilde{w}\rangle$, namely,

$$\dot{N} = 2{\rm Tr}\,(\hat{H}_A\hat{W}) + {\rm Tr}\,\hat{F}. \qquad (12)$$

This indicates that $\hat{H}_A$ determines the loss of quanta, or dissipation of DW. [In particular, the term $\mu_{\rm dw}$ in Eq. (10b) is responsible for DW dissipation to the external environment, whereas the term $[\hat{U}'', \hat{p}_x\hat{p}_{\rm D}^{-2}]_-/(2i)$ destroys DW quanta while conserving the total enstrophy, as will be discussed in Sec. III E.] Also, $\hat{H}_H$ determines conservative dynamics of DW and thus can be understood as the *drifton Hamiltonian.* (The non-Hermitian operator $\hat{H}$ will be attributed as the generalized Hamiltonian.) Notice that the distinction between dissipation and Hamiltonian effects remains unambiguous even beyond the GO approximation.

Equation (11) can be understood as a generalized von Neumann equation akin to the one that commonly emerges in quantum mechanics. A standard approach to such equation is to project it on the phase space using the Weyl transform. (Readers who are not familiar with the Weyl calculus are encouraged to read Appendix A before continuing further.) Hence, we proceed as follows.

C. Wigner–Moyal equation

Let us introduce W as the Weyl symbol of $\hat{W}$, i.e.,

$$W(\mathbf{x}, \mathbf{p}, t) \doteq \int {\rm d}^2s\, e^{-i\mathbf{p}\cdot\mathbf{s}}\,\langle\mathbf{x} + \mathbf{s}/2|\hat{W}|\mathbf{x} - \mathbf{s}/2\rangle\,, \qquad (13)$$

which is real because $\hat{W}$ is Hermitian. In quantum mechanics, a similar construct is known as the Wigner function [25], so one can readily identify the physical meaning of W. Specifically, in the regime when the ray approximation applies and dissipation is negligible, $W/(2\pi)^2$ represents the phase-space probability density of driftons [the numerical coefficient comes from Eq. (A4)], while beyond the GO limit it can be considered as a *generalization* of this probability density [26]. Using the fact that our $\widetilde{w}(\mathbf{x}, t)$ is real, one can also cast W as

$$W(\mathbf{x}, \mathbf{p}, t) \doteq \int {\rm d}^2s\, e^{-i\mathbf{p}\cdot\mathbf{s}}\,\widetilde{w}\Big(\mathbf{x} + \frac{\mathbf{s}}{2}, t\Big)\,\widetilde{w}\Big(\mathbf{x} - \frac{\mathbf{s}}{2}, t\Big)\,, \qquad (14)$$

which also implies

$$W(\mathbf{x}, \mathbf{p}, t) = W(\mathbf{x}, -\mathbf{p}, t). \qquad (15)$$

One can interpret the right-hand side of Eq. (14) as the local spatial spectrum of the correlation function of w. Hence, W will be called the *DW spectral function.*

By applying the Weyl transform to Eq. (11), one gets the following pseudodifferential equation:

$$\partial_t W = \{\{H_H, W\}\} + [[H_A, W]] + F. \qquad (16)$$

Here $\{\{\cdot,\cdot\}\}$ and $[[\cdot,\cdot]]$ are Moyal's "sine bracket" [Eq. (A10)] and "cosine bracket" [Eq. (A12)]. The functions H_H, H_A, and F are the Weyl symbols of $\hat{H}_H$, $\hat{H}_A$, and $\hat{F}$, respectively. In particular, using Eq. (A5) and the fact that U is independent of x, one gets

$$H_H = -\beta p_x/p_{\rm D}^2 + Up_x + [[U'', p_x/p_{\rm D}^2]]/2, \qquad (17)$$

$$H_A = \{\{U'', p_x/p_{\rm D}^2\}\}/2 - \mu_{\rm dw}, \qquad (18)$$

where $p_{\rm D}^2 \doteq p^2 + L_{\rm D}^{-2}$. By analogy with quantum mechanics, we call Eq. (16) a Wigner–Moyal equation.

Next, let us consider the zonal average of this equation,

$$\partial_t\overline{W} = \{\{H_H, \overline{W}\}\} + [[H_A, \overline{W}]] + \overline{F}, \qquad (19)$$

where $\overline{W} = \overline{W}(y, \mathbf{p}, t)$. We adopt the ergodic assumption, namely, that the zonal average is equivalent to the ensemble average [denoted $\langle\langle\ldots\rangle\rangle$] over realizations of the random force $\widetilde{\xi}$ [15]. To calculate $\overline{F} = \langle\langle F\rangle\rangle$, consider integrating Eq. (8) on a time interval (t_0, t). The result can be written as $|\widetilde{w}_t\rangle = |\widetilde{w}_{t_0}\rangle + |\delta\widetilde{w}_t\rangle + \int_{t_0}^t {\rm d}t'\,|\widetilde{\xi}_{t'}\rangle$, where the indexes denote the times at which functions are evaluated, and $|\delta\widetilde{w}_t\rangle \doteq -i\int_{t_0}^t {\rm d}t'\,\hat{H}\,|\widetilde{w}_{t'}\rangle$. We will assume

$$\langle\langle\widetilde{\xi}(\mathbf{x}, t)\widetilde{\xi}(\mathbf{x}', t')\rangle\rangle = \delta(t - t')\;\Xi[(y + y')/2, \mathbf{x} - \mathbf{x}'], \quad (20)$$

where Ξ is some function [13, 17]. Since $|\delta\widetilde{w}_t\rangle$ can be affected by $|\widetilde{\xi}_{t'}\rangle$ only if $t' < t$, one has $\langle\langle|\widetilde{\xi}_t\rangle\langle\delta\widetilde{w}_t|\rangle\rangle = 0$. Hence,

$$\overline{F}(y, \mathbf{p}) = \int {\rm d}^2s\, e^{-i\mathbf{p}\cdot\mathbf{s}}\langle\langle\langle\mathbf{x} + \mathbf{s}/2|\widetilde{\xi}_t\rangle\,\langle\widetilde{w}_t|\mathbf{x} - \mathbf{s}/2\rangle\rangle\rangle + {\rm c.c.}$$

$$= \int {\rm d}^2s\;\Xi(y, \mathbf{s})\cos(\mathbf{p}\cdot\mathbf{s}), \qquad (21)$$

where c.c. denotes "complex conjugate." In other words, once the correlation function Ξ of $\widetilde{\xi}$ is specified, $\overline{F}$ can be readily calculated as the Fourier image of Ξ.

This concludes the calculation of the functions that determine the evolution of $\overline{W}$ through Eq. (19). However, these functions generally depend on U, so an additional equation for U is needed to make the theory self-consistent. This equation is derived as follows.

D. Equation for the zonal-flow velocity

Returning to Eq. (6b), we rewrite the nonlinear term as

$$\begin{aligned}\widetilde{v}_x\widetilde{v}_y &= -(\partial_y\widetilde{\psi})(\partial_x\widetilde{\psi}) \\ &= -\langle \mathbf{x}|\,\hat{p}_y\,|\widetilde{\psi}\rangle\,\langle\widetilde{\psi}|\,\hat{p}_x\,|\mathbf{x}\rangle \\ &= -\langle \mathbf{x}|\hat{p}_y\hat{p}_{\mathrm{D}}^{-2}\hat{W}\hat{p}_{\mathrm{D}}^{-2}\hat{p}_x|\mathbf{x}\rangle \\ &= -\int \frac{d^2p}{(2\pi)^2}\,\frac{p_y}{p_{\mathrm{D}}^2}\star W\star\frac{p_x}{p_{\mathrm{D}}^2},\end{aligned} \tag{22}$$

where we used Eq. (A3) in the last step. After introducing the averaged vorticity density $\overline{W}$, Eq. (6b) becomes

$$\partial_t U + \mu_{\mathrm{zf}}U = \frac{\partial}{\partial y}\int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{p_y}{p_{\mathrm{D}}^2}\star\overline{W}\star\frac{p_x}{p_{\mathrm{D}}^2}. \tag{23}$$

Since $\overline{W}$ is independent of x and satisfies the condition (15), Eq. (23) can also be written as

$$\partial_t U + \mu_{\mathrm{zf}}U = \frac{\partial}{\partial y}\int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{1}{p_{\mathrm{D}}^2}\star p_xp_y\overline{W}\star\frac{1}{p_{\mathrm{D}}^2}. \tag{24}$$

The combination of Eqs. (19) and (24) forms a closed set of equations that can be used to calculate the dynamics of $\overline{W}$ and U self-consistently.

