Princeton Plasma Physics Laboratory

PPPL-5182

Energetically-consistent collisional gyrokinetics

J. W. Burby,[1] A. J. Brizard,[2] and H. Qin1,[3]

1)Princeton Plasma Physics Laboratory, Princeton, New Jersey 08543, USA

2)Department of Physics, Saint Michael's College, Colchester, Vermont 05439, USA

3)Dept. of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China

July 2015

Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

Princeton Plasma Physics Laboratory
Report Disclaimers

PPPL Report Availability

Energetically-consistent collisional gyrokinetics

J. W. Burby,[1] A. J. Brizard,[2] and H. Qin[1,3]

[1)] *Princeton Plasma Physics Laboratory, Princeton, New Jersey 08543, USA*

[2)] *Department of Physics, Saint Michael's College, Colchester, Vermont 05439, USA*

[3)] *Dept. of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

(Dated: 31 July 2015)

We present a formulation of collisional gyrokinetic theory with exact conservation laws for energy and canonical toroidal momentum. Collisions are accounted for by a nonlinear gyrokinetic Landau operator. Gyroaveraging and linearization do not destroy the operator's conservation properties. Just as in ordinary kinetic theory, the conservation laws for collisional gyrokinetic theory are selected by the limiting collisionless gyrokinetic theory.

Introduction.— One of the greatest unsolved problems in the theory of magnetically-confined plasmas is understanding and controlling the turbulent flux of particles and heat into a fusion reactor's wall [1]. It is believed that the predominant cause of these fluxes is low-frequency fluctuating electromagnetic fields with wavelengths on the order of the gyroradius. While a collisionless gyrokinetic model of these fluctuating fields has been developed that is fully consistent with the First Law of Thermodynamics (for a recent review see Ref. 2), this energetically-consistent model has the serious flaw of ignoring collisions altogether.

In order to accurately describe irreversible plasma transport processes, the effects of collisions must be incorporated into gyrokinetic theory. Previous work on linear gyrokinetic collision operators[3–5] assumed a strict two-scale separation between a large-scale equilibrium distribution function F_o and a small-scale fluctuating part $\delta F = F - F_o$. Conservation properties of the collision operator in Ref. 3, for example, were discussed in the gyroBohm limit. Here, we will focus on nonlinear gyrokinetic collision operators for a global full-F approach that do not make this split, and that can thus investigate more completely the possible effects of finite $\epsilon = \rho_i/L$ in experiments, such as corrections to gyroBohm scaling and non-local turbulence spreading (see footnote 5 on p. 427 in Ref. 2.)

When finite-ϵ effects are accounted for, preserving exact conservation properties, and therefore ensuring consistency with the First Law of Thermodynamics, is a nontrivial unsolved problem. The collision operators in Refs. 3 and 4, for example, were obtained by transforming a particle-space collision operator with exact conservation properties into the lowest-order guiding center coordinates. While this approach guarantees the existence of energy and momentum-like quantities that annihilate the collision operator, these same quantities are not conserved by the full-F collisionless gyrokinetic system, and therefore fail to be conserved by the full-F collisional system. More generally, existing gyrokinetic collision operators are not energetically consistent in a full-F formalism because: (a) the gyrocenter coordinate transformation, and therefore any collision operator transformed into gyrocenter coordinates, is only known as an asymptotic expansion in the gyrokinetic ordering parameter ϵ; and (b) replacing the asymptotic expansion of such an operator with a truncated power series destroys exact conservation laws. See section IV of Ref. 6, from which we borrowed the term "energetic consistency," for the analogous discussion of truncation and conservation laws in collisionless gyrokinetics. The purpose of this Letter is to present the first collisional formulation of global full-F gyrokinetics with exact conservation laws.

Refining the example set in Ref. 7, we will show that the key to obtaining this formulation of collisional gyrokinetics is expressing the collision operator in terms of single-particle Poisson brackets "as much as possible."

Electrostatic Model. — For the sake of simplicity, our discussion will focus on quasi-neutral electrostatic gyrokinetics (for instance, see Ref. 8 for a meticulous and explicit account of the theory). However, the ideas behind our discussion apply equally-well to electromagnetic gyrokinetics (for example, see Ref. 9.) Our primary result consists of an expression for the non-linear Landau operator in gyrocenter coordinates that is corrected by small terms to ensure exact energy and momentum conservation [see Eq. (25).] These correction terms are analogous to the $B_{\|}^{*}$-denominators in the Hamiltonian guiding center theory introduced by Littlejohn[10]; they do not increase the theory's order of accuracy, but they are essential to include for the sake of ensuring exact energy and momentum conservation.

