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Recently a variational integrator for ideal magnetohydrodynamics in Lagrangian labeling has been developed using discrete exterior calculus. Its built-in frozen-in equation makes it optimal for studying current sheet formation. We use this scheme to study the Hahm-Kulsrud-Taylor problem, which considers the response of a 2D plasma magnetized by a sheared field under mirrored sinusoidal boundary forcing. We obtain equilibrium solutions that preserve the topology of the initial field exactly, with a fluid mapping that is non-differentiable. Unlike previous studies that examine the current density output, we identify a singular current sheet from the fluid mapping. The results are benchmarked with an unconventional Grad-Shafranov solver.

Introduction. Current sheet formation has long been an issue of interest in plasma physics. In toroidal fusion plasmas, closed field lines exist at rational surfaces. It is believed that current singularities are inevitable when rational surfaces are subject to resonant perturbations [1–10], which jeopardizes the existence of 3D equilibria with nested flux surfaces. In the solar corona, field lines are tied into the boundaries and do not close on themselves. Yet Parker [11, 12] argued that there would still be current sheets forming frequently, and the subsequent field line reconnections can lead to substantial heating. This theory has stayed controversial to this day [13–26].

Albeit inherently a dynamical problem, current sheet formation is usually treated by examining magnetostatic equilibria for simplicity. The justification is, if there exists no smooth equilibrium for an initially smooth magnetic field to relax to, current sheets must form during the relaxation. In this context the plasma is supposed to be perfectly-conducting, so the equilibrium needs to preserve the topology of the initial field. This topological constraint is difficult to explicitly describe and attach to the magnetostatic equilibrium equation, and to enforce it is a major challenge for studying current sheet formation, either analytically or numerically.

It turns out this difficulty can be overcome by adopting Lagrangian labeling, where the frozen-in equation is built-in to the equilibrium equation, instead of the commonly used Eulerian labeling. Analytically this was first shown by Zweibel and Li [14]. Numerically, a Lagrangian relaxation scheme has been developed using conventional finite difference [27], and extensively used to study current sheet formation [22–26, 28]. It has later been found that its current density output can violate charge conservation ($\nabla \cdot \mathbf{j} = 0$), and mimetic discretization has been applied to fix it [29, 30].

Recently, a variational integrator for ideal magnetohydrodynamics (MHD) in Lagrangian labeling [31] has been developed using discrete exterior calculus [32]. It is derived in a geometric and field-theoretic manner such that the many of the conservation laws of ideal MHD are automatically inherited. Here we present the first results of applying this novel scheme to studying current sheet formation.

We consider a problem first proposed by Taylor and studied by Hahm and Kulsrud (the HKT problem from here on), where a 2D plasma in a sheared magnetic field is subject to mirrored sinusoidal boundary perturbation [3]. It was originally designed to study forced magnetic reconnection induced by resonant perturbation on a rational surface. In the context of studying current sheet formation, we refer to finding a topologically constrained equilibrium solution to this problem as the ideal HKT problem in this paper. Zweibel and Li's [14] linear solution to this problem contains a current sheet, but also discontinuous displacement which is unphysical. It has remained unclear whether the nonlinear solution to the problem is ultimately singular or smooth.

We study how the nonlinear numerical solution to the ideal HKT problem converges with increasing spatial resolution, and find the fluid mapping along the neutral line non-differentiable. This is strong evidence that there exists no smooth solution to the ideal HKT problem. Unlike previous studies that depend heavily on the current density diagnostic that is more vulnerable to numerical inaccuracies [22–26, 28], we identify a singular current sheet from the quadratic fluid mapping normal to the neutral line. Stimulated by these results, we adopt an unconventional Grad-Shafranov solver with the guide field advected rather than prescribed [19] to benchmark on the problem.

The HKT problem. The HKT problem originally considers a 2D incompressible plasma magnetized by an equilibrium field $B_y = \epsilon x$ with constant shear ϵ. The boundaries at $x = \pm a$ are then subject to mirrored sinusoidal perturbation so that $x = \pm(a - \delta \cos ky)$. Two branches of solution to the new equilibrium were obtained by solving the magnetostatic equilibrium equation,

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p, \tag{1}$$

where $\mathbf{B}$ is the magnetic flux density and p the pressure.

