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A tutorial introduction to the statistical theory of turbulent plasmas, a half century after Kadomtsev's *Plasma Turbulence* and the resonance-broadening theory of Dupree and Weinstock

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In honour of the 50th anniversary of the influential review/monograph on plasma turbulence by B. B. Kadomtsev as well as the seminal works of T. H. Dupree and J. Weinstock on resonance-broadening theory, an introductory tutorial is given about some highlights of the statistical-dynamical description of turbulent plasmas and fluids, including the ideas of nonlinear incoherent noise, coherent damping, and self-consistent dielectric response. The statistical closure problem is introduced. Incoherent noise and coherent damping are illustrated with a solvable model of passive advection. Self-consistency introduces turbulent polarization effects that are described by the dielectric function $\mathcal{D}$. Dupree's method of using $\mathcal{D}$ to estimate the saturation level of turbulence is described, then it is explained why a more complete theory that includes nonlinear noise is required. The general theory is best formulated in terms of Dyson equations for the covariance C and an infinitesimal response function R, which subsumes $\mathcal{D}$. An important example is the direct-interaction approximation (DIA). It is shown how to use Novikov's theorem to develop an $\boldsymbol{x}$-space approach to the DIA that is complementary to the original $\boldsymbol{k}$-space approach of R. H. Kraichnan. A dielectric function is defined for arbitrary quadratically nonlinear systems, including the Navier–Stokes equation, and an algorithm for determining the form of $\mathcal{D}$ in the DIA is sketched. The independent insights of Kadomtsev and Kraichnan about the problem of the DIA with random Galilean invariance are described. The mixing-length formula for drift-wave saturation is discussed in the context of closures that include nonlinear noise (shielded by $\mathcal{D}$). The role of R in the calculation of the symmetry-breaking (zonostrophic) instability of homogeneous turbulence to the generation of inhomogeneous mean flows is addressed. The second-order cumulant expansion and the stochastic structural stability theory are also discussed in that context. Various historical research threads are mentioned and representative entry points to the literature are given. Some outstanding conceptual issues are enumerated.

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CONTENTS

1. Introduction

The first steps toward theories of plasma turbulence occurred in the early 1960s with the development of quasilinear theory (Vedenov *et al.* 1962; Drummond & Pines 1962). It was, of course, appreciated that the quasilinear approximation had a restricted domain of applicability. For calculation of the lowest-order nonlinear corrections, it was natural to use regular perturbation theory, such as in the work of Kadomtsev & Petviashvili (1963); however, the way forward to an analytical description of strong turbulence was not very clear. It was therefore a significant event when Kadomtsev (1920–1998[1]) published his review/monograph on plasma turbulence [in Russian, as an article in Vol. 4 of *Problems in Plasma Theory* (1964); in English translation, as a book entitled *Plasma Turbulence* (1965)[2]], which addressed not only quasilinear and weak-turbulence theory but also sophisticated results about strong turbulence. Because of the diversity and unfamiliarity of many of the ideas, it was a challenge to understand. A reviewer wrote (Drummond 1966),

> "It is not an easy book to read. The method of presentation is in most cases extremely brief and often it is not clear how the conclusions follow from the discussion. [It] is such that it is also difficult to distinguish between those results which are speculative and those which have been more rigorously derived. . . .
>
> "The third chapter, which is the most speculative, contains a description of what is apparently recent work in the theory of strong turbulence. Although some ideas are presented clearly, it is next to impossible to follow in detail and the conclusions reached are evidently of a speculative character."

Nevertheless, Kadomtsev's insights were seminal and the reviewer ultimately concluded, "it should be required reading for any serious student in this field"—and so it served. It was a bible for multiple generations of plasma theorists. And although some of the conclusions may have appeared at the time to be speculative, in fact Kadomtsev's discussion contained, among many other notable ideas, a deep and entirely correct insight about a deficiency of the leading theory of strong turbulence, the *direct-interaction*

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[1]A memorial article that surveys Kadomtsev's entire career is by Velikhov *et al.* (1998).

[2]In the present article, page references to Kadomtsev refer to the 1965 English translation.

approximation (DIA) of Kraichnan (1959). It is ironic that Kadomtsev's contribution is better known in fields other than plasma physics (Monin & Yaglom 1971, Vol. 2, p. 307). I shall explain the issue in Sec. 5.1.

Shortly after the 1965 Western publication of Kadomtsev's book, Dupree (1966) published the paper "A perturbation theory for strong plasma turbulence." Subsequently the formalism, including further significant advances by Dupree (1967) and Weinstock (1968, 1969, 1970), came to be known as *resonance-broadening theory* (RBT); that work was also extremely influential.[3]

It is now a half century since those watershed events. Over that time the theory of turbulence in both fluids and plasmas has developed substantially, and a beginning researcher interested in turbulence will find that much conceptual background is assumed. Fluid turbulence theory has been addressed in multiple books (Leslie 1973; Frisch 1995; Lesieur 1997; McComb 1990, 2014) and articles (Eyink & Sreenivasan 2006; Falkovich 2006; Boffetta & Ecke 2012); the plasma literature is sparser although a review by Yoshizawa *et al.* (2001) ranges broadly over many conceptual and practical topics for both fluids and plasmas, a comprehensive review of analytical statistical turbulence theory with a focus on plasma issues is available (Krommes 2002), and the book by Diamond *et al.* (2010) provides an introductory survey of various approaches to plasma turbulence.[4] But absorbing all of this material is a formidable undertaking. The purpose of this article is to provide some help by introducing in a tutorial[5] way a few highlights of the statistical description of turbulence, using Kadomtsev's review and the resonance-broadening theory as springboards. Whereas all of the discussion should be of interest to plasma theorists, some other parts apply equally to fluids and to plasmas, thus serving to unify those specialties.

1.1. *The role of analytical theory*

This article addresses *analytical* turbulence theory. It is a reasonable question to ask, Why should one care? The experimental situation has improved dramatically since the 1960s at both laboratory and (pre-)reactor scales, and we now have refined diagnostics and data-analysis capabilities that provide direct windows into nature. Certainly the biggest advance over the last half century has been the explosive rise in computing power,[6]

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[3]As is often true in research, the works by Kadomtsev, Dupree, and Weinstock were not without flaws. However, the more significant point is that early on those scientists brought powerful insights and techniques to an important problem. For examples of technical discussion of some issues, see Catto (1978*a*), Krommes (1981), Weinstock & Catto (1981), and additional remarks later in this article.