As a first step, we review how the energy conservation law is discussed in collisionless kinetic theory. The governing equations of collisionless electrostatic kinetic theory are the Vlasov-Poisson equations,

$$\partial_t f_s + \{f_s, H_s\} = 0 \tag{1}$$

$$\Delta\varphi = -4\pi\rho(f), \tag{2}$$

where f_s is the species-s distribution function, φ is the electrostatic potential, $\rho(f) = \sum_s e_s \int f_s \, d\boldsymbol{p}$ is the charge density, $H_s = p^2/2m_s + e_s\varphi$, and $\{\cdot,\cdot\}$ is the standard canonical Poisson bracket. Equations (1)-(2) conserve the total energy

$$\mathcal{E} = \sum_s \int \frac{p^2}{2m_s} f_s \, dz + \left\langle \varphi,\, \rho(f) + \frac{1}{8\pi}\Delta\varphi \right\rangle, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard L^2-pairing of functions on configuration space and $dz = d\boldsymbol{x}\, d\boldsymbol{p}$. Note that the second term in Eq. (3) is equal to the total energy stored in the electric field $\frac{1}{8\pi}\int \nabla\varphi \cdot \nabla\varphi \, d\boldsymbol{x}$; this follows from applying integration by parts together with the assumption that φ decays sufficiently rapidly to ensure $\int \nabla \cdot (\varphi\nabla\varphi)\, d\boldsymbol{x} = 0$. Because binary collisions conserve energy, Eq. (3) must also be conserved in collisional kinetic theory. In particular, if the Vlasov-Poisson equations are modified by the addition of a bilinear collision operator,

$$\partial_t f_s + \{f_s, H_s\} = \sum_{\bar{s}} C_{s\bar{s}}(f_s, f_{\bar{s}}) \tag{4}$$

$$\Delta\varphi = -4\pi\rho(f), \tag{5}$$

then $C_{s\bar{s}}$ must be chosen to satisfy the condition

$$\begin{aligned} 0 =& \frac{d\mathcal{E}}{dt} = \sum_s \int H_s\, \partial_t f_s\, dz + \left\langle \partial_t\varphi, \rho(f) + \frac{1}{4\pi}\Delta\varphi \right\rangle \\ =& \sum_{s,\bar{s}} \int H_s C_{s\bar{s}}(f_s, f_{\bar{s}})\, dz. \end{aligned} \tag{6}$$

Note that $\rho - \Delta\varphi/4\pi = 0$. Because this identity must hold for an arbitrary multi-species distribution function, the collision operator therefore has to satisfy the well-known identities

$$\int H_s C_{s\bar{s}}(f_s, f_{\bar{s}})\, dz + \int H_{\bar{s}} C_{\bar{s}s}(f_{\bar{s}}, f_s)\, d\bar{z} = 0, \tag{7}$$

which express the fact that the energy gained by species s due to collisions with species $\bar{s}$ is precisely the energy lost by species $\bar{s}$ due to collisions with species s. The non-linear Landau operator (summation rule is implied),

$$C_{s\bar{s}}(f_s, f_{\bar{s}}) = -\frac{\Gamma_{s\bar{s}}}{2}\{x_i, \gamma_i^{s\bar{s}}\},, \tag{8}$$

where x_i is the i'th cartesian component of the position vector $\boldsymbol{x}$, satisfies the identities (7), and therefore defines an energetically-consistent collisional kinetic theory. Here $\Gamma_{s\bar{s}} = 4\pi e_s^2 e_{\bar{s}}^2 \ln\Lambda$; the 3-component vector $\boldsymbol{\gamma}^{s\bar{s}}$ is

$$\gamma_i^{s\bar{s}}(z) = \int \delta(\boldsymbol{x} - \bar{\boldsymbol{x}})\, \mathbb{Q}^{s\bar{s}}(z, \bar{z})\, \boldsymbol{A}_{s\bar{s}}(z, \bar{z})\, d\bar{z}; \tag{9}$$

the 3×3 matrix $\mathbb{Q}^{s\bar{s}}$ is given by

$$\mathbb{Q}^{s\bar{s}}(z, \bar{z}) = \frac{1}{W_{s\bar{s}}(z, \bar{z})}\mathbb{P}[\boldsymbol{W}_{s\bar{s}}(z, \bar{z})], \tag{10}$$