E. Main equations and conservation laws

Let us slightly change the notation and summarize the above equations in the following form:

$$\partial_t\overline{W} = \{\{\mathcal{H},\overline{W}\}\} + [[\Gamma,\overline{W}]] + \overline{F} - 2\mu_{\mathrm{dw}}\overline{W}, \tag{25a}$$

$$\partial_t U + \mu_{\mathrm{zf}}U = \frac{\partial}{\partial y}\int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{1}{p_{\mathrm{D}}^2}\star p_xp_y\overline{W}\star\frac{1}{p_{\mathrm{D}}^2}. \tag{25b}$$

As a reminder, $\overline{W}(y,\mathbf{p},t)$ is the zonal-averaged spectral (or Wigner) function that describes DW turbulence, and $U(y,t)$ is the ZF velocity. Also, $\overline{F}=\overline{F}(y,\mathbf{p})$ is determined by the correlation function of the external noise $\widetilde{\xi}$ (Sec. III C). We have also introduced $\mathcal{H}\doteq H_H$ and $\Gamma\doteq H_A+\mu_{\mathrm{dw}}$, or, explicitly,

$$\mathcal{H}(y,\mathbf{p},t) = -\beta p_x/p_{\mathrm{D}}^2 + p_xU + [[U'',p_x/p_{\mathrm{D}}^2]]/2, \tag{26a}$$

$$\Gamma(y,\mathbf{p},t) = \{\{U'',p_x/p_{\mathrm{D}}^2\}\}/2. \tag{26b}$$

In Appendix B, we also present a spectral representation of these equations that can be used for a numerical implementation of the Wigner–Moyal formulation.

The function $\mathcal{H}$ can be understood as the Weyl symbol of the drifton Hamiltonian, whereas Γ determines dissipation of DW quanta that is caused specifically by DW interaction with ZF. This is explained as follows. Since Eqs. (25) are *exact* within the quasilinear approximation (modulo the ergodic assumption), they inherit the same conservation laws as the original quasilinear model given by Eqs. (6). Specifically, for isolated systems ($\overline{F}=0$ and $\mu_{\mathrm{dw,zf}}=0$), Eqs. (25) and (26) exactly conserve the *total* enstrophy and energy [Eqs. (5)]

$$\mathcal{Z} = \mathcal{Z}_{\mathrm{dw}} + \mathcal{Z}_{\mathrm{zf}}, \quad \mathcal{E} = \mathcal{E}_{\mathrm{dw}} + \mathcal{E}_{\mathrm{zf}} \tag{27}$$

rather than their DW and ZF components. (A direct proof is given in Appendix C 1.) For completeness, we present expressions for these components:

$$\mathcal{Z}_{\mathrm{dw}} \doteq \frac{1}{2}\int \mathrm{d}^2x\,\widetilde{w}^2 = \frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\,\overline{W}, \tag{28a}$$

$$\mathcal{Z}_{\mathrm{zf}} \doteq \frac{1}{2}\int \mathrm{d}y\,\overline{w}^2 = \frac{1}{2}\int \mathrm{d}y\,(U')^2, \tag{28b}$$

$$\mathcal{E}_{\mathrm{dw}} \doteq -\frac{1}{2}\int \mathrm{d}^2x\,\widetilde{w}\widetilde{\psi} = \frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\,\frac{\overline{W}}{p_{\mathrm{D}}^2}, \tag{28c}$$

$$\mathcal{E}_{\mathrm{zf}} \doteq -\frac{1}{2}\int \mathrm{d}y\,\overline{w}\overline{\psi} = \frac{1}{2}\int \mathrm{d}y\,U^2, \tag{28d}$$

where we used Eqs. (A4) and (A14) to derive the second set of equalities. In particular, notice that, according to Eqs. (28a) and (A4), the DW enstrophy $\mathcal{Z}_{\mathrm{dw}}$ and the total number of DW quanta $N\doteq \mathrm{Tr}\,\hat{W}$ are the same up to a constant factor.

The conservative equations (25) and (26), which we attribute as the Wigner–Moyal formulation of DW-ZF interactions, constitute the main result of our work. On one hand, this formulation can be understood as an alternative representation of the CE2 since it is derived from the same quasilinear model. On the other hand, the Wigner–Moyal formulation is arguably more intuitive than the CE2, namely, for two reasons: (i) Like in the tWKE, driftons are treated as particles, except now they are *quantumlike* particles, i.e., have nonzero wavelengths; hence, one is not constrained to the GO limit. (ii) Also, the separation between Hamiltonian effects and dissipation remains transparent and unambiguous even beyond the GO approximation. The Wigner–Moyal formulation also elucidates the link between the WKE formalism and the CE2 and helps make approximations rigorous by making them systematic. Below, these and other applications are discussed in further detail.

IV. GROWTH RATE OF ZONAL FLOWS

To demonstrate the convenience of the proposed theory, let us apply it to rederive the rate of the zonostropic

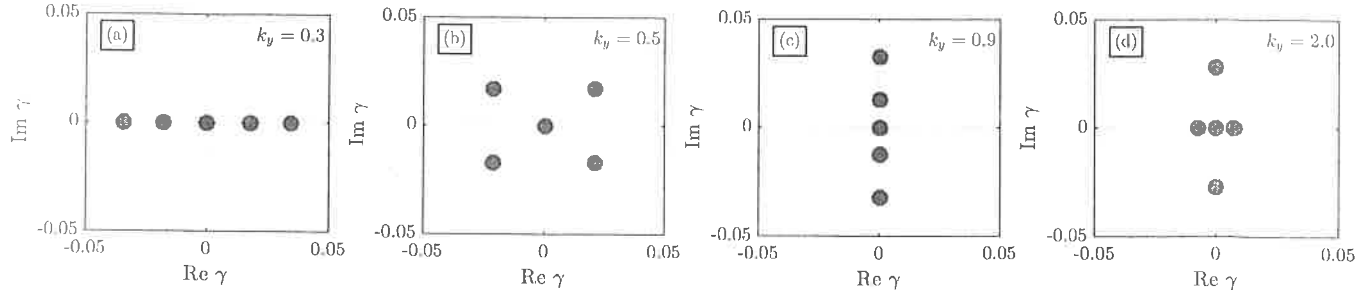


FIG. 1: Numerical solutions of the dispersion relation (34) for different k_y with fixed $k_x = 1$, $q = 0.1$, $\beta = 1$, $L_{\mathrm{D}} = 1$, and $\mathscr{N} = 1$. The solutions shown in subfigures (a)-(d) correspond to $k_y = 0.3$, $k_y = 0.5$, $k_y = 0.9$, and $k_y = 2.0$, respectively. In the interval $0.33 \lesssim |k_y| \lesssim 0.82$, the solutions γ can be complex valued. [Similar regimes can also be observed in the barotropic limit ($L_{\mathrm{D}} \to \infty$) using the same forcing and same fixed parameters.]

instability, i.e., the growth rate of weak ZF. Suppose a homogeneous equilibrium with zero ZF velocity and some DW spectral function $\mathscr{W}(\mathbf{p})$. [As pointed out in Sec. III C, the corresponding $\mathscr{W}(\mathbf{p})/(2\pi)^2$ represents the phase-space probability distribution of driftons.] Consider small perturbations to this equilibrium, namely,

$$U = \delta U(y, \mathbf{p}, t), \quad \delta U = \mathrm{Re}\,(U_q e^{iqy+\gamma t}),$$
$$\overline{W} = \mathscr{W}(\mathbf{p}) + \delta\overline{W}(y, \mathbf{p}, t), \quad \delta\overline{W} = \mathrm{Re}\,[\overline{W}_q(\mathbf{p}) e^{iqy+\gamma t}].$$