[4]Although there is a small amount of overlap between the book of Diamond *et al.* and the present article, the goals and emphases are rather different. I focus more on the challenges of a systematic analytical description, and I cover only a restricted set of topics but discuss some of those in more detail than do Diamond *et al.*: I give a lengthy discussion of a solvable random oscillator model, I treat the renormalized dielectric function thoroughly, and I provide a complete statement and (new) technical approach to the DIA. I also include some topics omitted from that book, including (i) the symmetry-breaking bifurcation from homogeneous turbulence to a state of inhomogeneous turbulence that includes steady zonal jets, and (ii) some modern closures (CE2 and S3T) that are well suited to considerations of the interactions between mean fields and turbulence.

[5]A lengthier and more introductory set of tutorial lectures on some plasma-physics topics is by Krommes (2006).

[6]Of the 215 references cited by Kadomtsev (1965 English version; some references in the Russian original differ), just one reported computer simulations—namely, the pioneering work by Dawson (1962) on a 1D model of the two-stream instability.

which has enabled entirely new methods of scientific discovery.[7] A companion tool has been the nonlinear gyrokinetic formalism (c. 1982), which has become indispensable for both analytical insights into, and the numerical simulation of, turbulence in magnetized plasmas. We have learned much more about plasma turbulence from both experiments and simulations than we have from sophisticated theory. Nevertheless, there are good arguments in favour of analytical approaches, as I discuss in the next several paragraphs.

It is worth emphasizing that before one attempts to build a technically sophisticated formalism, one should understand the dimensional and scaling consequences of the governing partial differential equations (PDEs), which are assumed to be given. Consider, for example, random motion in one spatial dimension and suppose that, in a statistical sense, the dynamics are entirely specified in terms of just two dimensional parameters σ and τ having the dimensions of space and time, respectively. I shall indicate random variables by a tilde. It follows that the mean-square displacement must obey $\langle \widetilde{x}^2 \rangle = f(t; \sigma, \tau)$, where f is an unknown function.[8] By dimensional and scaling analysis (Barenblatt 1996; Krommes 2002, Appendix B), it follows that f must have the form $f(t; \sigma, \tau) = \sigma^2 g(t/\tau)$, where g is a dimensionless function of a dimensionless argument. Although g can be measured in experiments or simulations, much deeper understanding follows from theories that predict the form of g, such as models of continuous-time random walks (Klafter *et al.* 1996; Metzler & Klafter 2000). One of many possibilities is $g(\overline{t}) = A\overline{t}^\alpha$, where the dimensionless constants A and α are not determined by scaling analysis. The value $\alpha = 1$ corresponds to the classical random walk, but subdiffusive or superdiffusive values of α are also possible in some interesting physical situations.

By generalizing from such examples, one can argue (Connor & Taylor 1977; Connor 1988) that when faced with a new problem in statistical theory one should always first extract the maximum amount of information accessible from scaling analysis of the governing equations. In some cases, that fully determines the form of the answer except for a numerical constant. More generally, it couches the answer in terms of various dimensionless functions. It is the role of experimental observation, numerical simulation, and analytical theory to determine the forms of those functions and to identify the physics that underlies them.

The basic challenge for any of those approaches, especially for theory, is that the dynamical equations are nonlinear. Nonlinearity leads to rich and interesting behaviour, including turbulent transport, fluctuation spectra with suggestive shapes, and probability density functions (PDFs) that are frequently non-Gaussian. Analytical turbulence theory provides methods to explain and predict that behaviour both qualitatively and quantitatively. Qualitatively, it has proven fruitful to discuss turbulence in the same language of random walks and (generalized) Langevin equations that has long enriched our understanding of classical Brownian motion. Quantitatively, statistical theory provides formulas for turbulence-induced transport fluxes in terms of weighted sums over fluctuation spectra, and it also leads to approximate equations for those spectra; the predictions can be directly compared with experimental measurements or computer simulations. More sophisticated techniques (not discussed in this article) can predict entire PDFs. Analytical methods have also been useful in addressing the generation

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[7]A few examples selected from a very long list are the works of Rogers *et al.* (2000), Wang *et al.* (2011), and Hatch *et al.* (2011*a*).

[8]The restriction to a statistical description is essential; σ and τ are properties of probability density functions descriptive of the dynamics. Motion in an individual realization would also depend at least on the initial condition $\widetilde{x}_0$, so the allowable form for $\widetilde{x}^2(t)$ would be much less constrained.

mechanisms of zonal flows, which are important in regulating the levels of turbulence and transport. In some cases, one can analytically predict the threshold for the onset of zonal flows and the characteristic scale of those flows, among other things. Furthermore, from analytical statistical descriptions one can gain unique insights into the mechanisms by which those flows are self-sustained.

The restricted goal of this article is to introduce some of the conceptual issues and technical challenges relating to the analytical, especially statistical, description of turbulence; that provides a language for discussion and a framework for thinking and modelling,[9] and it can lead to quantitative predictions. It is remarkable how much the pioneers understood of this subject 50 years ago, and that deserves to be celebrated; it is also instructive to see how far we have (or have not) come.

1.2. *The plan of the article*

Broadly speaking, one can say that the two key concepts in statistical turbulence theory (introduced in Sec. 2) are nonlinearity-induced *incoherent noise* and *coherent damping*, which are generalizations of analogous effects in the simple Langevin description of classical Brownian motion. Both of those are already present in systems involving passive advection, as I shall illustrate with an exactly solvable model in Sec. 2.3. Self-consistency adds additional complications, including the concept of a dielectric function $\mathcal{D}(\boldsymbol{k}, \omega)$. The dielectric function is a fundamental object in plasma physics. In its lowest-order (linearized-Vlasov) approximation, it is treated exhaustively in many textbooks. Yet not one of those discusses how to systematically treat dielectric shielding when the system is turbulent, which is the case almost universally encountered in practice. Then $\mathcal{D}$ becomes a functional of the turbulence level, so it is sometimes referred to as the *nonlinear dielectric function*. Failure by some to appreciate that nonlinear corrections to $\mathcal{D}$ are essential in steady-state turbulence has led to confusion.

It is a common belief that the existence of a nontrivial $\mathcal{D}$ importantly distinguishes the electromagnetic, polarizable plasma from the neutral fluid. Plasma physicists proudly lay claim to $\mathcal{D}$ as a special feature of their field, with its potpourri of linear waves. In fact, however, even the incompressible Navier–Stokes turbulent fluid possesses a well-defined $\mathcal{D}$, as I shall show. The consequences of this result are unclear because $\mathcal{D}$ is subsumed by the infinitesimal response function R with which statistical turbulence theory is usually formulated. However, it does imply that turbulent plasmas are not conceptually as special as one might have thought, and it lends itself to a unified discussion of the spectral balances in both plasmas and fluids.