The one with no magnetic islands along the neutral line $x = 0$ reads

$$B_y = \epsilon[x + \mathrm{sgn}(x)ka\delta \cosh kx \cos ky/\sinh ka]. \quad (2)$$

Note that the sign function $\mathrm{sgn}(x)$ introduces a jump in B_y, namely a current sheet at the neutral line. However, it can be shown that this solution in fact introduces residual islands with width of $O(\delta)$ on both sides of the neutral line [7, 8], therefore the field line topology of this solution is different from that of the initial field. This highlights the difficulty in enforcing the topological constraint when one studies current sheet formation with Eq. (1).

However, the topological constraint can be naturally enforced if one adopts Lagrangian labeling, which traces the motion of the fluid elements in terms of a continuous mapping from the initial position $\mathbf{x}_0$ to the current position $\mathbf{x}(\mathbf{x}_0, t)$. In this formulation, the advection (continuity, adiabatic, and frozen-in) equations are [33]

$$\rho\, \mathrm{d}^3x = \rho_0\, \mathrm{d}^3x_0 \Rightarrow \rho = \rho_0/J, \quad (3a)$$

$$p/\rho^\gamma = p_0/\rho_0^\gamma \Rightarrow p = p_0/J^\gamma, \quad (3b)$$

$$B_i\, \mathrm{d}S_i = B_{0i}\, \mathrm{d}S_{0i} \Rightarrow B_i = x_{ij}B_{0j}/J, \quad (3c)$$

where $x_{ij} = \partial x_i/\partial x_{0j}$, $J = \det(x_{ij})$ is the Jacobian, and $\rho_0 = \rho(\mathbf{x}_0, 0)$, $p_0 = p(\mathbf{x}_0, 0)$, $\mathbf{B}_0 = \mathbf{B}(\mathbf{x}_0, 0)$ are the initial mass density, pressure and magnetic flux density respectively, and γ is the adiabatic index. They reflect the fact that in ideal MHD, mass, entropy, and magnetic flux are advected by the motion of the fluid elements. These equations are built-in to the ideal MHD Lagrangian and the subsequent Euler-Lagrange equation [33],

$$\rho_0\ddot{x}_i - B_{0j}\frac{\partial}{\partial x_{0j}}\left(\frac{x_{ik}B_{0k}}{J}\right) + \frac{\partial J}{\partial x_{ij}}\frac{\partial}{\partial x_{0j}}\left(\frac{p_0}{J^\gamma} + \frac{x_{kl}x_{km}B_{0l}B_{0m}}{2J^2}\right) = 0. \quad (4)$$

This is the momentum equation, the only ideal MHD equation in Lagrangian labeling. Without time dependence it becomes an equilibrium equation. Unlike Eq. (1), it automatically satisfies the topological constraint required in studying current sheet formation. The problem simply becomes whether there can be singular solution to such equilibrium equation, given smooth initial and boundary conditions. If the initial magnetic field $\mathbf{B}_0$ is smooth, any singularity in the equilibrium field $\mathbf{B}$ should trace back to that in the fluid mapping $\mathbf{x}(\mathbf{x}_0)$.

Zweibel and Li [14] first adopted the advantageous Lagrangian labeling to study current sheet formation. Their linear solution to the ideal HKT problem reads

$$x = x_0 - A\frac{\sinh kx_0}{x_0}\cos ky_0, \quad (5a)$$

$$y = y_0 - A\left(\frac{\sinh kx_0}{kx_0^2} - \frac{\cosh kx_0}{x_0}\right)\sin ky_0, \quad (5b)$$

where $A = \mathrm{sgn}(x_0)a\delta/\sinh ka$. This solution agrees with Eq. (2) linearly and also contains a current sheet. But at the neutral line x is discontinuous, which means the fluid is either overlapped or torn up, but neither scenario is consistent with the ideal MHD model. The failure at the neutral line is expected from the linear solution since the linear assumption breaks down there.

It is worth mentioning that instead of enforcing incompressibility ($J = 1$), Zweibel and Li used a guide field $B_{0z} = \sqrt{1 - \epsilon^2x_0^2}$ such that the unperturbed equilibrium is force-free. Their solution (5) turns out to be linearly incompressible. In fact, even near the neutral line, the plasma should still be rather incompressible since the guide field dominates there. Therefore the physics of the ideal HKT problem will not be affected by such alteration in setup, which we shall adopt in our numerical studies.