Here, the constant q serves as the modulation wave number, and the constant γ is the instability rate to be found. The linearization of Eq. (25a) leads to

$$\begin{aligned}(\partial_t + 2\mu_{\mathrm{dw}})\delta\overline{W} &+ \{\{\beta p_x/p_{\mathrm{D}}^2, \delta\overline{W}\}\} \\ &= \{\{p_x \delta U, \mathscr{W}\}\} + \{\{[[\delta U'', p_x/p_{\mathrm{D}}^2]]/2, \mathscr{W}\}\} \\ &\quad + [[\{\{\delta U'', p_x/p_{\mathrm{D}}^2\}\}/2, \mathscr{W}]], \quad (29)\end{aligned}$$

where we substituted Eqs. (26). The brackets can be calculated using Eqs. (A17). Hence, we obtain

$$\begin{aligned}&\left[i(\gamma + 2\mu_{\mathrm{dw}}) + \beta p_x \left(\frac{1}{p_{\mathrm{D},+q}^2} - \frac{1}{p_{\mathrm{D},-q}^2}\right)\right] \overline{W}_q \\ &= (\mathscr{W}_{+q} - \mathscr{W}_{-q}) \left[\frac{p_x q^2}{2}\left(\frac{1}{p_{\mathrm{D},+q}^2} + \frac{1}{p_{\mathrm{D},-q}^2}\right) - p_x\right] U_q \\ &\quad + \frac{p_x q^2}{2}(\mathscr{W}_{+q} + \mathscr{W}_{-q})\left(\frac{1}{p_{\mathrm{D},+q}^2} - \frac{1}{p_{\mathrm{D},-q}^2}\right) U_q, \quad (30)\end{aligned}$$

where we assume the notation $A_{\pm q} \doteq A(\mathbf{p} \pm \mathbf{e}_y q/2)$ for any A. Solving for $\overline{W}_q$ in terms of U_q leads to

$$\begin{aligned}\overline{W}_q &= \frac{i p_x p_{\mathrm{D},+q}^2 p_{\mathrm{D},-q}^2}{(\gamma + 2\mu_{\mathrm{dw}}) p_{\mathrm{D},+q}^2 p_{\mathrm{D},-q}^2 + 2i\beta q p_x p_y} \\ &\times \left[\mathscr{W}_{+q}\left(1 - \frac{q^2}{p_{\mathrm{D},+q}^2}\right) - \mathscr{W}_{-q}\left(1 - \frac{q^2}{p_{\mathrm{D},-q}^2}\right)\right] U_q.\end{aligned}$$

Then, Eq. (25b) yields

$$(\gamma + \mu_{\mathrm{zf}}) e^{iqy} U_q = \frac{\partial}{\partial y} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{1}{p_{\mathrm{D}}^2} \star p_x p_y e^{iqy} \overline{W}_q \star \frac{1}{p_{\mathrm{D}}^2}.$$

Due to Eq. (A16), this can be simplified as follows:

$$(\gamma + \mu_{\mathrm{zf}}) U_q = iq \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{p_x p_y}{p_{\mathrm{D},+q}^2 p_{\mathrm{D},-q}^2} \overline{W}_q. \quad (31)$$

After substituting the expression for $\overline{W}_q$, one gets

$$\begin{aligned}\gamma + \mu_{\mathrm{zf}} &= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{q p_x^2 p_y}{(\gamma + 2\mu_{\mathrm{dw}}) p_{\mathrm{D},+q}^2 p_{\mathrm{D},-q}^2 + 2i\beta q p_x p_y} \\ &\times \left[\mathscr{W}_{-q}\left(1 - \frac{q^2}{p_{\mathrm{D},-q}^2}\right) - \mathscr{W}_{+q}\left(1 - \frac{q^2}{p_{\mathrm{D},+q}^2}\right)\right]. \quad (32)\end{aligned}$$

As expected, this dispersion relation coincides with that obtained using the CE2 formalism [17]. Notably, the dependence of the integrand on $\mathscr{W}_{\pm q}$ makes the expression similar to dispersion relations that emerge in quantum mechanics; for instance, cf. Ref. [27, Sec. 40].

As a side note, it is commonly thought that ZF only grow in situ, i.e., $\mathrm{Re}\,\gamma > 0$ with $\mathrm{Im}\,\gamma = 0$. There have been questions over whether it is possible to have unstable zonal modes at nonzero $\mathrm{Im}\,\gamma$ [28]. Here we show, by presenting an example, that the answer is yes. Specifically, let us consider

$$\begin{aligned}\mathscr{W} &= (2\pi)^2 \mathscr{N}[\delta(p_x - k_x)\delta(p_y - k_y) + \delta(p_x + k_x)\delta(p_y + k_y) \\ &+ \delta(p_x + k_x)\delta(p_y - k_y) + \delta(p_x - k_x)\delta(p_y + k_y)]/4 \quad (33)\end{aligned}$$

and assume $\mu_{\mathrm{dw,zf}} = 0$ for simplicity. After integrating, the dispersion relation (32) can be cast as follows:

$$\begin{aligned}0 &= \gamma\Bigg[1 - \frac{\mathscr{N} k_x^2}{v_{gy}^2 q k_{\mathrm{D}}^4}\left(1 - \frac{q^2}{k_{\mathrm{D}}^2}\right) \\ &\times \sum_{n \in \{-1,1\}} \frac{n(k_y + nq/2) k_{\mathrm{D},+2nq}^2 / k_{\mathrm{D}}^2}{X^2 k_{\mathrm{D},+2nq}^4 / k_{\mathrm{D}}^4 + (k_y + nq/2)^2 / k_y^2}\Bigg], \quad (34)\end{aligned}$$

where $v_{gy} \doteq 2\beta k_x k_y / k_{\mathrm{D}}^4$ is the DW group velocity in the absence of zonal flows, and $X \doteq \gamma / q v_{gy}$.

Numerical solutions of Eq. (34) are presented in Fig. 1. Although the solutions for γ are real at small enough k_y, they become complex over some interval of k_y. This is

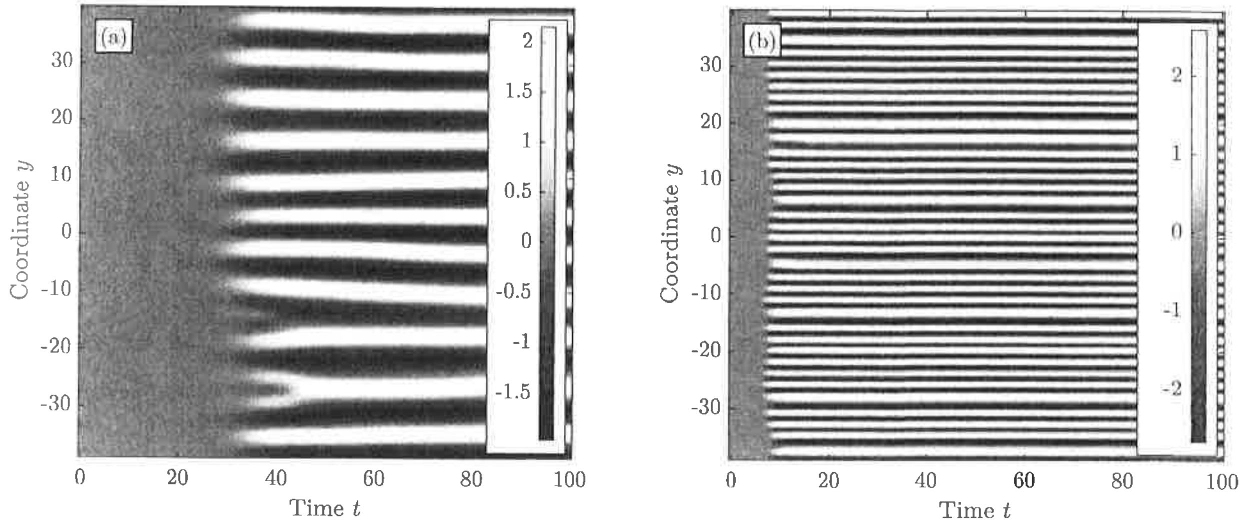


FIG. 2: The ZF velocity $U(y,t)$ obtained by numerically integrating the WKE (41) for $\mathcal{H}$ and Γ of two types: (a) our model [Eqs. (42)]; (b) the tWKE model [Eqs. (43)]. Both simulations used the same parameters and initial conditions. Small initial values for $\overline{W}$ and U were randomly assigned such that Eq. (15) was satisfied. The parameters used are: $\beta = 1$, $L_{\rm D} = 1$, $\mu_{\rm dw,zf} = 0.1$, and $\overline{F} = 4\pi\delta(|\mathbf{p}| - 1)$. Equation (41a) was discretized in a $[-39, 39] \times [-2, 2] \times [-4, 4]$ phase space using a discontinuous-Galerkin (DG) method [29] on a uniformly-spaced Cartesian grid with $80 \times 24 \times 48$ cells, while Eq. (41b) was discretized on a subset of this grid. Time advancement was done using an explicit third-order strong-stability-preserving Runge-Kutta algorithm [30]. The solution was expanded locally in each cell as a sum of piecewise polynomials of degree one. At cell interfaces, an upwind numerical flux was used in Eq. (41a) and a centered numerical flux was used in Eq. (41b). Higher-order spatial derivatives such as U'' and U''' were computed using the Recovery-based DG method [31]. For numerical stability, a small amount of hyperviscosity [16] was added in the simulations.