I shall discuss in Sec. 3 the definition and meaning of the nonlinear dielectric function of a turbulent plasma, then note that $\mathcal{D}$ cannot be practically evaluated without a statistical approximation or "closure." I shall sketch some of Dupree's early work on the subject. However, the limitations of his approach can only be appreciated in the context of a more complete formalism. The general theory is formulated in terms of a response function R that subsumes $\mathcal{D}$, so it is never necessary to evaluate $\mathcal{D}$ independently. Nevertheless, introducing $\mathcal{D}$ aids in heuristic understanding. Although fluid-turbulence theorists focus

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[9]One use of analytical modelling is to provide workable numerical algorithms for large-eddy simulations that are more computationally efficient than straightforward high-resolution direct numerical simulations. An example is the elimination of a shell of the largest wave vectors in favour of a simple eddy viscosity that acts on the resolved scales. Although the potential payoff for such advances is large, space limitations preclude discussion of this difficult area here. Some representative papers are by Smagorinsky (1963), Germano *et al.* (1991), Lilly (1992), and Morel *et al.* (2011, 2012); further words and references can be found in Krommes (2002, Sec. 7.3).

on R whereas many plasma physicists (still strongly influenced by Dupree's work) are inclined to think in terms of $\mathcal{D}$, in fact R and $\mathcal{D}$ are intimately related, as I shall explain.

An important self-consistent closure is the DIA. It provides coupled equations that balance the excitation due to nonlinearly produced incoherent noise with classical dissipation and nonlinear coherent damping, including a proper shielding of the noise by dielectric polarization. In Sec. 4 I shall present a new approach to the DIA that exploits the Furutsu–Novikov theorem (Appendix B). I shall also describe an algorithm that defines the direct-interaction approximation to $\mathcal{D}$ for arbitrary quadratically nonlinear systems. With the DIA in hand, I shall briefly explain in Sec. 5 its difficulty with random Galilean invariance, understood independently by Kadomtsev and by Kraichnan. I shall also emphasize the important role of nonlinear noise in the spectral balance for saturated turbulence, and I shall discuss the famous mixing-length formula for the saturation level of drift waves, illustrating with the aid of an exactly solvable model that includes nonlinear noise. Kadomtsev well appreciated the content of the DIA, but it is instructive to take a fresh look with the hindsight of 50 years of experience.

A different perspective on the role of infinitesimal response arises when one considers the instability of homogeneous turbulence to the generation of mean fields. I shall discuss that in Sec. 6.2, where I describe how R figures in contemporary research on the generation of zonal flows. That leads to a brief discussion of the so-called *second-order cumulant expansion* (CE2) and the closely related *stochastic structural stability theory* (S3T) that are currently popular for studies of the interactions between turbulent eddies and mean (e.g., zonal) fields. Understanding those processes involves the concept of *Reynolds stress*, which I shall describe.

Virtually all of the ideas and concepts discussed in the mid-1960s have been much further developed over the last half century, and new ones have been introduced. I emphasize that this article is not a comprehensive review of that huge body of work; it is an introductory tutorial. However, in Sec. 7 I shall mention selected highlights and provide some entry points to the literature.

I shall enumerate some open problems in Sec. 8, which concludes the body of the article. Three appendices are also included: drift-wave terminology is defined in Appendix A, a simple proof of Novikov's theorem is given in Appendix B, and a list of notation is provided in Appendix C.

Much of practical plasma physics involves the use of basic tools such as dimensional analysis, simple model building at the cartoon level, and back-of-the-envelope estimates of scaling laws, saturation levels, and transport coefficients. In the face of a rich phenomenology, such elementary approaches are essential, and they can carry one a long way. Some of the simplest arguments found in Kadomtsev's review, such as saturation at the mixing-length level, continue to influence one's thinking about plasma turbulence. Nevertheless, in the last half century the field has advanced significantly beyond mere hand waving. It is important to appreciate what has already been learned, not only to avoid reinventing the wheel but also to acquire the tools and perspective to build on existing methodology in ways that are presently unforeseen.

This article is intended to be accessible to advanced graduate students. No sophisticated mathematics is used.[10] The ideas, however, are profound. Even if one reproduces every step of the mathematics, it is unlikely that the full implications of the necessarily brief discussion will be appreciated on a first reading. A half century of ideas (a full

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[10]Familiarity with basic statistical objects, including probability density functions and cumulants, is helpful (I shall give some references); the most complicated operation is probably functional differentiation, which should be familiar from elementary variational calculus.

century, if one includes research on neutral-fluid turbulence[11]) cannot be absorbed in one sitting. But I hope that this article will provide historical perspective and an entrée to the intriguing edifice that is modern turbulence theory.

2. Statistical descriptions of turbulence; the statistical closure problem

The basic goals of a statistical theory of turbulence are to (i) predict experimentally measurable quantities such as the mean field (e.g., profiles of density, temperature, and flow), two-point fluctuation spectra, or transport coefficients; and (ii) understand the physical processes that underlie the observations. These require one to consider the famous statistical closure problem, which arises because the PDEs of interest are nonlinear.

2.1. *The statistical closure problem*

The fundamental PDE of many-body plasma physics is the *Klimontovich equation* (Klimontovich 1967, written here for zero magnetic field $\boldsymbol{B}$):

$$\partial_t \widetilde{N}_s(\boldsymbol{x}, \boldsymbol{v}, t) + \boldsymbol{v} \cdot \boldsymbol{\nabla} \widetilde{N} + \widetilde{\boldsymbol{E}} \cdot \boldsymbol{\partial} \widetilde{N} = 0, \tag{2.1}$$

where

$$\widetilde{N}_s(\boldsymbol{x}, \boldsymbol{v}, t) \doteq \frac{1}{\overline{n}_s} \sum_{i \in s} \delta(\boldsymbol{x} - \widetilde{\boldsymbol{x}}_i(t)) \delta(\boldsymbol{v} - \widetilde{\boldsymbol{v}}_i(t)) \tag{2.2}$$

is the phase-space microdensity of species s, $\doteq$ denotes a definition, $\overline{n}$ is the mean density, $\boldsymbol{\partial} \doteq (q/m)\partial/\partial\boldsymbol{v}$, q and m are charge and mass, and $\widetilde{\boldsymbol{E}}$ is the microscopic electric field. In electrostatics, to which this article is restricted for simplicity, it is sufficient to write $\widetilde{\boldsymbol{E}} = -\boldsymbol{\nabla}\widetilde{\phi}$ and to determine $\widetilde{\boldsymbol{E}}$ or $\widetilde{\phi}$ from Poisson's equation:

$$\boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{E}} = -\nabla^2 \widetilde{\phi} = 4\pi\widetilde{\rho} = 4\pi \sum_{\overline{s}} (\overline{n}q)_{\overline{s}} \int \mathrm{d}\overline{\boldsymbol{v}} \, \widetilde{N}_{\overline{s}}(\boldsymbol{x}, \overline{\boldsymbol{v}}, t). \tag{2.3}$$