where $\mathbb{P}(\boldsymbol{\xi}) \equiv \mathbb{I} - \hat{\boldsymbol{\xi}}\hat{\boldsymbol{\xi}}$ is the orthogonal projection onto the plane perpendicular to the vector $\boldsymbol{\xi}$; the velocity difference $\boldsymbol{W}_{s\bar{s}}$ is given by

$$\boldsymbol{W}_{s\bar{s}}(z, \bar{z}) = \{\boldsymbol{x}, H_s\}(z) - \{\boldsymbol{x}, H_{\bar{s}}\}(\bar{z}); \tag{11}$$

and the vector

$$\boldsymbol{A}_{s\bar{s}}(z,\bar{z}) = f_s(z)\{\boldsymbol{x}, f_{\bar{s}}\}(\bar{z}) - \{\boldsymbol{x}, f_s\}(z)f_{\bar{s}}(\bar{z}). \tag{12}$$

When comparing this form of the Landau operator to more conventional expressions, it is useful to note that $\{\boldsymbol{x}, g\} = \partial_{\boldsymbol{p}} g$, where g is any function on phase space, which leads to the identity

$$\boldsymbol{v} - \bar{\boldsymbol{v}} = \{\boldsymbol{x}, H_s\}(z) - \{\boldsymbol{x}, H_{\bar{s}}\}(\bar{z}). \tag{13}$$

Moreover, the identities (7) follow immediately from the fact that the velocity difference $\boldsymbol{W}_{s\bar{s}}$ is a null-eigenvector of the matrix $\mathbb{Q}^{s\bar{s}}$.

Electrostatic Gyrokinetic Model.— In order to apply this same argument to gyrokinetic theory, we start with the gyrokinetic Vlasov-Poisson system

$$\partial_t F_s + \{F_s, H_s^{\rm gy}\}_s^{\rm gc} = 0 \tag{14}$$

$$\nabla\cdot\boldsymbol{P} = \rho(F). \tag{15}$$

Here, F_s is the gyrocenter distribution function; $\rho(F) = \sum_s e_s \int F_s\,\mathcal{J}_s dv_\parallel\, d\mu\, d\theta$; $\mathcal{J}_s$ is the guiding center Jacobian; and θ is the gyrophase; φ is the electrostatic potential; $\{\cdot,\cdot\}_s^{\rm gc}$ is the guiding center Poisson bracket;

$$H_s^{\rm gy} = H_s^{\rm gc} + e_s\,\langle\psi\rangle + \frac{e_s^2}{2}\langle\{\tilde{\psi},\tilde{\Psi}\}_s^{\rm gc}\rangle \equiv K_s(\boldsymbol{E}) + e_s\varphi \tag{16}$$

is the gyrocenter Hamiltonian; $\psi(z) = \varphi(\boldsymbol{X}+\boldsymbol{\rho}_{os})$, where $\boldsymbol{\rho}_{os}$ is the lowest-order guiding-center gyroradius; $\langle\cdot\rangle$ denotes the gyroaverage; $\tilde{\Psi}$ denotes the gyroangle antiderivative of $\tilde{\psi} \equiv \psi - \langle\psi\rangle$; $K_s(\boldsymbol{E})$ is the gyrocenter kinetic energy; $\boldsymbol{P} = -\,\delta\mathcal{K}/\delta\boldsymbol{E}$ is the gyrocenter polarization density[11]; $\mathcal{K} = \sum_s\int F_s K_s(\boldsymbol{E})\,dz_s^{\rm gc}$; and $dz_s^{\rm gc}$ denotes the guiding center Liouville volume element. These equations govern collisionless quasineutral electrostatic gyrokinetic theory when the $E\times B$ speed is much less than the thermal speed, and they conserve the total energy,

$$\mathcal{E}^{\rm gy} = \sum_s\int F_s H_s^{\rm gy}\,dz_s^{\rm gc}, \tag{17}$$

exactly, which can be verified directly or derived from this collisionless system's formulation as a Lagrangian field theory[9] (which we will not present here.) Note that the quasineutrality equation (15) implies that this system governs plasma dynamics on time scales long compared to the period of plasma oscillations (this is discussed, for instance, in section IV C in Ref. 9). Also note that the gyrocenter Hamiltonian (16) agrees with the dt-component of Eq. (16) in Ref. 12 modulo geometric terms that appear in H_s^{gc} and the guiding center Poisson bracket. The relevant geometric terms in H_s^{gc} can be found explicitly in Ref. 13, in either Eq. (31) or Eq. (35). If Eq. (31) is used, the guiding center Poisson bracket must be the one defined by the Lagrange 1-form given in that reference's Eq. (30). If Eq. (35) is used instead, the guiding center Poisson bracket must follow from Eq. (33). In general, the guiding center Hamiltonian and Poisson bracket must be derived from the same guiding center Lagrangian.