transparent in the limit $q \ll k$, in which Eq. (34) simplifies to

$$0 = X\left[1 - 8\alpha\frac{X^2 - 1}{(X^2+1)^2}\right] + \mathcal{O}(q^2), \tag{35}$$

where

$$\alpha \doteq \frac{\mathscr{N} k_x^2}{8 v_{gy}^2 k_{\rm D}^4}\left(1 - \frac{4k_y^2}{k_{\rm D}^2}\right). \tag{36}$$

One may consider this as the GO limit of Eq. (34). Equation (35) predicts four nontrivial solutions for X, which are given by $X^2 = -1 + 4\alpha \pm 4[\alpha(\alpha-1)]^{1/2}$. Different regimes for the solutions can be deduced. When $\alpha \geq 1$, the solutions γ are purely real. For the parameters in Fig. 1, this regime corresponds to $|k_y| \lesssim 0.33$. In the interval $0 < \alpha < 1$ corresponding to $0.33 \lesssim |k_y| \lesssim 0.82$, γ is complex valued. In the interval $-1/8 \leq \alpha \leq 0$ which corresponds to $0.82 \lesssim |k_y| \lesssim 1.07$, the solutions are purely imaginary. Finally, in the interval $\alpha < -1/8$ corresponding to $|k_y| \gtrsim 1.07$, two solutions γ are purely imaginary, and two other solutions are purely real. The different regimes identified by solving Eq. (35) are consistent with the observed numerical solutions of the exact dispersion relation (34). In the next section, we will explore the GO limit of the DW-ZF interactions in more detail.

V. GEOMETRICAL-OPTICS LIMIT AND THE WAVE KINETIC EQUATION

Let us assume that the characteristic wavelengths for zonal flows and drift waves are $\lambda_{\rm zf}$ and $\lambda_{\rm dw}$, respectively, and

$$\epsilon \doteq \max\left(\frac{\lambda_{\rm dw}}{\lambda_{\rm zf}}, \frac{L_{\rm D}}{\lambda_{\rm zf}}\right) \ll 1. \tag{37}$$

Hence, the following estimates will be adopted:

$$\begin{aligned} \partial_y \overline{W} &\backsim \lambda_{\rm zf}^{-1}\overline{W}, \quad \partial_{\mathbf{p}}\overline{W} \backsim \lambda_{\rm dw}\overline{W}, \\ \partial_y H &\backsim \lambda_{\rm zf}^{-1} H, \quad \partial_{\mathbf{p}} H \backsim L_{\rm D} H, \end{aligned} \tag{38}$$

where H denotes both $\mathcal{H}$ and Γ. (The latter estimate is given for the *maximum* of $\partial_{\mathbf{p}} H$, which is realized at $p \sim L_D^{-1}$ [32].) This gives

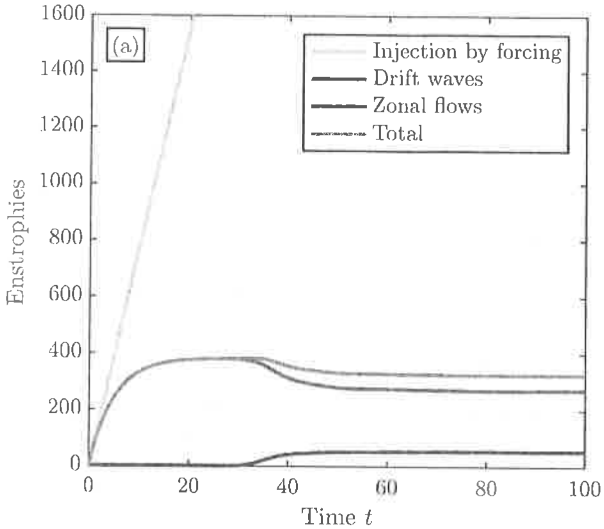
$$\frac{\partial^n H}{\partial y^n}\frac{\partial^n \overline{W}}{\partial p_y^n} \backsim \left(\frac{\lambda_{\rm dw}}{\lambda_{\rm zf}}\right)^n H\overline{W} \lesssim \epsilon^n H\overline{W}, \tag{39}$$

$$\frac{\partial^n H}{\partial p_y^n}\frac{\partial^n \overline{W}}{\partial y^n} \backsim \left(\frac{L_{\rm D}}{\lambda_{\rm zf}}\right)^n H\overline{W} \lesssim \epsilon^n H\overline{W}. \tag{40}$$

Then, using the lowest-order approximations of the Moyal products (Appendix A), Eqs. (25) reduce to

$$\partial_t \overline{W} = \{\mathcal{H}, \overline{W}\} + 2\Gamma\overline{W} + \overline{F} - 2\mu_{\rm dw}\overline{W}, \tag{41a}$$

$$\partial_t U + \mu_{\rm zf}\, U = \frac{\partial}{\partial y}\int \frac{{\rm d}^2 p}{(2\pi)^2}\frac{p_x p_y \overline{W}}{p_{\rm D}^4}, \tag{41b}$$



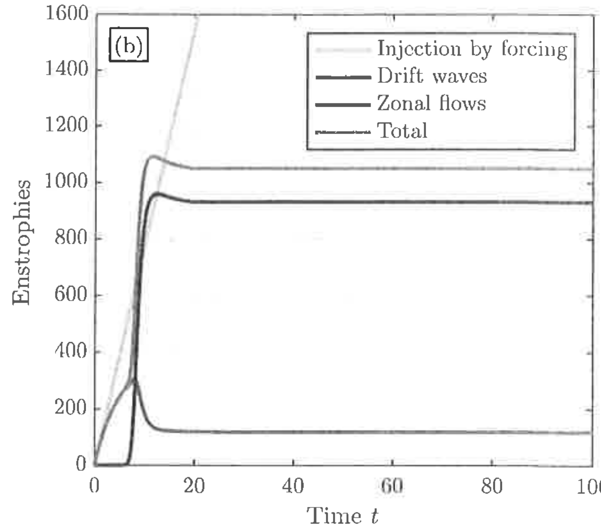


FIG. 3: The total, DW, and ZF enstrophies obtained by numerically integrating the WKE (41) for $\mathcal{H}$ and Γ of two types: (a) our model [Eqs. (42)]; (b) the tWKE model [Eqs. (43)]. The yellow lines show the total enstrophy that one would get due to the external force $\bar{F}$ at $\mu_{\rm dw,zf} = 0$. The initial conditions and simulation parameters are the same as in Fig. 2.

where $\{\cdot,\cdot\}$ is the canonical Poisson bracket (A8), and

$$\mathcal{H} \simeq -\beta p_x/p_{\rm D}^2 + p_x U + p_x U''/p_{\rm D}^2, \tag{42a}$$

$$\Gamma \simeq \{U'', p_x/p_{\rm D}^2\}/2 = -p_x p_y U'''/p_{\rm D}^4. \tag{42b}$$

One may recognize Eq. (41a) as a variation of the WKE, so we attribute Eqs. (41) and (42) as the *WKE limit* of our general theory. Clearly, $\mathcal{H}$ acts as the drifton ray Hamiltonian, while Γ acts as the corresponding dissipation rate. [The factors of two in Eq. (41a) are due to the fact that $\bar{W}$ is quadratic in the DW amplitude.] In other words, $\omega(y,\mathbf{p},t) \doteq \mathcal{H} + i\Gamma - i\mu_{\rm dw}$ can be viewed as the local complex frequency of DW with given wave vector $\mathbf{p}$.