The functions $\widetilde{\boldsymbol{x}}(t)$ and $\widetilde{\boldsymbol{v}}(t)$ are the random particle trajectories that evolve from certain initial conditions. The quantities $\widetilde{\boldsymbol{x}}$, $\widetilde{\boldsymbol{v}}$, and hence $\widetilde{N}$ and $\widetilde{\boldsymbol{E}}$ are random because it is assumed that at some initial time the phase-space locations of the particles are sampled from some ensemble; i.e., $\widetilde{N}$ describes the phase-space dynamics of one particular *realization* of that ensemble. An *observable* quantity is the ensemble or statistical average, denoted by $\langle \ldots \rangle$, of a random variable. For example, the so-called one-particle distribution function is[12] $f_s(\boldsymbol{x}, \boldsymbol{v}, t) \doteq \langle \widetilde{N}_s(\boldsymbol{x}, \boldsymbol{v}, t) \rangle$. One is also interested in higher-order statistics of the fluctuation $\delta N \doteq \widetilde{N} - \langle \widetilde{N} \rangle$, such as $\langle \delta N(t)\delta N(t') \rangle$. In classical, near-equilibrium plasmas, that two-point correlation function is used to derive the Balescu–Lenard collision operator (usually approximated in practice by the Landau operator[13]) that describes the effects of particle discreteness in the limit of weak coupling

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[11]See the early articles on fluid turbulence reproduced in Friedlander & Topper (1961).

[12]This formula is a physical case of the general result from probability theory that a PDF $P(x)$ for a random variable $\widetilde{x}$ is the average of a delta function that pins the observer coordinate x to $\widetilde{x}$: $P(x) = \int_{-\infty}^{\infty} \mathrm{d}\overline{x}\, \delta(x - \overline{x})P(\overline{x}) = \langle \delta(x - \widetilde{x}) \rangle$. The $f_s(\boldsymbol{x}, \boldsymbol{v}, t)$ defined here differs inessentially from a PDF because it is normalized such that $V^{-1} \int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}\, f_s(\boldsymbol{x}, \boldsymbol{v}, t) = 1$, where V is the volume of the system.

[13]The plasma collision operators are discussed in most textbooks; an early treatise is by Montgomery & Tidman (1964).

$[\epsilon_{\mathrm{P}} \doteq 1/(n\lambda_{\mathrm{D}}^3) \ll 1$, where λ_{D} is the Debye length[14]]. But even in the continuum limit $\epsilon_{\mathrm{P}} \to 0$, turbulent fluctuations of the phase-space fluid can occur (Davidson 1972). The goal is to determine their effects on observable spectra and turbulent transport.[15]

For neutral fluids, the paradigmatic PDE is the Navier–Stokes equation constrained by incompressibility:

$$\partial_t \widetilde{\boldsymbol{u}}(\boldsymbol{x},t) + \widetilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\widetilde{\boldsymbol{u}} = -\rho_{\mathrm{m}}^{-1}\boldsymbol{\nabla}\widetilde{P} + \mu\nabla^2\widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{f}}^{\mathrm{ext}}(\boldsymbol{x},t), \qquad (2.4a)$$

$$\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{u}} = 0. \qquad (2.4b)$$

Here ρ_{m} is the mass density, μ is the kinematic viscosity (here assumed to be constant), and $\widetilde{\boldsymbol{f}}^{\mathrm{ext}}$ is a solenoidal external random forcing.[16] The pressure $\widetilde{P}$ in Eq. (2.4a) is determined in terms of $\widetilde{\boldsymbol{u}}$ by taking the divergence of Eq. (2.4a), enforcing Eq. (2.4b), and solving the resulting Poisson equation. The stochastic velocity $\widetilde{\boldsymbol{u}}$ is random either because the initial velocities are distributed randomly or because of the random forcing. Because a single realization varies erratically in space and time, one is interested in the ensemble average $\boldsymbol{u}(\boldsymbol{x},t) \doteq \langle\widetilde{\boldsymbol{u}}(\boldsymbol{x},t)\rangle$ and in higher-order statistics such as the two-point correlation function $\langle\delta\boldsymbol{u}(\boldsymbol{x},t)\delta\boldsymbol{u}(\boldsymbol{x}',t')\rangle$, the Fourier transform of which with respect to difference variables is the energy spectrum in wave number and frequency. The mean flow $\boldsymbol{u}(\boldsymbol{x},t)$ is analogous to the plasma $f_s(\boldsymbol{x},\boldsymbol{v},t)$ in the sense that both are the ensemble averages of random variables. (Unlike f, however, $\boldsymbol{u}$ is not a PDF.)

Operationally, ensemble averaging is well defined. One is given some probability density function(al) of initial conditions and/or random forcing, and averages are taken with that PDF. A question is whether the result of the averaging has something useful to say about the observable macroscopic world. In particular, whereas Newton's laws of motion are time reversible, ensemble averaging (often together with certain asymptotic limiting procedures) typically introduces irreversibility (one manifestation of which one sees in the laboratory as turbulent transport). The use of ensembles and the origins of irreversibility

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[14]The Debye length is defined in terms of the Debye wave number k_{D} according to $\lambda_{\mathrm{D}} \doteq k_{\mathrm{D}}^{-1}$, where $k_{\mathrm{D}}^2 \doteq \sum_s k_{\mathrm{D}s}^2$ and $k_{\mathrm{D}s}^2 \doteq 4\pi(nq^2/T)_s$.