The equations governing collisional gyrokinetic theory are given by adding a bilinear collision operator to the gyrokinetic Vlasov-Poisson equations,

$$\partial_t F_s + \{F_s, H_s^{\mathrm{gy}}\}_s^{\mathrm{gc}} = \sum_{\bar{s}} C_{s\bar{s}}^{\mathrm{gy}}(F_s, F_{\bar{s}}) \tag{18}$$

$$\nabla \cdot \boldsymbol{P} = \rho(F). \tag{19}$$

Because the conservation laws of ordinary collisional kinetic theory are consistent with those of collisionless kinetic theory, the gyrokinetic collision operator $C_{s\bar{s}}^{\mathrm{gy}}$ must not alter the conservation of $\mathcal{E}^{\mathrm{gy}}$. Thus,

$$\begin{aligned} 0 =& \frac{d\mathcal{E}^{\mathrm{gy}}}{dt} = \sum_s \int H_s^{\mathrm{gy}} \partial_t F_s \, dz_s^{\mathrm{gc}} + \Big\langle \rho(F) - \nabla \cdot \boldsymbol{P},\, \partial_t \varphi \Big\rangle \\ =& \sum_{s,\bar{s}} \int H_s^{\mathrm{gy}} C_{s\bar{s}}^{\mathrm{gy}}(F_s, F_{\bar{s}}) \, dz_s^{\mathrm{gc}}. \end{aligned} \tag{20}$$

Note that $\rho - \nabla \cdot \boldsymbol{P} = 0$. This identity will be satisfied for a general multi-species gyrocenter distribution function if and only if

$$\int H_s^{\mathrm{gy}} C_{s\bar{s}}^{\mathrm{gy}}(F_s, F_{\bar{s}}) \, dz_s^{\mathrm{gc}} + \int H_{\bar{s}}^{\mathrm{gy}} C_{\bar{s}s}^{\mathrm{gy}}(F_{\bar{s}}, F_s) \, d\bar{z}_{\bar{s}}^{\mathrm{gc}} = 0, \tag{21}$$

which is the gyrokinetic version of Eq. (7). The identities (21) must be satisfied exactly by any energetically-consistent gyrokinetic collision operator.

An energetically-consistent collision operator — While Eq. (21) imposes important qualitative constraints, they cannot determine the form of the gyrokinetic collision operator by themselves. A quantitative constraint is necessary as well. To this end, it is important that the gyrokinetic collision operator agrees with the the transformation of the particle-space

Landau operator[14] into gyrocenter coordinates, at least up to some desired order in the gyrokinetic ordering parameter ϵ. Is it possible to satisfy these qualitative and quantitative constraints simultaneously? The answer is "yes".

We have discovered an accurate gyrokinetic collision operator that is consistent with the conservation laws of collisionless gyrokinetic theory, and therefore the first law of thermodynamics. The form of the operator is suggested by the somewhat-peculiar presentation of the particle-space Landau operator given earlier, i.e. Eq. (8). Let $\boldsymbol{y}_s = \boldsymbol{X} + \boldsymbol{\rho}_{os}$ and define the gyrocenter velocity difference

$$\boldsymbol{W}^{\mathrm{gy}}_{s\bar{s}}(z,\bar{z}) = \{\boldsymbol{y}_s, H^{\mathrm{gy}}_s\}^{\mathrm{gc}}_s(z) - \{\boldsymbol{y}_{\bar{s}}, H^{\mathrm{gy}}_{\bar{s}}\}^{\mathrm{gc}}_{\bar{s}}(\bar{z}), \tag{22}$$

the associated 3×3 matrix

$$\mathbb{Q}^{s\bar{s}}_{\mathrm{gy}}(z,\bar{z}) = \frac{1}{W^{\mathrm{gy}}_{s\bar{s}}(z,\bar{z})}\mathbb{P}[\boldsymbol{W}^{\mathrm{gy}}_{s\bar{s}}(z,\bar{z})], \tag{23}$$

and the vector

$$\boldsymbol{A}^{\mathrm{gy}}_{s\bar{s}}(z,\bar{z}) = F_s(z)\{\boldsymbol{y}_{\bar{s}}, F_{\bar{s}}\}^{\mathrm{gc}}_{\bar{s}}(\bar{z}) - \{\boldsymbol{y}_s, F_s\}^{\mathrm{gc}}_s(z)F_{\bar{s}}(\bar{z}). \tag{24}$$