Notice that our WKE differs from the tWKE, which assumes a simpler dispersion of DW, namely,

$$\mathcal{H} = -\beta p_x/p_{\rm D}^2 + p_x U, \quad \Gamma = 0. \tag{43}$$

Although the difference is only in the high-order derivatives of U, these terms remain important for various reasons. For example, in the Hamiltonian $\mathcal{H}$, U'' can be comparable to β (as is sometimes the case in geophysics [2]). Also, consider the following. In isolated systems, the tWKE is $\partial_t \bar{W} = \{\mathcal{H}, \bar{W}\}$, so it conserves DW quanta, or, in other words, the DW enstrophy $\mathcal{Z}_{\rm dw}$ [Eq. (28a)]. At the same time, the ZF enstrophy $\mathcal{Z}_{\rm zf}$ [Eq. (28b)] generally evolves, so the total enstrophy $\mathcal{Z} = \mathcal{Z}_{\rm dw} + \mathcal{Z}_{\rm zf}$ does too. This is in contradiction with the gHME, which conserves $\mathcal{Z}$, and can lead to overestimating the ZF velocity and shear generated by DW turbulence [33]. In contrast to the tWKE, our theory is free from such issues, because Eqs. (41) and (42) exactly conserve both $\mathcal{Z}$ and $\mathcal{E}$ (Appendix C 2). Note that, in order to retain this conservation property, it is necessary to keep both U''' and U'' in Eqs. (42). In this sense, *Eqs. (41) and (42) represent the simplest GO model that is physically meaningful in the nonlinear regime.* This is in agreement with Ref. [9], where a similar conclusion was made based on comparing the linear zonostrophic instability rate predicted by the CE2. (As a note on terminology, Ref. [9] refers to the tWKE [Eqs. (41) and (43)] as the *Asymptotic WKE*, i.e., the limit obtained when one assumes the ZFs are asymptotically large scale. Also, Ref. [9] refers to Eqs. (41) and (42) as *CE2-GO.*)

The numerical results presented in Figs. 2-4 illustrate the importance of the difference between our WKE and the tWKE [subfigures (a) and (b), respectively]. As seen in Fig. 2, while our WKE model predicts ZF with a particular $\lambda_{\rm zf}$, the scale of ZF predicted by tWKE is determined by nothing but the grid size that is used in simulations. This is because the tWKE predicts that the rate of the zonostrophic instability γ (Sec. IV) scales linearly with the ZF wave number q, so ZF are produced at the largest q that is allowed [9].

Consider also the enstrophy plots in Fig. 3. To aid our discussion, we added plots of the enstrophy $\mathcal{Z}_{\rm ext}$ that the external forcing $\bar{F}$ injects into the DW-ZF system, namely, $\mathcal{Z}_{\rm ext} = (t/2)(2\pi)^{-2}\int dy\, d^2p\, \bar{F}$. Within our model, the total enstrophy $\mathcal{Z}$ remains always smaller than $\mathcal{Z}_{\rm ext}$, which is natural, since the simulation is done for $\mu_{\rm dw,zf} > 0$. In contrast, the tWKE model predicts that $\mathcal{Z}$ can surpass $\mathcal{Z}_{\rm ext}$, which is unphysical. In addition, the values of the ZF and total enstrophies predicted by the tWKE are several times larger than those predicted by our model.

For the sake of completeness, Fig. 4 also presents the corresponding energies and the energy $\mathcal{E}_{\rm ext}$ introduced by the external force, $\mathcal{E}_{\rm ext} = (t/2)(2\pi)^{-2}\int dy\, d^2p\, \bar{F}/p_{\rm D}^2$. In both cases, $\mathcal{E}(t) \leq \mathcal{E}_{\rm ext}(t)$, which is in agreement with the fact that both models conserve the total energy of an isolated system. Still, the tWKE predicts very differ-

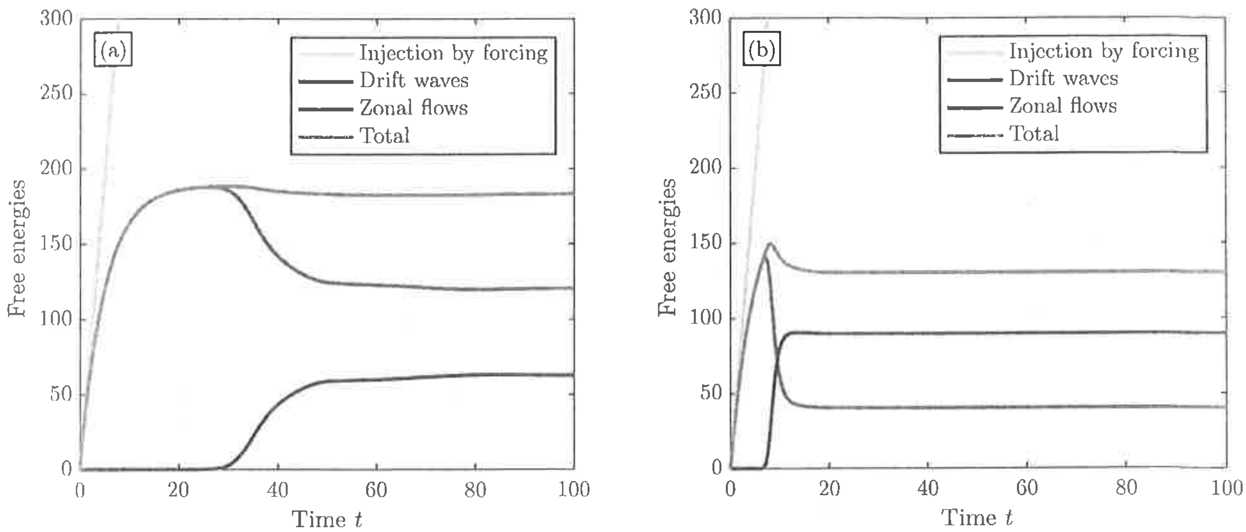


FIG. 4: The total, DW, and ZF energies obtained by numerically integrating the WKE (41) for $\mathcal{H}$ and Γ of two types: (a) our model [Eqs. (42)]; (b) the tWKE model [Eqs. (43)]. The yellow lines show the total energy that one would get due to the external force $\overline{F}$ at $\mu_{\mathrm{dw,zf}} = 0$. The initial conditions and simulation parameters are the same as in Fig. 2.

ent results quantitatively, even though the tWKE model [Eqs. (43)] is seemingly close to ours [Eqs. (42)].

VI. CONCLUSIONS

The goal of this paper was to propose a theory of DW-ZF interactions that would be more accurate than the tWKE and, simultaneously, more intuitive than the CE2. We adopted the same model [Eqs. (6)] that was previously applied to derive the CE2. Then, we manipulated it using the Weyl calculus to produce a phase-space formulation of DW-ZF interactions. The resulting theory [Eqs. (25) and (26)] is akin to a quantum kinetic theory and involves a pseudodifferential Wigner–Moyal equation. To facilitate its numerical implementation in the future, we also presented an integral representation of our main equations (Appendix B).

On one hand, this Wigner–Moyal formulation can be understood as an alternative representation to the CE2 since both models use the same assumptions. For example, we show that it leads to the same linear growth rate of weak ZF as that obtained from the CE2 (Sec. IV). On the other hand, the Wigner–Moyal formulation is arguably more intuitive than the CE2, namely, for two reasons: (i) it permits treating driftons as particles (i.e., as objects traveling in phase space), except now they are *quantumlike* particles with nonzero wavelengths; and (ii) the separation between Hamiltonian effects and dissipation remains unambiguous even beyond the GO limit.

Compared to the tWKE, the new approach is more precise, also for two reasons: (i) it captures effects beyond the GO limit; and (ii) even in the GO limit, it predicts corrections to the tWKE that emerge from the newly found corrections to the drifton dispersion (Sec. V). These corrections are essential as they allow DW-ZF enstrophy exchange, which is not included in the tWKE. By deriving the GO limit from first principles, we eliminated this discrepancy and arrived at a theory that exactly conserves the total enstrophy (as opposed to the DW enstrophy conservation predicted by the tWKE) and the total energy, in agreement with the underlying gHME. We also illustrated the substantial difference between the GO limit of our theory and the tWKE using numerical simulations.

This work can be expanded at least in two directions. First, the difference between the Wigner–Moyal formulation and the newly proposed WKE can be assessed quantitatively using numerical simulations. Second, the analytic methods we proposed here can be extended to other turbulence models, such as in Refs. [34, 35]. The anticipated benefit is that more accurate equations would be derived that would respect fundamental conservation laws that existing theories may be missing otherwise.