Numerical results. The numerical scheme we use is a recently developed variational integrator for ideal MHD [31]. It is obtained by discretizing Newcomb's Lagrangian for ideal MHD in Lagrangian labeling [33] on a moving mesh. Using discrete exterior calculus [32], the momentum equation (4) is spatially discretized into a conservative many-body form $M_i\ddot{\mathbf{x}}_i = -\partial V/\partial\mathbf{x}_i$, where M_i and $\mathbf{x}_i$ are the mass and position of the ith vertex respectively. The Verlet method is then used for temporal discretization such that the scheme is symplectic and momentum preserving [34]. The scheme inherits built-in advection equations from the continuous formulation, so error and dissipation associated with solving them are avoided. It has been shown that the scheme can handle prescribed singular current sheets without suffering from artificial field line reconnection. Such capability of enforcing the frozen-in law makes it an optimal tool for studying current sheet formation. In order to obtain topologically constrained equilibria, friction is added to the momentum equation to dynamically relax the system.

For the ideal HKT problem, we choose a symmetrically structured triangular mesh. Thanks to the mirror symmetry of the problem, we only need to simulate a quarter of the domain, $[0, a] \times [0, \pi/k]$. At $x_0 = a$ it is constrained that $x = a - \delta\cos ky$. The vertices are allowed to move tangentially along but not normal to the boundaries. These boundary conditions are exactly consistent with the original HKT setup. The parameters we choose are $\epsilon = 1$, $\rho_0 = 1$, $a = 0.5$, $k = 2\pi$, and $\delta = 0.1$. We use a large perturbation so that the nonlinear effect is more significant and easier to resolve. The vertices are distributed uniformly in y but non-uniformly in x to devote more resolution to the region near the neutral line. The system starts from a smoothly perturbed configuration consistent with the boundary conditions and relaxes to equilibrium. In Fig. 1 we plot the field line configuration of the equilibrium.

An observation from Fig. 1 is that $B_y(x, 0)$ becomes a finite constant approaching the neutral line. To better illustrate the origin of such tangential discontinuity, we

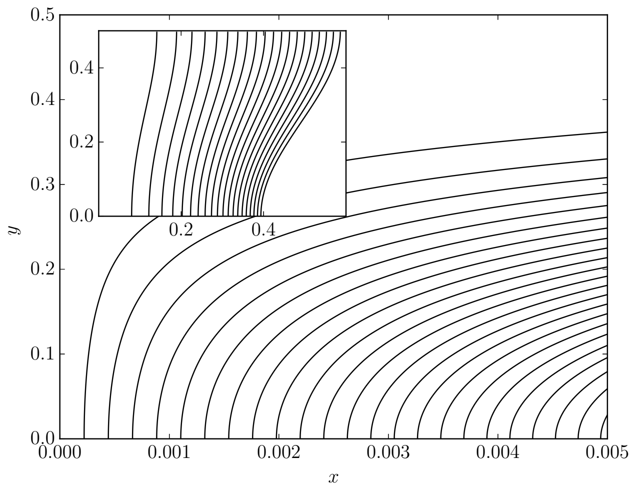

FIG. 1. Equilibrium field line configuration in the vicinity of the neutral line, and the entire domain (inset). The field lines are equally spaced along $y = 0$ near the neutral line.

review a simple yet instructive 1D problem [28], where an exact nonlinear solution with current sheet is available. Consider the same sheared field $B_{0y} = \epsilon x_0$ as in the HKT problem, but the plasma is compressible, and there is no guide field or pressure. The boundaries at $x_0 = \pm a$ are perfectly conducting rigid walls. The system is not in equilibrium and will collapse towards a topologically constrained equilibrium with a quadratic fluid mapping, $x = x_0|x_0|/a$. The Jacobian $J = 2|x_0|/a$ is zero at the neutral line, where the equilibrium field $B_y = B_{0y}/J = \epsilon a\,\mathrm{sgn}(x)/2$ yields a current sheet. As we shall show next, the current sheet in the ideal HKT problem develops from the same ingredients, sheared initial field and quadratic fluid mapping.