[15]For turbulent plasmas with $\epsilon_{\mathrm{P}} \neq 0$, the theory becomes extremely complex and some of the formalism used in this paper is either wrong or incomplete because a fundamental assumption about Gaussian initial conditions is violated. Rose (1979) has elegantly discussed the general case, but for this introduction I shall mostly be content to ignore particle discreteness. (See, however, the discussion of discreteness noise in Sec. 7.1.4.) Distinguish the limit $\epsilon_{\mathrm{P}} \to 0$ from the completely reversible case $\epsilon_{\mathrm{P}} = 0$. Collisional dissipation remains in the limit, which is singular, and that is important to retain in order that turbulent steady states can be achieved. Further discussion of this singular limit is given by Krommes & Hu (1994).

[16]External forcing can represent the effects of other modes not described by the basic PDE (such as the influence of baroclinic instabilities on the barotropic vorticity equation commonly used in meteorology), or it can provide a stirring that replaces the macroscopic instabilities that would arise if one imposed nontrivial boundary conditions.

are subtle topics.[17] A readable article on the subject as applied to many-body theory is by Lebowitz (1993), who says (the italics are his),

> "*We neither have nor do we need ensembles when we carry out observations* . . . . What we do need and can expect to have is typical behavior. Ensembles are merely mathematical tools, useful for computing typical behavior as long as the dispersion in the quantities of interest is sufficiently small.

> "There is no such typicality with respect to ensembles describing the time evolution of a system with only a few degrees of freedom. This is an essential difference (unfortunately frequently overlooked or misunderstood) between the irreversible and the chaotic behavior of Hamiltonian systems. The latter, which can be observed already in systems consisting of only a few particles, will not have a unidirectional time behavior in any particular realization. Thus if we had only a few hard spheres in a box, we would get plenty of chaotic dynamics and very good ergodic behavior, but we could not tell the time order of any sequence of snapshots."

Having thus warned the reader, I shall continue with the use of ensemble averages, which are technically convenient. In practice they can often be equated to some sort of time average, providing a link between the processing of simulation data and the analytical theory.

Both of Eqs. (2.1) and (2.4) are quadratically nonlinear, so they are special cases of the generic quadratic PDE

$$\partial_{t_1} \widetilde{\psi}(1) = U_2(1,2)\widetilde{\psi}(2) + \frac{1}{2}U_3(1,2,3)\widetilde{\psi}(2)\widetilde{\psi}(3) + \widetilde{f}^{\text{ext}}(1), \tag{2.5}$$

where 1 stands for the complete set of independent variables,[18] the U's are specified coupling coefficients [for examples, see Eqs. (3.10a) and (3.10b)], U_3 can be assumed to be symmetric in its last two arguments, an integration–summation convention over repeated arguments is assumed, and $\widetilde{f}^{\text{ext}}$ is external random forcing. It is conventional to choose $\widetilde{f}^{\text{ext}}$ to be Gaussian for simplicity, so it is completely specified by its mean $\langle \widetilde{f}^{\text{ext}} \rangle$ and covariance $F^{\text{ext}}(1,1') \doteq \langle \delta f^{\text{ext}}(1)\delta f^{\text{ext}}(1') \rangle$. There are two basic approaches to determining the statistics of $\widetilde{\psi}$: either formulate approximate coupled equations for a few low-order cumulants, or ambitiously try to find an equation for the entire PDF[19]

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[17]For example, the Liouville equation for the N-particle PDF of a many-body system is reversible; so is the cumulant hierarchy that follows from the Euler equation (the Navier–Stokes equation with viscosity set to zero), as discussed by Orszag (1977). An authoritative overview on the origins of macroscopic irreversibility for many-body systems is by Lebowitz (2007); related discussion can be found in the beautiful book by Cercignani (1998). Plasma physicists encounter an analogous situation in the description of Landau damping. Hammett *et al.* (1993) discuss how a perturbation in the lowest (density) moment of a distribution function can propagate reversibly through an infinite chain of higher moments while the density itself exhibits irreversible behaviour (the origin of which is often described as phase mixing). If one does not want to deal with the infinite chain, one can introduce irreversibility by inserting damping at some point in a finitely truncated chain; that is analogous to the difficulty of arranging an initial many-body state precisely in a time-reversal experiment. This is the basic mechanism underlying fluid closures that capture Landau damping (Hammett & Perkins 1990; Hammett *et al.* 1992).

[18]For example, in a fluid problem $\widetilde{\psi}(1) \equiv \widetilde{\psi}(\boldsymbol{x}_1, t_1)$.

[19]The complete multivariate statistics of a random variable $\widetilde{\psi}(\boldsymbol{x}, t)$ are captured by a probability density *functional* $\mathcal{P}[\psi]$, which includes information about all multipoint cumulants in both space and time. As a special case, one can consider an ordinary probability density *function* $P(\psi, \boldsymbol{x}, t)$, which contains information about only the equal-time statistics at some specific $\boldsymbol{x}$ and t; $P(\psi)$ is independent of $\boldsymbol{x}$ and t in a homogeneous steady state. Even determination of the reduced function $P(\psi)$ is generally entirely nontrivial because the given PDE usually couples the

of $\widetilde{\psi}$. The cumulant approach is easier and traditional, and it is the one I shall describe here. An article that summarizes aspects of both approaches is by Kraichnan (1991).

The simplest statistical quantity is the mean field, which exactly obeys

$$\partial_{t_1} \langle \psi \rangle(1) = U_2(1,2)\langle \psi \rangle(2) + \frac{1}{2}U_3(1,2,3)\langle \psi \rangle(2)\langle \psi \rangle(3)$$
$$+ \frac{1}{2}U_3(1,2,3)C(2,3) + \langle f^{\text{ext}} \rangle(1). \tag{2.6}$$

This already exhibits a closure problem: the mean field is determined by the covariance $C(2,3) \doteq \langle \delta\psi(2)\delta\psi(3) \rangle$, which is unknown. To obtain an equation for C, subtract Eq. (2.6) from Eq. (2.5) to find that the fluctuations obey

$$\partial_{t_1} \delta\psi(1) = U_2(1,2)\delta\psi(2) + U_3(1,2,3)\langle \psi \rangle(2)\delta\psi(3)$$
$$+ \frac{1}{2}U_3(1,2,3)[\delta\psi(2)\delta\psi(3) - C(2,3)] + \delta f^{\text{ext}}(1). \tag{2.7}$$