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Appendix A: Weyl calculus

This appendix summarizes our conventions for the Weyl transform. (For more information, see the excellent reviews in Refs. [36–39].) The n-dimensional Weyl symbol $A(\mathbf{x},\mathbf{p})$ of any given operator $\hat{A}$ is defined as

$$A(\mathbf{x},\mathbf{p}) \doteq \int \mathrm{d}^n s\, e^{-i\mathbf{p}\cdot\mathbf{s}}\, \langle \mathbf{x}+\mathbf{s}/2|\hat{A}|\mathbf{x}-\mathbf{s}/2\rangle . \quad \text{(A1)}$$

We shall refer to this description of the operators as a *phase-space representation*, since Weyl symbols are functions of the $2n$-dimensional ray phase space $(\mathbf{x},\mathbf{p})$. Conversely, the inverse Weyl transformation is

$$\hat{A} = \frac{1}{(2\pi)^n} \int \mathrm{d}^n x\, \mathrm{d}^n p\, \mathrm{d}^n s\, e^{-i\mathbf{p}\cdot\mathbf{s}} \times A(\mathbf{x},\mathbf{p})\, |\mathbf{x}-\mathbf{s}/2\rangle\, \langle \mathbf{x}+\mathbf{s}/2| . \quad \text{(A2)}$$

In particular, notice that, for any operator $\hat{A}$, its matrix elements in the coordinate representation, $A(\mathbf{x},\mathbf{x}') \doteq \langle \mathbf{x}|\hat{A}|\mathbf{x}'\rangle$, can be expressed as

$$A(\mathbf{x},\mathbf{x}') = \frac{1}{(2\pi)^n} \int \mathrm{d}^n p\, e^{-i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} A\left(\frac{\mathbf{x}+\mathbf{x}'}{2},\mathbf{p}\right),$$

so $A(\mathbf{x},\mathbf{p})$ can be understood as a spectrum of $A(\mathbf{x},\mathbf{x}')$. In particular,

$$A(\mathbf{x},\mathbf{x}) = \int \frac{\mathrm{d}^n p}{(2\pi)^n}\, A(\mathbf{x},\mathbf{p}) . \quad \text{(A3)}$$

Other properties of the Weyl transform that we use in this paper are as follows:

- For any operator $\hat{A}$, its trace can be expressed as

$$\operatorname{Tr} \hat{A} = \frac{1}{(2\pi)^n} \int \mathrm{d}^n x\, \mathrm{d}^n p\, A(\mathbf{x},\mathbf{p}). \quad \text{(A4)}$$

- If $A(\mathbf{x},\mathbf{p})$ is the Weyl symbol of $\hat{A}$, then $A^*(\mathbf{x},\mathbf{p})$ is the Weyl symbol of $\hat{A}^\dagger$. As a corollary, the Weyl symbol of a Hermitian operator is real.

- For any $\hat{C} = \hat{A}\hat{B}$, the corresponding Weyl symbols satisfy [20, 21]

$$C(\mathbf{x},\mathbf{p}) = A(\mathbf{x},\mathbf{p}) \star B(\mathbf{x},\mathbf{p}). \quad \text{(A5)}$$

  Here, $\star$ is the *Moyal product*, which is given by

$$A(\mathbf{x},\mathbf{p}) \star B(\mathbf{x},\mathbf{p}) \doteq A(\mathbf{x},\mathbf{p}) e^{i\hat{\mathcal{L}}/2} B(\mathbf{x},\mathbf{p}), \quad \text{(A6)}$$

  and $\hat{\mathcal{L}}$ is the *Janus operator*, which is given by

$$\hat{\mathcal{L}} \doteq \overleftarrow{\partial_\mathbf{x}} \cdot \overrightarrow{\partial_\mathbf{p}} - \overleftarrow{\partial_\mathbf{p}} \cdot \overrightarrow{\partial_\mathbf{x}} \equiv \{\overleftarrow{\cdot},\overrightarrow{\cdot}\}. \quad \text{(A7)}$$

  The arrows indicate the directions in which the derivatives act, and $A\hat{\mathcal{L}}B = \{A,B\}$ is the canonical Poisson bracket, namely,

$$\{A,B\} \doteq (\partial_\mathbf{x} A)\cdot(\partial_\mathbf{p} B) - (\partial_\mathbf{p} A)\cdot(\partial_\mathbf{x} B). \quad \text{(A8)}$$

- The Moyal product is associative, i.e.,

$$A \star B \star C \doteq (A \star B) \star C = A \star (B \star C). \quad \text{(A9)}$$

- The anti-symmetrized Moyal product defines the so-called *Moyal bracket*

$$\{\{A,B\}\} \doteq -i\,(A \star B - B \star A) = 2A\sin(\hat{\mathcal{L}}/2)B. \quad \text{(A10)}$$

  Because of the latter equality, this bracket is also called the *sine bracket*. In the ray approximation,

$$\{\{A,B\}\} \simeq \{A,B\}. \quad \text{(A11)}$$

- The symmetrized Moyal product is defined as

$$[[A,B]] \doteq A \star B + B \star A = 2A\cos(\hat{\mathcal{L}}/2)B. \quad \text{(A12)}$$

  Because of the latter equality, this bracket is also called the *cosine bracket*. In the ray approximation,

$$[[A,B]] \simeq 2AB. \quad \text{(A13)}$$

- Assuming that fields vanish at infinity rapidly enough, the phase-space integral of the Moyal product of two symbols equals the integral of the regular product of these symbols; i.e.,

$$\int \mathrm{d}^n x\, \mathrm{d}^n p\, A \star B = \int \mathrm{d}^n x\, \mathrm{d}^n p\, AB. \quad \text{(A14)}$$

  As a corollary,

$$\int \mathrm{d}^n x\, \mathrm{d}^n p\, \{\{A,B\}\} = 0, \quad \text{(A15a)}$$

$$\int \mathrm{d}^n x\, \mathrm{d}^n p\, [[A,B]] = 2\int \mathrm{d}^n x\, \mathrm{d}^n p\, AB. \quad \text{(A15b)}$$

- For constant $\mathbf{k}$, one has

$$\begin{aligned} A(\mathbf{p}) \star e^{i\mathbf{q}\cdot\mathbf{x}} &= A(\mathbf{p}) e^{\overleftarrow{\partial}_\mathbf{p}\cdot(\mathbf{q}/2)} e^{i\mathbf{q}\cdot\mathbf{x}} \\ &= A(\mathbf{p}+\mathbf{q}/2) e^{i\mathbf{q}\cdot\mathbf{x}}. \end{aligned} \quad \text{(A16)}$$

- As a corollary, one has

$$\begin{aligned} &\{\{A(\mathbf{p}), e^{i\mathbf{q}\cdot\mathbf{x}}\}\} \\ &\quad = \frac{1}{i}\left[A\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right) - A\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right)\right] e^{i\mathbf{q}\cdot\mathbf{x}}, \quad \text{(A17a)} \\ &[[A(\mathbf{p}), e^{i\mathbf{q}\cdot\mathbf{x}}]] \\ &\quad = \left[A\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right) + A\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right)\right] e^{i\mathbf{q}\cdot\mathbf{x}}. \quad \text{(A17b)} \end{aligned}$$

- For constant $\mathbf{k}$ and $\mathbf{q}$, one can also show that

$$\begin{aligned} &A(\mathbf{p}) e^{i\mathbf{k}\cdot\mathbf{x}} \star B(\mathbf{p}) e^{i\mathbf{q}\cdot\mathbf{x}} \\ &\quad = A(\mathbf{p}+\mathbf{q}/2) B(\mathbf{p}-\mathbf{k}/2) e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}. \end{aligned} \quad \text{(A18)}$$

Appendix B: Spectral representation of the Wigner–Moyal formulation

To facilitate numerical implementations of our theory in the future, we propose an integral form of Eqs. (25) using a spectral representation. (Numerical simulations of the CE2 theory are presented in Refs. [10–14].) The assumed notation for the Fourier representation of any $A(y, \mathbf{p}, t)$ will be

$$A(y, \mathbf{p}, t) = \int \frac{\mathrm{d}q}{2\pi}\, A_q(\mathbf{p}, t) e^{iqy}. \tag{B1}$$

We start by rewriting Eqs. (A17) as

$$\begin{aligned}\mathcal{H} &= -\frac{\beta p_x}{p_\mathrm{D}^2} + \int \frac{\mathrm{d}q}{2\pi}\left(p_x e^{iqy} - \frac{q^2}{2}[[e^{iqy}, p_x p_\mathrm{D}^{-2}]]\right) U_q \\ &= -\frac{\beta p_x}{p_\mathrm{D}^2} + \int \frac{\mathrm{d}q}{2\pi}\left[p_x - \frac{q^2}{2}\left(\frac{p_x}{p_{\mathrm{D},-q}^2} + \frac{p_x}{p_{\mathrm{D},+q}^2}\right)\right] U_q e^{iqy},\end{aligned}$$

where $p_{\mathrm{D},\pm q}^2 \doteq p_\mathrm{D}^2(\mathbf{p} \pm \mathbf{e}_y q/2)$. This leads to

$$\mathcal{H}_q = -2\pi\delta(q)\frac{\beta p_x}{p_\mathrm{D}^2} + p_x\left[1 - \frac{q^2}{2}\left(\frac{1}{p_{\mathrm{D},-q}^2} + \frac{1}{p_{\mathrm{D},+q}^2}\right)\right] U_q. \tag{B2}$$