We check how the equilibrium solutions converge with increasing spatial resolution, from 64^2 to 128^2, 256^2, and 512^2. For solutions with higher resolution, we only show the part in the vicinity of the neutral line, since they converge very well away from it. In Fig. 2(a), we plot the equilibrium fluid mapping normal to the neutral line at $y_0 = 0$, namely $x(x_0, 0)$. For the part the solutions converge near $x_0 = 0$, quadratic power law $x \sim x_0^2$ can be observed. As discussed the 1D case above, together with a sheared field $B_{0y} \sim x_0$, such a mapping leads to a magnetic field $B_y = B_{0y}/(\partial x/\partial x_0) \sim \mathrm{sgn}(x_0)$ (note that $J = (\partial x/\partial x_0)(\partial y/\partial y_0)$ at $y_0 = 0$) which is discontinuous at $x_0 = 0$, as plotted in the inset of Fig. 2(a).

Despite the remarkable resemblance on the mechanism of current sheet formation, there is a key distinction between the 1D collapse and the ideal HKT problem. For the former, the plasma is infinitely compressible at the neutral line, and the equilibrium fluid mapping is continuous and differentiable. In fact, if there is guide field or pressure, no matter how small, to supply finite compressibility that prevents the Jacobian from reaching zero, the topologically constrained equilibrium would be smooth with no current sheet [28]. In the ideal HKT problem, the plasma is (close to) incompressible. This is confirmed by our numerical solutions which show $J \approx 1 + O(\delta^2)$. As a result, the equilibrium fluid mapping turns out to be non-differentiable.

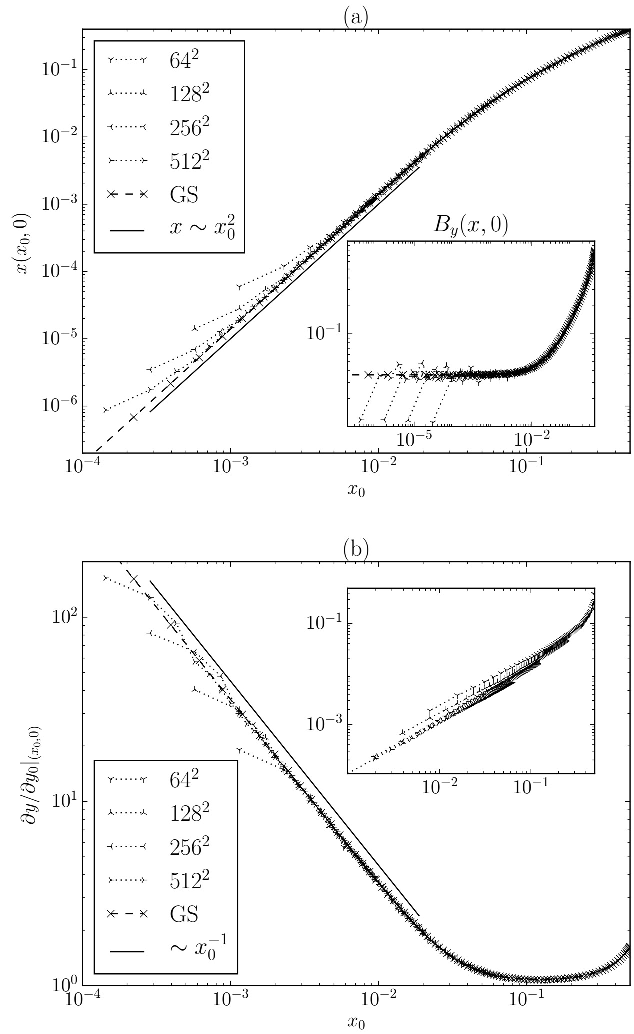

FIG. 2. Numerical solutions of $x(x_0, 0)$ (a), $B_y(x, 0)$ (inset of a) and $\partial y/\partial y_0|_{(x_0,0)}$ (b) for different resolutions (dotted lines). The converged parts agree with the results obtained with an unconventional Grad-Shafranov solver (dashed lines). Near the neutral line $x(x_0, 0)$ shows quadratic power law, $B_y(x, 0)$ becomes discontinuous, while $\partial y/\partial y_0|_{(x_0,0)}$ shows x_0^{-1} power law. The solutions do not converge for the few vertices closest the neutral line. In the inset of (b), the final versus initial distance to $(0, 0.5)$ for the vertices on the neutral line, i. e. $0.5 - y(0, y_0)$ vs. $0.5 - y_0$ for different resolutions are shown to not converge.