Upon multiplying Eq. (2.7) by $\delta\psi(1')$ and averaging, one finds that $C(1,1')$ obeys[20]

$$\partial_{t_1} C(1,1') = U_2(1,2)C(2,1') + U_3(1,2,3)\langle \psi \rangle(2)C(3,1')$$
$$+ \frac{1}{2}U_3(1,2,3)T(2,3,1') + \langle \delta f^{\text{ext}}(1)\delta\psi(1') \rangle, \tag{2.8}$$

where the three-point correlation function is

$$T(2,3,1') \doteq \langle \delta\psi(2)\delta\psi(3)\delta\psi(1') \rangle. \tag{2.9}$$

Quantities such as $\langle \psi \rangle$, C, or T are *cumulants*, a central concept.[21] An important property is that multipoint cumulants of statistically independent quantities vanish. We see that n-point cumulants are driven by $(n+1)$-point cumulants; the *cumulant hierarchy* is unclosed. This is the classical *statistical closure problem* (Kraichnan 1962; Leslie 1973; Orszag 1977; McComb 1990, 2014; Krommes 2002, and references therein). To close the system at the covariance level, one must express the triplet correlation function T and the cross-correlation of δf^{ext} and $\delta\psi$ in terms of one-point (the mean field) and two-point quantities (such as C; a two-point response function R will also be important).

Statistical closure would not be a problem if the PDF of $\widetilde{\psi}$ were Gaussian. General PDFs can usually be represented in terms of an infinite set of cumulants;[22] Gaussians are special because they possess just two cumulants—namely, the mean and the variance. A theorem states that the sum of two Gaussian variables is again Gaussian, so if the dynamics were linear (i.e., $U_3 = 0$), $\widetilde{\psi}$ would be Gaussian if it evolved from Gaussian initial conditions under Gaussian forcing. Unfortunately, nonlinearity confounds this result. The product of two Gaussian variables is not Gaussian, so even if the initial conditions were Gaussian, non-Gaussian statistics would develop after just one time step.[23] Once non-Gaussianity arises, it is compounded during each subsequent

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space point $\boldsymbol{x}$ to adjacent points via integro-differential operators, but $P(\psi)$ does not contain information about multipoint statistics. PDF methods are challenging and state of the art.

[20]Equation (2.8) has the form of a certain (functional) derivative of Eq. (2.6). This is a special case of a *generating-functional formalism*; see the review by Krommes (2002), who cites the original references.

[21]Some helpful references on cumulants are by Kubo (1962) and McCullagh & Kolassa (2009). A succinct discussion that is oriented toward turbulence applications is given by Krommes & Parker (2015).

[22]Some counterexamples are mentioned by Krommes & Parker (2015).

[23]For simplicity of discussion and the application of simple random-variable theory, it is imagined

time interval. After long times a statistically steady PDF can develop, but it will be non-Gaussian in a self-consistent way that is difficult to determine analytically.[24] The inevitable development of non-Gaussian statistics in nonlinear PDEs is the essence of the closure problem.

Over roughly the last century, enormous efforts have been expended toward obtaining satisfactory statistical closures. A famous example is the DIA; it will be discussed in Sec. 4. Some simpler approximations have enjoyed good successes for the description of the interactions of mean fields with turbulence; see Sec. 6.3.

2.2. *Interlude: Langevin equations*

Although statistical closures for realistic nonlinear PDEs such as the Klimontovich equation are difficult and nontrivial, a good deal of the physics embodied in such equations can be understood with simple intuition. A good example is the use of Langevin equations (Lemons & Gythiel 1997; Gardiner 2004) to describe the statistical properties of classical Brownian motion. I shall include a few words of introduction here so that later I can interpret certain exact and closure results in terms of generalized Langevin descriptions.

2.2.1. Classical Brownian motion

In physics, classical Brownian motion refers to the random motion of a heavy "test" particle (mass M) moving in a thermal bath (temperature T) of light particles. The original model was motivated by observations[25] (Brown 1828) of the jittery motion of particles ejected by pollen grains suspended in water; in plasmas, the description applies equally well to an ion moving in a sea of electrons. In the simplest Langevin model, Newton's second law of motion for the velocity of an unmagnetized test particle is written approximately (in 1D for simplicity) as[26]

$$\dot{\widetilde{v}} + \nu\widetilde{v} = \widetilde{a}(t). \tag{2.10}$$

Here the time axis has been assumed to be coarse grained into intervals that are much larger than the time for a single interaction between the test particle and a bath particle. Then the effects of the microscopic interactions have been replaced by a coarse-grained force divided into two physically distinct parts: $\widetilde{a}$ is a centred Gaussian random variable that describes the random or incoherent kicks due to the light particles, and $-\nu\widetilde{v}$ describes the coherent drag experienced by the test particle as it pushes through the sea. Of course, that drag arises microscopically from discrete interactions with the bath particles, so both ν and the statistics of $\widetilde{a}$ can ultimately be determined in terms of microscopic statistical dynamics. Because of the coarse graining, it is conventional to take the fluctuation δa to be delta correlated:

$$\langle \delta a(t)\delta a(t') \rangle = 2D_v \delta(t - t'). \tag{2.11}$$

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here that the time axis is discretized. Clearly, however, non-Gaussian statistics also develop in continuous time under the action of nonlinearity.

[24] An important and readable introduction to the technical issues relating to the determination of non-Gaussian statistics is given in the first section of Martin *et al.* (1973).

[25] Further history and references are given by Metzler & Klafter (2000).

[26] Here I present a physicist's approach to the Langevin model. More mathematically precise representations in terms of the Itō stochastic calculus are sometimes useful but are not required for the discussion in this article.

For short times ($\nu t \ll 1$), one finds from Eq. (2.10) that $\langle \delta v^2 \rangle = 2D_v t$, so D_v is the diffusion coefficient for a simple random walk (Rudnick & Gaspari 2004) in velocity space. At long times one finds (Wang & Uhlenbeck 1945)

$$\lim_{t \to \infty} \langle \delta v^2 \rangle \equiv \lim_{t \to \infty} C(t,t) = D_v/\nu, \tag{2.12}$$

which describes the fluctuation level arising from a steady-state balance between forcing and dissipation; in that same limit, one finds the two-time correlation function to be

$$C(t,t') \doteq \langle \delta v(t) \delta v(t') \rangle = (D_v/\nu)\mathrm{e}^{-\nu|t-t'|}. \tag{2.13}$$