The energetically-consistent gyrokinetic Landau operator is given by

$$C^{\mathrm{gy}}_{s\bar{s}}(F_s, F_{\bar{s}}) = -\frac{\Gamma_{s\bar{s}}}{2}\{y_{s\,i}, \gamma^{s\bar{s}}_{\mathrm{gy}\,i}\}^{\mathrm{gc}}_s, \tag{25}$$

where

$$\boldsymbol{\gamma}^{s\bar{s}}_{\mathrm{gy}}(z) = \int \delta^{\mathrm{gy}}_{s\bar{s}}(z,\bar{z})\mathbb{Q}^{s\bar{s}}_{\mathrm{gy}}(z,\bar{z})\boldsymbol{A}^{\mathrm{gy}}_{s\bar{s}}(z,\bar{z})\,d\bar{z}^{\mathrm{gc}}_{\bar{s}}, \tag{26}$$

and $\delta^{\mathrm{gy}}_{s\bar{s}}(z,\bar{z}) = \delta(\boldsymbol{y}_s(z) - \boldsymbol{y}_{\bar{s}}(\bar{z}))$. Note that this operator depends explicitly on the electric field through the gyrocenter Hamiltonians that appear in Eq. (22). We also note that by including $\boldsymbol{\rho}_{os}$ in $\boldsymbol{y}_s$, this gyrocenter collision operator includes the classical diffusion terms reported on in Ref. 7 in that Reference's Eqs. (51)-(56). Using a straightforward, but tedious argument that will be given in a forthcoming publication, we have shown that this operator agrees with the Landau operator transformed into gyrocenter coordinates (which is also discussed, for instance, in Ref. 5) with leading-order accuracy.

Because the proofs are simple, we will now show explicitly that the gyrokinetic Landau-Poisson system (18) defined in terms of the collision operator (25) has exact conservation laws for energy and momentum and produces entropy. We hope to convey the similarity of this demonstration with the analogous demonstration for the ordinary Landau-Poisson system (4)-(5). However, a word of caution is in order here. It is essential that the guiding center Poisson brackets that appear in Eq. (25) be genuine Poisson brackets (i.e., the brackets must satisfy the Leibniz and Jacobi identities[15]). Dropping terms from a bracket that satisfies these properties will destroy the gyrokinetic Landau-Poisson system's exact conservation laws. The correct approach to deriving the guiding center Poisson bracket is described in detail in Sec. II C of Ref. 16.

Energy conservation — Proving that the gyrokinetic Landau operator (25) satisfies the identities (21) is very similar to proving that the particle-space Landau operator satisfies the identities (7). Setting $\dot{\mathcal{E}}_{s\bar{s}} = \int H_s^{\mathrm{gy}} C_{s\bar{s}}^{\mathrm{gy}}(F_s, F_{\bar{s}})\, dz_s^{\mathrm{gc}}$, it is simple to verify that

$$\dot{\mathcal{E}}_{s\bar{s}} + \dot{\mathcal{E}}_{\bar{s}s} = \frac{\Gamma_{s\bar{s}}}{2} \iint (\boldsymbol{W}_{s\bar{s}}^{\mathrm{gy}})^{\dagger} \mathbb{Q}_{\mathrm{gy}}^{s\bar{s}} \boldsymbol{A}_{s\bar{s}}^{\mathrm{gy}} \delta_{s\bar{s}}^{\mathrm{gy}}\, d\bar{z}_{\bar{s}}^{\mathrm{gc}}\, dz_s^{\mathrm{gc}}, \tag{27}$$

where all two-point quantities in the integrand are evaluated at $(z, \bar{z})$ and $\cdot^{\dagger}$ denotes the ordinary matrix transpose. Because $\mathbb{Q}_{\mathrm{gy}}^{s\bar{s}}$ is a symmetric matrix with null eigenvector $\boldsymbol{W}_{s\bar{s}}^{\mathrm{gy}}$, the right-hand-side of this equation vanishes exactly. Thus the gyrokinetic Landau operator (25) satisfies the identities (21) exactly, and the gyrokinetic Landau-Poisson system (18) has an exact energy conservation law, $d\mathcal{E}^{\mathrm{gy}}/dt = 0$.