Similarly,

$$\begin{aligned}\Gamma &= -\int \frac{\mathrm{d}q}{2\pi}\frac{q^2}{2}\{\{e^{iqy}, p_x p_\mathrm{D}^{-2}\}\} U_q \\ &= -\int \frac{\mathrm{d}q}{2\pi}\frac{q^2}{2i}\left(\frac{p_x}{p_{\mathrm{D},-q}^2} - \frac{p_x}{p_{\mathrm{D},+q}^2}\right) U_q,\end{aligned} \tag{B3}$$

so one obtains

$$\Gamma_q = \frac{i}{2}\left(\frac{1}{p_{\mathrm{D},-q}^2} - \frac{1}{p_{\mathrm{D},+q}^2}\right) p_x q^2 U_q. \tag{B4}$$

Also, using Eq. (A18), we obtain

$$\begin{aligned}&\{\{\mathcal{H}, \overline{W}\}\} \\ &= \int \frac{\mathrm{d}r\,\mathrm{d}s}{(2\pi)^2}\{\{\mathcal{H}_r(\mathbf{p}, t)e^{iry}, \overline{W}_s(\mathbf{p}, t)e^{isy}\}\} \\ &= \int \frac{\mathrm{d}r\,\mathrm{d}s}{(2\pi)^2}\frac{1}{i}\left(\mathcal{H}_{r,+s}\overline{W}_{s,-r} - \mathcal{H}_{r,-s}\overline{W}_{s,+r}\right) e^{i(r+s)y},\end{aligned}$$

where $A_{r,\pm s} \doteq A_r(\mathbf{p} \pm \mathbf{e}_y s/2, t)$ for any $A_r(\mathbf{p}, t)$. Also,

$$\begin{aligned}&[[\Gamma, \overline{W}]] \\ &= \int \frac{\mathrm{d}r\,\mathrm{d}s}{(2\pi)^2}[[\Gamma_r(\mathbf{p}, t)e^{iry}, \overline{W}_s(\mathbf{p}, t)e^{isy}]] \\ &= \int \frac{\mathrm{d}r\,\mathrm{d}s}{(2\pi)^2}\left(\Gamma_{r,+s}\overline{W}_{s,-r} + \Gamma_{r,-s}\overline{W}_{s,+r}\right) e^{i(r+s)y}.\end{aligned}$$

By inserting these into Eq. (25a), we obtain

$$\begin{aligned}\partial_t \overline{W}_q &= \overline{F}_q - 2\mu_\mathrm{dw}\overline{W}_q \\ &+ \int \frac{\mathrm{d}r}{2\pi}[(\Gamma_{r,+q-r} - i\mathcal{H}_{r,+q-r})\overline{W}_{q-r,-r} \\ &\qquad + (\Gamma_{r,+r-q} + i\mathcal{H}_{r,+r-q})\overline{W}_{q-r,r}].\end{aligned} \tag{B5}$$

Also, using that

$$\begin{aligned}&\frac{\partial}{\partial y}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{1}{p_\mathrm{D}^2} \star p_x p_y \overline{W} \star \frac{1}{p_\mathrm{D}^2} \\ &= \frac{\partial}{\partial y}\int \frac{\mathrm{d}^2 p\,\mathrm{d}q}{(2\pi)^3}\frac{1}{p_\mathrm{D}^2} \star p_x p_y \overline{W}_q e^{iqy} \star \frac{1}{p_\mathrm{D}^2} \\ &= \frac{\partial}{\partial y}\int \frac{\mathrm{d}^2 p\,\mathrm{d}q}{(2\pi)^3}\frac{p_x p_y}{p_{\mathrm{D},+q}^2 p_{\mathrm{D},-q}^2}\overline{W}_q e^{iqy} \\ &= \frac{i}{2}\int \frac{\mathrm{d}^2 p\,\mathrm{d}q}{(2\pi)^3}\left(\frac{1}{p_{\mathrm{D},-q}^2} - \frac{p_x}{p_{\mathrm{D},+q}^2}\right) p_x \overline{W}_q e^{iqy},\end{aligned} \tag{B6}$$

one gets the following representation of Eq. (25b):

$$\partial_t U_q + \mu_\mathrm{zf} U_q = \frac{i}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\left(\frac{1}{p_{\mathrm{D},-q}^2} - \frac{1}{p_{\mathrm{D},+q}^2}\right) p_x \overline{W}_q. \tag{B7}$$

Equations (B2), (B4), (B5), and (B7) constitute the spectral representation of our Wigner–Moyal formulation.

Appendix C: Conservation of the total enstrophy and energy

Here, we prove the conservation of the total enstrophy $\mathcal{Z}$ and the total energy $\mathcal{E}$ for isolated systems ($Q = 0$) for the Wigner-Moyal model and the WKE model.

1. Wigner–Moyal model

First, consider the Wigner-Moyal model [Eqs. (25) and (26)]. For the enstrophy, we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathcal{Z}}{\mathrm{d}t} &= \int \mathrm{d}y\,(\partial_y U)(\partial_y \partial_t U) + \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\mathrm{d}y\,\partial_t \overline{W}\\
&= \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\mathrm{d}y\left[2U'''\left(\frac{1}{p_{\mathrm{D}}^2}\star p_x p_y \overline{W}\star\frac{1}{p_{\mathrm{D}}^2}\right)+\{\{\mathcal{H},\overline{W}\}\}+[[\Gamma,\overline{W}]]\right]\\
&= \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\mathrm{d}y\left[2U'''\left(\frac{1}{p_{\mathrm{D}}^2}\star p_x p_y \overline{W}\star\frac{1}{p_{\mathrm{D}}^2}\right)+2\Gamma\overline{W}\right],\end{aligned} \tag{C1}$$

where we used Eqs. (A15). To evaluate the remaining terms, we use the Fourier representations of $\overline{W}(y,\mathbf{p},t)$ and $U(y,t)$ as defined via Eqs. (B1). Specifically, after substituting Eq. (26b) for Γ, we obtain

$$\frac{\mathrm{d}\mathcal{Z}}{\mathrm{d}t} = \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}k}{2\pi}\,\mathrm{d}y\,U_q\overline{W}_k\left[-2iq^3\left(\frac{1}{p_{\mathrm{D}}^2}\star p_x p_y e^{iky}\star\frac{1}{p_{\mathrm{D}}^2}\right)e^{iqy}-q^2\{\{e^{iqy},p_x p_{\mathrm{D}}^{-2}\}\}e^{iky}\right]. \tag{C2}$$

The Moyal products and the brackets can be evaluated using Eqs. (A16) and (A17). Using the notation $A_{\pm q}\doteq A(\mathbf{p}\pm\mathbf{e}_y q/2)$ for any $A(\mathbf{p})$, one then obtains

$$\begin{aligned}\frac{\mathrm{d}\mathcal{Z}}{\mathrm{d}t} &= \frac{1}{2i}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}k}{2\pi}\,U_q\overline{W}_k\left[\frac{2p_x p_y q^3}{p_{\mathrm{D},+k}^2 p_{\mathrm{D},-k}^2}-p_x q^2\left(\frac{1}{p_{\mathrm{D},-q}^2}-\frac{1}{p_{\mathrm{D},+q}^2}\right)\right]\int \mathrm{d}y\,e^{i(k+q)y}\\
&= -i\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}q}{2\pi}\,\mathrm{d}k\,U_q\overline{W}_k p_x p_y q^3\left(\frac{1}{p_{\mathrm{D},+k}^2 p_{\mathrm{D},-k}^2}-\frac{1}{p_{\mathrm{D},+q}^2 p_{\mathrm{D},-q}^2}\right)\delta(k+q)\\
&= 0.\end{aligned} \tag{C3}$$

For the energy, one has

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} &= \int \mathrm{d}y\,U(\partial_t U) + \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\mathrm{d}y\,\frac{\partial_t \overline{W}}{p_{\mathrm{D}}^2}\\
&= \frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\mathrm{d}y\left[2U\frac{\partial}{\partial y}\left(\frac{1}{p_{\mathrm{D}}^2}\star p_x p_y \overline{W}\star\frac{1}{p_{\mathrm{D}}^2}\right)+\frac{1}{p_{\mathrm{D}}^2}\{\{\mathcal{H},\overline{W}\}\}+\frac{1}{p_{\mathrm{D}}^2}[[\Gamma,\overline{W}]]\right]\\
&= -\frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\mathrm{d}y\left[2U'\left(\frac{1}{p_{\mathrm{D}}^2}\star p_x p_y \overline{W}\star\frac{1}{p_{\mathrm{D}}^2}\right)-\frac{1}{p_{\mathrm{D}}^2}\left\{\left\{p_x U+\left[\left[U'',p_x p_{\mathrm{D}}^{-2}\right]\right]/2,\overline{W}\right\}\right\}-\frac{1}{p_{\mathrm{D}}^2}[[\{\{U'',p_x p_{\mathrm{D}}^{-2}\}\}/2,\overline{W}]]\right].\end{aligned} \tag{C4}$$