This depends on only the time difference $\tau \doteq t - t'$, so one can Fourier transform[27] with respect to τ:

$$\widehat{C}(\omega) = 2(D_v/\nu)\,\mathrm{Re}\,\widehat{g}(\omega), \tag{2.14}$$

where $g(\tau) \doteq H(\tau)\mathrm{e}^{-\nu\tau}$ is Green's function[28] for the left-hand side of Eq. (2.10). Equations (2.13) and (2.14) are called *fluctuation–dissipation relations*, where *relation* (as opposed to *theorem*) implies that one has not yet asserted a connection between forcing and dissipation. If one does so by invoking microscopic statistical equilibration according to $\frac{1}{2}M\langle \delta v^2 \rangle = \frac{1}{2}T$, one is led to the Einstein (1905) relation $D_v/\nu = T/M \equiv V_\mathrm{t}^2$. Then $C(\tau) = V_\mathrm{t}^2 \exp(-\nu|\tau|)$, or

$$\widehat{C}(\omega) = 2V_\mathrm{t}^2\,\mathrm{Re}\,\widehat{g}(\omega). \tag{2.15}$$

This is the simplest example of a fluctuation–dissipation *theorem* (Kubo 1966; Martin 1968; Zwanzig 2001). Such theorems are derived most fundamentally by studying the linear response of a Hamiltonian system perturbed from a Gibbsian thermal equilibrium.

This superficial treatment of Langevin equations merely scratches the surface. For example, because the statistics of all dependent variables of a Gaussianly forced linear Langevin system are jointly Gaussian, a variety of exact results (including the complete form of the PDF) can be obtained; see the articles by Fox & Uhlenbeck (1970) and Fox (1978). A wealth of information about linear and nonlinear Langevin equations is contained in the book by Zwanzig (2001). Some aspects of stochastic differential equations are reviewed by van Kampen (1976).

The fact that the statistics of the Brownian particle follow from a Langevin *amplitude* equation (an equation for a random variable, not a mean quantity) has profound implications. It guarantees that the statistics calculated from the model must obey all of the *realizability inequalities* (Kraichnan 1980; van Kampen 1981) that stem from the requirement that the PDF be non-negative. The simplest realizability inequality is $\langle \widetilde{v}^2 \rangle \geqslant \langle \widetilde{v} \rangle^2$; i.e., $C \equiv C(t,t) \doteq \langle \delta v^2 \rangle \geqslant 0$. In the latter form, this is obvious if one has an explicit solution for the real quantity $\delta v(t)$, as in the linear Langevin model (2.10). But in closure theory one usually formulates approximate evolution equations for cumulants like C rather than for random variables: $\partial_t C = f(C)$, where f can be a complicated nonlinear function. In general, there is no guarantee that solutions

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[27]My Fourier integral transform convention is $f(\boldsymbol{x},t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathrm{d}\omega\,(2\pi)^{-d} \int \mathrm{d}\boldsymbol{k}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\omega t} \times$ $\widehat{f}(\boldsymbol{k},\omega)$, where d is the dimensionality of space. For two-point quantities such as covariances, the transform is with respect to the difference variables in space and/or time. Later I will also use a discrete transform in space, denoted by a wave-vector subscript instead of argument: $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \widehat{f}_{\boldsymbol{k}}$. Passage to the continuum representation can be effected by approximating $\sum_{\boldsymbol{k}} \approx (\delta k)^{-d} \int \mathrm{d}\boldsymbol{k}$, where $\delta k \doteq 2\pi/L$ is the mode spacing in a box of side $L \to \infty$. Thus $\widehat{f}_{\boldsymbol{k}} = L^{-d}\widehat{f}(\boldsymbol{k})$.

[28]The unit step function (Heaviside function) $H(\tau)$ ensures causality. It is defined by $H(\tau) = 0$ if $\tau < 0$, $\frac{1}{2}$ if $\tau = 0$, or 1 if $\tau > 0$.

of such nonlinear equations preserve the non-negativity of C for all time. If they do not, catastrophes such as finite-time blowup can ensue (Ogura 1962a,b, 1963; Bowman *et al.* 1993). However, positive-semidefiniteness is guaranteed if it is known that the $\partial_t C$ equation follows rigorously from an underlying amplitude equation for a random variable. That is the case, for example, for the DIA; see Eq. (4.29) and the discussion nearby.

2.2.2. The Taylor formula for a diffusion coefficient

Note that Eq. (2.11) can be inverted:

$$D_v = \int_0^\infty d\tau \, \langle \delta a(\tau) \delta a(0) \rangle. \tag{2.16}$$

This is the simplest example of the *Taylor formula* (Taylor 1921) for the diffusion coefficient D_z of a test particle or fluid element evolving in variable z according to $\dot{\tilde{z}} = \widetilde{V}$:

$$D_z = \int_0^\infty d\tau \, C_{VV}(\tau), \tag{2.17}$$

where

$$C_{VV}(\tau) \doteq \langle \delta V(\tau) \delta V(0) \rangle \tag{2.18}$$

is called the *Lagrangian correlation function*. Here $\widetilde{V}(\tau) \equiv \widetilde{V}(\widetilde{\boldsymbol{x}}(\tau), \widetilde{\boldsymbol{v}}(\tau), \tau)$; i.e., it is measured along the random phase-space trajectory $\{\widetilde{\boldsymbol{x}}(\tau), \widetilde{\boldsymbol{v}}(\tau)\}$. Whether diffusion happens in position space ($z = x$) or velocity space ($z = v$) depends on which variable suffers random kicks. In the case of an unmagnetized Brownian particle, the generalized "velocity" of the general theory is actually acceleration; diffusion occurs in velocity space (for times much shorter than a collision time).[29] For a magnetized test particle moving across a magnetic field, $\widetilde{V}$ would be the $\boldsymbol{E} \times \boldsymbol{B}$ velocity and diffusion would occur in $\boldsymbol{x}$ space.