Toroidal momentum conservation — We will prove that if the background magnetic field is axisymmetric, then the gyrokinetic Landau-Poisson system conserves the total toroidal momentum

$$P_\phi = \sum_s \int p_{\phi s} F_s\, dz_s^{\mathrm{gc}}, \tag{28}$$

where $p_{\phi s}$ is the guiding center canonical toroidal momentum. In general the expression for $p_{\phi s}$ will depend on one's choice of guiding center representation. However, given the guiding center Lagrange 1-form ϑ in a particular representation (e.g. Eq. (33) in Ref. 13,) $p_\phi = \vartheta_{\boldsymbol{X}} \cdot R^2 \nabla\phi$, where $\vartheta_{\boldsymbol{X}}$ is the $d\boldsymbol{X}$-component of ϑ, R is the major radius, and ϕ is the toroidal angle. If the background magnetic field has different or additional symmetries, a similar proof of the conservation of the corresponding total momentum can easily be constructed. The time derivative of Eq. (28) yields

$$\frac{dP_\phi}{dt} = \sum_{s,\bar{s}} \int p_{\phi s} C_{s\bar{s}}^{\mathrm{gy}}(F_s, F_{\bar{s}})\, dz_s^{\mathrm{gc}} = \sum_{s,\bar{s}} \dot{P}_{\phi s\bar{s}}, \tag{29}$$

where P_ϕ is conserved exactly by the gyrokinetic Vlasov-Poisson system, as is discussed in detail in Ref. 6. Here, we find

$$\dot{P}_{\phi s\bar{s}} + \dot{P}_{\phi \bar{s}s} = \\ \frac{\Gamma_{s\bar{s}}}{2} \iint (\{\boldsymbol{y}_s, p_{\phi s}\}^{\mathrm{gc}}_s - \{\boldsymbol{y}_{\bar{s}}, p_{\phi\bar{s}}\}^{\mathrm{gc}}_{\bar{s}})^\dagger \mathbb{Q}^{s\bar{s}}_{\mathrm{gy}} \boldsymbol{A}^{\mathrm{gy}}_{s\bar{s}} \delta^{\mathrm{gy}}_{s\bar{s}} \, d\bar{z}^{\mathrm{gc}}_{\bar{s}} \, dz^{\mathrm{gc}}_s. \tag{30}$$

Now using the fact that $p_{\phi s}$ is the generator of infinitesimal toroidal rotations, i.e. $\{h, p_{\varphi s}\}^{\mathrm{gc}}_s = \partial_\phi h$ for any function on phase space h, we can see that $\{\boldsymbol{y}_s, p_{\phi s}\}^{\mathrm{gc}}_s = e_z \times \boldsymbol{y}_s$, where e_z is the unit vector along the axis of rotation. Therefore the vector quantity $(\{\boldsymbol{y}_s, p_{\phi s}\}^{\mathrm{gc}}_s - \{\boldsymbol{y}_{\bar{s}}, p_{\phi\bar{s}}\}^{\mathrm{gc}}_{\bar{s}})\, \delta^{\mathrm{gy}}_{s\bar{s}} = e_z \times (\boldsymbol{y}_s - \boldsymbol{y}_{\bar{s}})\, \delta^{\mathrm{gy}}_{s\bar{s}} = 0$, which follows from standard δ-function properties. This shows that $\dot{P}_{\phi s\bar{s}} + \dot{P}_{\phi\bar{s}s} = 0$, which in turn implies total toroidal momentum conservation $dP_\phi/dt = 0$.

Entropy production — As we have discussed, these conservation laws ensure that the gyrokinetic Landau-Poisson system is consistent with the the First Law of Thermodynamics. On the other hand, they do not directly imply that the gyrokinetic Landau-Poisson system is consistent with the Second Law of Thermodynamics. To verify that entropy is indeed a non-decreasing function of time, we have computed the time derivative of $S = -\sum_s \int F_s \ln F_s \, dz^{\mathrm{gc}}_s$ and found

$$\frac{dS}{dt} = \frac{\Gamma_{s\bar{s}}}{2} \iint \frac{1}{F_s F_{\bar{s}}} (\boldsymbol{A}^{\mathrm{gy}}_{s\bar{s}})^\dagger \mathbb{Q}^{s\bar{s}}_{\mathrm{gy}} \boldsymbol{A}^{\mathrm{gy}}_{s\bar{s}} \delta^{\mathrm{gy}}_{s\bar{s}} \, d\bar{z}^{\mathrm{gc}}_{\bar{s}} \, dz^{\mathrm{gc}}_s. \tag{31}$$

Key in deriving this identity is the fact that any function of F_s is conserved by the collisionless gyrokinetic system. Because $\mathbb{Q}^{s\bar{s}}_{\mathrm{gy}}$ is a positive semi-definite matrix and the distribution function is positive[17], the right-side of Eq. (31) is non-negative, which is the desired result.