Here, we used the fact that the Taylor expansion of Eq. (A10) for the Moyal bracket $\{\{\overline{W},\beta p_x/p_{\mathrm{D}}^2\}\}/p_{\mathrm{D}}^2$ consists of total derivatives on y, so its integral over y is zero. The other terms can be expressed as follows. First of all,

$$\begin{aligned}&\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\mathrm{d}y\left[2U'\left(\frac{1}{p_{\mathrm{D}}^2}\star p_x p_y \overline{W}\star\frac{1}{p_{\mathrm{D}}^2}\right)-\frac{1}{p_{\mathrm{D}}^2}\{\{p_x U,W\}\}\right]\\
&= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}k}{2\pi}\,\mathrm{d}y\,U_q\left[2iq\overline{W}_k\left(\frac{1}{p_{\mathrm{D}}^2}\star p_x p_y e^{iky}\star\frac{1}{p_{\mathrm{D}}^2}\right)e^{iqy}-\frac{1}{p_{\mathrm{D}}^2}\{\{p_x e^{iqy},\overline{W}_k\}\}e^{iky}\right]\\
&= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}k}{2\pi}\,U_q\left[i\overline{W}_k\frac{2p_x p_y q}{p_{\mathrm{D},+k}^2 p_{\mathrm{D},-k}^2}-\frac{p_x}{ip_{\mathrm{D}}^2}\left(\overline{W}_{k,-q}-\overline{W}_{k,+q}\right)\right]\int \mathrm{d}y\,e^{i(k+q)y}\\
&= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{\mathrm{d}q}{2\pi}\,\mathrm{d}k\,iU_q\overline{W}_k\left[\frac{2p_x p_y q}{p_{\mathrm{D},+k}^2 p_{\mathrm{D},-k}^2}+p_x\left(\frac{1}{p_{\mathrm{D},+q}^2}-\frac{1}{p_{\mathrm{D},-q}^2}\right)\right]\delta(k+q)\\
&= 0.\end{aligned} \tag{C5}$$

Also, using Eq. (A18), we obtain

$$
\begin{aligned}
&\int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\,\frac{1}{2p_\mathrm{D}^2}\Big(\{\{[[U'',p_xp_\mathrm{D}^{-2}]],\overline{W}\}\}+[[\{\{U'',p_xp_\mathrm{D}^{-2}\}\},\overline{W}]]\Big)\\
&=-\int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{\mathrm{d}q}{2\pi}\,\frac{\mathrm{d}k}{2\pi}\,\mathrm{d}y\,\frac{U_qq^2}{2p_\mathrm{D}^2}\Big(\{\{[[e^{iqy},p_xp_\mathrm{D}^{-2}]],\overline{W}_ke^{iky}\}\}+[[\{\{e^{iqy},p_xp_\mathrm{D}^{-2}\}\},\overline{W}_ke^{iky}]]\Big)\\
&=-\int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{\mathrm{d}q}{2\pi}\,\frac{\mathrm{d}k}{2\pi}\,\mathrm{d}y\,\frac{U_qp_xq^2}{2p_\mathrm{D}^2}\Big(\{\{(p_{\mathrm{D},+q}^{-2}+p_{\mathrm{D},-q}^{-2})e^{iqy},\overline{W}_ke^{iky}\}\}-i[[(p_{\mathrm{D},+q}^{-2}-p_{\mathrm{D},-q}^{-2})e^{iqy},\overline{W}_ke^{iky}]]\Big)\\
&=\int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{\mathrm{d}q}{2\pi}\,\frac{\mathrm{d}k}{2\pi}\,\mathrm{d}y\,\frac{U_qp_xq^2}{2ip_\mathrm{D}^2}\Big(\frac{\overline{W}_{k,q}}{p_{\mathrm{D},+q-k}^2}-\frac{\overline{W}_{k,-q}}{p_{\mathrm{D},+q+k}^2}+\frac{\overline{W}_{k,q}}{p_{\mathrm{D},-q-k}^2}-\frac{\overline{W}_{k,-q}}{p_{\mathrm{D},-q+k}^2}\\
&\qquad\qquad+\frac{\overline{W}_{k,q}}{p_{\mathrm{D},+q-k}^2}+\frac{\overline{W}_{k,-q}}{p_{\mathrm{D},+q+k}^2}-\frac{\overline{W}_{k,q}}{p_{\mathrm{D},-q-k}^2}-\frac{\overline{W}_{k,-q}}{p_{\mathrm{D},-q+k}^2}\Big)e^{i(k+q)y}\\
&=-i\int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{\mathrm{d}q}{2\pi}\,\mathrm{d}k\,U_q\overline{W}_kp_xq^2\Big(\frac{1}{p_{\mathrm{D},-q}^2p_{\mathrm{D},-k}^2}-\frac{1}{p_{\mathrm{D},+q}^2p_{\mathrm{D},+k}^2}\Big)\delta(k+q)\\
&=0.
\end{aligned}
\tag{C6}
$$

By substituting Eqs. (C5) and (C6) into Eq. (C4), one obtains $\dot{\mathcal{E}}=0$.

2. WKE model

Now let us consider the WKE model [Eqs. (41) and (42)]. For the enstrophy, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{Z}}{\mathrm{d}t}&=\int \mathrm{d}y\,(\partial_yU)(\partial_y\partial_tU)+\frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\,\partial_t\overline{W}\\
&=\frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\left(\frac{2p_xp_y}{p_\mathrm{D}^4}\,U'''\overline{W}+\{\mathcal{H},\overline{W}\}+2\Gamma\overline{W}\right)\\
&=\frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\left(\frac{2p_xp_y}{p_\mathrm{D}^4}U'''+2\Gamma\right)\overline{W}\\
&=0,
\end{aligned}
\tag{C7}
$$

where we used the fact that the phase-space integral of the Poisson bracket is zero and also substituted Eq. (42b). For the energy, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}&=\int \mathrm{d}y\,U(\partial_tU)+\frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\,\frac{\partial_t\overline{W}}{p_\mathrm{D}^2}\\
&=-\frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\frac{1}{p_\mathrm{D}^2}\left(\frac{2p_xp_y}{p_\mathrm{D}^2}\,U'\overline{W}-\{\mathcal{H},\overline{W}\}-2\Gamma\overline{W}\right)\\
&=-\frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\frac{1}{p_\mathrm{D}^2}\left(\frac{2p_xp_y}{p_\mathrm{D}^2}\,U'\overline{W}-\left\{p_xU+p_xp_\mathrm{D}^{-2}U'',\overline{W}\right\}+\frac{2p_xp_y}{p_\mathrm{D}^4}U'''\overline{W}\right),
\end{aligned}
\tag{C8}
$$

where we used that the integral of $\{\overline{W},\beta p_x/p_\mathrm{D}^2\}/p_\mathrm{D}^2$ over y is zero because this term can be written as a total derivative on y. Finally,

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}&=-\frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\frac{1}{p_\mathrm{D}^2}\left(\frac{2p_xp_y}{p_\mathrm{D}^2}U'\overline{W}-p_xU'\partial_{p_y}\overline{W}-\frac{p_x}{p_\mathrm{D}^2}\,U'''\,\partial_{p_y}\overline{W}+\frac{2p_xp_y}{p_\mathrm{D}^4}\,U''\partial_y\overline{W}+\frac{2p_xp_y}{p_\mathrm{D}^4}\,U'''\overline{W}\right)\\
&=-\frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\mathrm{d}y\frac{1}{p_\mathrm{D}^2}\left(\frac{2p_xp_y}{p_\mathrm{D}^2}\,U'\overline{W}-\frac{2p_xp_y}{p_\mathrm{D}^2}U'\overline{W}-\frac{4p_xp_y}{p_\mathrm{D}^4}\,U'''\overline{W}-\frac{2p_xp_y}{p_\mathrm{D}^4}\,U'''\overline{W}+\frac{2p_xp_y}{p_\mathrm{D}^4}\,U'''\overline{W}\right)\\
&=0.
\end{aligned}
\tag{C9}
$$

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