At $y_0 = 0$, the converged power law $x \sim x_0^2$ suggests that $\partial x/\partial x_0 \sim x_0$ would vanish as x_0 approaches 0. To ensure incompressibility, there should be $\partial y/\partial y_0 \sim x_0^{-1}$

which would diverge at $x_0 = 0$. This is shown in Fig. 2(b). Physically, this means the fluid elements on the neutral line are infinitely compressed in the normal direction (x), while infinitely stretched in the tangential direction (y).

However, it is improbable to numerically resolve a diverging x_0^{-1} power law at $x_0 = 0$. As a result, the numerical solutions $x(x_0, 0)$ and $\partial y/\partial y_0|_{(x_0,0)}$ both deviate from the converged power law for the few vertices closest to the neutral line. This deviation does not disappear with increasing resolution. From the inset of Fig. 2(b) it can be seen that the vertices on the neutral line get more packed at $(0, 0.5)$ as the resolution increases, showing that the solutions do not converge on the neutral line. For solutions with higher resolution, we only show the part closest to $(0, 0.5)$.

These numerical results are benchmarked with the solutions from a Grad-Shafranov (GS) solver. In this solver the guide field is determined self-consistently from flux conservation [19], unlike conventional ones where it is prescribed as a flux function. This feature makes the solver capable for studying the ideal HKT problem. As shown in Fig. 2, the GS results are in excellent agreement with the converged part of those obtained with the Lagrangian scheme. Since the fluid mapping is inferred rather than directly solved for, the GS solver is able to achieve better agreement with the x_0^{-1} power law shown in Fig. 2(b). However, it should be pointed out that the applicability of the GS solver is limited to 2D problems with nested flux surfaces, whereas the Lagrangian scheme can be readily generalized to 3D problems with complex magnetic topology.

Discussion. A most straightforward conclusion we can draw from the numerical solutions to the ideal HKT problem is that there exists no smooth equilibrium fluid mapping. However, this does not necessarily conclude whether there is genuine current singularity. In the context of studying current sheet formation, one needs to take the extra step and confirm that. This is exactly what we have done in this paper for the ideal HKT problem, by identifying a singular current sheet from the numerical solutions.

In previous studies that use similar Lagrangian relaxation methods [22–26, 28], current singularities are identified by examining whether the maximum current density diverges with increasing spatial resolution. However, involving second-derivatives, the output of current density is generally far less reliable than that of the fluid mapping, especially where the mesh is highly distorted. Since any singularity in current density should trace back to that in the more fundamental fluid mapping, we choose to identify current singularities by examining the latter. In this paper, the current sheet we find originates from the quadratic fluid mapping normal to the neutral line. In this sense, we consider our numerical evidence for current sheet formation in 2D to be the strongest in the extant literature.

It is also worthwhile to compare our result with the recent work of Loizu et al. that also studies the ideal HKT problem [10], but in the context of finding well-defined ideal MHD equilibria with nested flux surfaces. For the original HKT setup, they find no such equilibrium. Then they introduce an alternate formulation to the problem, which in our terminology is equivalent to making the initial magnetic field discontinuous, $B_{0y} = \epsilon[x_0 + \mathrm{sgn}(x_0)\alpha]$, where α is a non-zero constant. Analytically, this would make the linear solution (5) continuous, such that smooth equilibrium fluid mapping becomes possible. We are able to get converged numerical solutions as well when such formulation is adopted. However, the results in this paper differ from those of Ref. [10] in that we begin with a smooth initial condition, rather than one with discontinuity, so as to observe the emergence of a current sheet.

Zweibel and Li [14] studied the ideal HKT problem as a variation of Parker's original model which considers a uniform field in 3D line-tied configuration [11]. Since a sheared field can be realized from a uniform field by sheared footpoint motion, it is more closely related to Parker's model than other variations that involve more complicated field topology such as magnetic-nulls [24–26]. The dynamics also stay simple since there are no violent instabilities like the coalescence instability [15]. Now that we have confirmed that there indeed is a current sheet in the 2D problem, naturally our next step is to find out whether it survives in 3D line-tined configuration. In fact, in Ref. [14] it is conjectured that current sheets would not form in the 3D ideal HKT problem.

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