Note that this proves one "half" of a gyrokinetic version of Boltzmann's H-theorem. The missing ingredient is a complete characterization of the distributions that satisfy $dS/dt = 0$, i.e. the gyrokinetic Maxwellians. Because the guiding center Poisson bracket is rather complicated, we have not yet found a complete characterization. However, we have verified that the distribution

$$F_{Ms} = \frac{1}{Z_s} \exp\left(-\frac{H^{\mathrm{gy}}_s}{T} \right), \tag{32}$$

where $Z_s = \int \exp(-H^{\mathrm{gy}}_s/T)\, dz^{\mathrm{gc}}_s$ is the partition function, maximizes the entropy. We leave the characterization of the most general gyrokinetic Maxwellian, which would be useful for the sake of deriving dissipative gyrofluid models with exact conservation laws[18], as a topic for future study.

Gyroaveraging — When the collision frequency is much smaller than the gyrofrequency[7], the full gyrokinetic Landau operator (25) can be replaced with that operator's gyroaverage, $\langle C^{\text{gy}}_{s\bar{s}}\rangle$. When this is done, the gyrokinetic Landau-Poisson system becomes the gyroaveraged Landau-Poisson system,

$$\partial_t F_s + \{F_s, H^{\text{gy}}_s\}^{\text{gc}}_s = \sum_s \langle C^{\text{gy}}_{s\bar{s}}(F_s, F_{\bar{s}})\rangle \tag{33}$$

$$\nabla\cdot\boldsymbol{P} = \rho(F), \tag{34}$$

where F_s is now interpreted as the gyroaveraged part of the distribution function. Because the functions H^{gy}_s and $p_{\phi s}$ are independent of the gyrophase, the proofs of energy and momentum conservation given earlier work with $C^{\text{gy}}_{s\bar{s}}$ replaced by $\langle C^{\text{gy}}_{s\bar{s}}\rangle$. Thus, the gyroaveraged Landau-Poisson system has exact energy and momentum conservation laws.

Linearization — Closely related to the gyroaveraged Landau-Poisson system is the collisionally-linear gyroaveraged Landau-Poisson system,

$$\partial_t F_s + \{F_s, H^{\text{gy}}_s\}^{\text{gc}}_s = \sum_{\bar{s}} \left(\delta C^{\text{test}}_{s\bar{s}} + \delta C^{\text{field}}_{s\bar{s}}\right), \tag{35}$$

$$\nabla\cdot\boldsymbol{P} = \rho(F), \tag{36}$$

where the linearized test-particle and field-particle collision operators are

$$\delta C^{\text{test}}_{s\bar{s}}(F_s) = \langle C^{\text{gy}}_{s\bar{s}}(F_s, F_{M\bar{s}})\rangle, \tag{37}$$

$$\delta C^{\text{field}}_{s\bar{s}}(F_{\bar{s}}) = \langle C^{\text{gy}}_{s\bar{s}}(F_{Ms}, F_{\bar{s}})\rangle. \tag{38}$$

This system of equations is obtained from the gyroaveraged Landau-Poisson system by assuming $F_s = F_{Ms} + \delta F_s$ and then dropping the non-linear term in the collision operator, $\langle C^{\text{gy}}_{s\bar{s}}(\delta F_s, \delta F_{\bar{s}})\rangle$. Note that $\langle C^{\text{gy}}_{s\bar{s}}(F_{Ms}, F_{M\bar{s}})\rangle = 0$[19]. Because the gyrokinetic Landau operator satisfies the identities (21), it is straightforward to prove that these equations have the same conservation laws for energy and momentum as the gyroaveraged Landau-Poisson system.

Concluding remarks — The key to deriving an energetically-consistent formulation of collisional gyrokinetics was first expressing the particle-space Landau operator in terms of Poisson brackets "as much as possible," which was an idea first championed by Brizard in

Ref. 7. In particular, the identity

$$\boldsymbol{v} - \bar{\boldsymbol{v}} = \{\boldsymbol{x}, H_s\}(z) - \{\boldsymbol{x}, H_{\bar{s}}\}(\bar{z}), \tag{39}$$

together with the close relationship between energy conservation and the null eigenvectors of the Q-matrix in the Landau operator, suggests that the appropriate definition of the gyrocenter velocity difference is given by Eq. (22). This idea, together with the procedure given earlier for determining the energetic consistency constraints, appears to be appropriate for deriving energetically-consistent collision operators for other reduced plasma models as well. In future work, we will report on the energy-conserving collisional formulations of electromagnetic gyrokinetics and oscillation center theory.

We note that, although the gyrokinetic Landau operator (25) and its linearized forms (37)-(38) may prove difficult to implement numerically, they identify the proper formalism for the inclusion of collisional transport in gyrokinetic theory. Hence, these gyrokinetic collision operators form the basis from which approximations can be implemented for practical applications. For instance, if one wishes to use a simplified guiding center Hamiltonian that neglects second-order geometry-dependent terms as in Ref. 12, then our work shows how the collision operator should be modified in order to preserve conservation laws.

Lastly, by setting $\varphi = 0$ in the above formulas, our results reduce to an energy-momentum-conserving guiding center collision operator. This operator would be ideally suited to incorporating collisions into orbit-following codes such as ORBIT [20]; see Ref. 21 for recent work on the Monte Carlo implementation of a 5D guiding center Fokker-Planck collision operator. All previous guiding center collision operators that have been applied in orbit-following codes either resort to *ad hoc* methods to ensure exact conservation laws[22], or else do not fully account for inhomogeneities in the magnetic field [23].

ACKNOWLEDGMENTS

This work was supported by DOE contracts DE-AC02-09CH11466 (JWB and HQ) and DE-SC0006721 (AJB).

REFERENCES

[1]M. Kikuchi and M. Azumi, Rev. Mod. Phys. **84**, 1807 (2012).

[2]A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007).

[3]I. G. Abel, M. Barnes, S. C. Cowley, W. Dorland, and A. A. Schekochihin, Phys. Plasmas **15**, 122509 (2008).

[4]B. Li and D. R. Ernst, Phys. Rev. Lett. **106**, 195002 (2011).

[5]J. Madsen, Phys. Rev. E **87**, 011101 (2013).

[6]B. Scott and J. Smirnov, Phys. Plasmas **17**, 112302 (2010).

[7]A. J. Brizard, Phys. Plasmas **11**, 4429 (2004).

[8]F. I. Parra and I. Calvo, Plasma Phys. Control. Fusion **53**, 045001 (2011).

[9]H. Sugama, Phys. Plasmas **7**, 466 (2000).

[10]R. G. Littlejohn, Phys. Fluids **24**, 1730 (1981).

[11]The functional derivative is defined by the relationship $\frac{d}{d\epsilon}\big|_0 \mathcal{K}(\boldsymbol{E}+\epsilon\delta\boldsymbol{E}) = \int \frac{\delta\mathcal{K}}{\delta\boldsymbol{E}}\cdot\delta\boldsymbol{E}\,d\boldsymbol{x}$, where $\delta\boldsymbol{E}$ is an arbitrary vector field.

[12]T. S. Hahm, Phys. Plasmas **31**, 2670 (1988).

[13]J. W. Burby, J. Squire, and H. Qin, Phys. Plasmas **20**, 072105 (2013).

[14]Necessary conditions for the use of the Landau operator are $\omega_c < \omega_p$ and $(\partial_t F)/(\omega_p F) < 1$. When these conditions are not satisfied, our discussion must be modified.

[15]R. Abraham and J. E. Marsden, *Foundations of Mechanics*, AMS Chelsea publishing (American Mathematical Soc., 2008).

[16]J. Cary and A. J. Brizard, Rev. Mod. Phys. **81**, 693 (2009).

[17]Positivity of the distribution function is also guaranteed by the positive semi-definiteness of $\mathbb{Q}_{\mathrm{gy}}^{s\bar{s}}$.

[18]J. Madsen, Phys. Plasmas **20**, 072301 (2013).

[19]Note that this identity does not contradict the message presented in Ref. 5. In that reference, the gyrokinetic Maxwellian is defined using only the lowest-order gyrocenter Hamiltonian.

[20]R. B. White and M. S. Chance, Phys. Fluids **27**, 2455 (1984).

[21]E. Hirvijoki, A. J. Brizard, A. Snicker, and T. Kurki-Suonio, Phys. Plasmas **20**, 092505 (2013).

[22]A. H. Boozer and G. Kuo-Petravic, Phys. Fluids **24**, 851 (1981).

[23]M. Tessarotto, R. B. White, and L. Zheng, Phys. Plasmas **1**, 951 (1994).

PRINCETON PLASMA PHYSICS LABORATORY

Princeton Plasma Physics Laboratory
Office of Reports and Publications

Managed by
Princeton University

under contract with the
U.S. Department of Energy
(DE-AC02-09CH11466)

P.O. Box 451, Princeton, NJ 08543
Phone: 609-243-2245
Fax: 609-243-2751

E-mail: publications@pppl.gov

Website: http://www.pppl.gov