If one defines a Lagrangian correlation time by

$$\tau_{\rm ac}^{\rm (L)} \doteq [C_{VV}(0)]^{-1} \int_0^\infty d\tau \, C_{VV}(\tau), \tag{2.19}$$

then one can write formula (2.17) as

$$D = \overline{V}^2 \tau_{\rm ac}^{\rm (L)}, \tag{2.20}$$

where $\overline{V} \doteq [C_{VV}(0)]^{1/2}$ is the rms level of the generalized velocity fluctuations. This formula is exact; the problem lies in determining $\overline{V}$ and $\tau_{\rm ac}^{\rm (L)}$. In general, relating Lagrangian correlation functions to more easily observable or calculable Eulerian ones is a difficult task that again requires one to face up to a closure problem. One does not see that difficulty in the linear Langevin model because the Langevin equation is coarse grained in time and D_v is simply prescribed. More realistically, one must "open up" the physics of the short time scale on which the random kicks occur and perform a nontrivial calculation of the Lagrangian correlation function; formula (2.17) then connects the short-time physics to the more macroscopic time scale on which diffusion is observed. This is what is done in the derivation of the Balescu–Lenard collision operator, and there are analogs of that procedure for turbulence.[30]

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[29]For times longer than a collision time, the integral $D_v(t) \doteq \int_0^t d\tau \, C_{\dot{v}\dot{v}}(\tau)$ asymptotically vanishes as $t \to \infty$. [$C_{\dot{v}\dot{v}}(\tau)$ has a long, negative tail.] Meanwhile, $D_x(t)$ asymptotes to a constant; diffusion occurs in x space. This can be understood by considering a new coarse graining of the time axis in units of the collisional mean free path.

[30]Mynick (1988) has taken the analogy literally and discussed a generalized Balescu–Lenard

2.3. *An exactly solvable model of passive advection*

There are actually two levels of difficulty in cumulant-based statistical closure. Problems arise even for passive advection. [In plasmas, that is frequently called *stochastic acceleration* (Orszag 1969), implying no backreaction of the particles on the electric field.] Self-consistency (in plasmas, determining the field from the particles rather than prescribing the field externally) adds more complications (polarization effects) and leads to the concept of a dielectric function $\mathcal{D}$.

2.3.1. The stochastic oscillator

Before turning to self-consistency, let us see how things work in a simple, exactly solvable passive model. The discussion introduces various important concepts and issues relating to random nonlinear systems, and I shall build on it in the next section, where I discuss the additional effects relating to self-consistency. Although there are important differences between passive and self-consistent problems, many points can be illustrated with passive models because although they are dynamically linear they are *stochastically nonlinear*—for example, Eq. (2.21) involves the product of the random variables $\widetilde{\boldsymbol{V}}$ and $\widetilde{\psi}$—and thus possess the closure problem (Kraichnan 1961). Thus consider the passive-advection equation

$$\partial_t \widetilde{\psi}(\boldsymbol{x}, t) + \nu \widetilde{\psi} + \widetilde{\boldsymbol{V}}(t) \cdot \boldsymbol{\nabla} \widetilde{\psi} = \widetilde{f}^{\text{ext}}(\boldsymbol{x}, t), \tag{2.21}$$

where ν is a positive constant, $\widetilde{\boldsymbol{V}}$ is a centred Gaussian random velocity with a specified two-time correlation function, and $\widetilde{f}^{\text{ext}}$ is a Gaussian random forcing that is homogeneous in space, stationary in time, and uncorrelated with $\widetilde{\boldsymbol{V}}$. For simplicity, I assume that $\widetilde{f}^{\text{ext}}$ is white noise, so its covariance F^{ext} has the form

$$F^{\text{ext}}(\boldsymbol{\rho}, \tau) \doteq \langle \delta f^{\text{ext}}(\boldsymbol{x} + \boldsymbol{\rho}, t + \tau) \delta f^{\text{ext}}(\boldsymbol{x}, t) \rangle \tag{2.22a}$$

$$= 2\varepsilon(\boldsymbol{\rho})\delta(\tau), \tag{2.22b}$$

where $\varepsilon(\boldsymbol{\rho})$ is given. To make analytical progress, I assume that $\widetilde{\boldsymbol{V}}$ is independent of $\boldsymbol{x}$. Then Eq. (2.21) can be Fourier transformed in space into

$$\partial_t \widetilde{\psi}(\boldsymbol{k}, t) + \nu \widetilde{\psi}(\boldsymbol{k}, t) + \mathrm{i}\widetilde{\omega}(\boldsymbol{k}, t)\widetilde{\psi}(\boldsymbol{k}, t) = \widetilde{f}^{\text{ext}}(\boldsymbol{k}, t), \tag{2.23}$$

with $\widetilde{\omega}(\boldsymbol{k}, t) \doteq \boldsymbol{k} \cdot \widetilde{\boldsymbol{V}}(t)$ and $\widehat{F}^{\text{ext}}(\boldsymbol{k}, \tau) = 2\widehat{\varepsilon}(\boldsymbol{k})\delta(\tau)$. Because the $\boldsymbol{k}$'s are uncoupled, one can drop the $\boldsymbol{k}$'s and consider the *stochastic oscillator* (Kubo 1959; Kraichnan 1961)

$$\partial_t \widetilde{\psi}(t) + \nu \widetilde{\psi} + \mathrm{i}\widetilde{\omega}(t)\widetilde{\psi} = \widetilde{f}^{\text{ext}}(t), \tag{2.24}$$

where the mean[31] $\overline{\omega}$ and the covariance $W(\tau)$ of $\widetilde{\omega}$ are prescribed. The normalized area under $W(\tau)$ defines an advection-related autocorrelation time according to

$$\tau_{\text{ac}}^{(W)} \doteq [W(0)]^{-1} \int_0^\infty \mathrm{d}\tau\, W(\tau). \tag{2.25}$$

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operator that includes turbulence. The general structure of turbulence theory has a more abstract analogy to the classical theory.

[31]The best way of interpreting $\overline{\omega}$ is as the free-streaming particle term $\boldsymbol{k} \cdot \boldsymbol{v}$ in linearized Vlasov theory. Do not think of it as a normal mode of oscillation; that will come later with the introduction of $\mathcal{D}$.

I shall consider two limits[32]—one in which $\widetilde{\boldsymbol{V}}$ is delta correlated, so $\langle \delta\widetilde{\boldsymbol{V}}(\tau)\delta\widetilde{\boldsymbol{V}}(0)\rangle = 2D\delta(\tau)\boldsymbol{I}$; and one in which it is time independent (but still random) with variance β^2:

$$W(\tau) \doteq \langle \delta\widetilde{\omega}(\tau)\delta\widetilde{\omega}(0)\rangle = \begin{cases} 2k^2 D\delta(\tau) & \text{if } \tau_{\mathrm{ac}}^{(W)} \to 0, \\ \beta^2 & \text{if } \tau_{\mathrm{ac}}^{(W)} \to \infty. \end{cases} \tag{2.26}$$

2.3.2. Exact solution of the stochastic oscillator

Being dynamically linear, the stochastic oscillator can be solved exactly. By doing so, I shall be able to introduce the concepts of a random Green's function or propagator[33] $\widetilde{g}(t;t')$ as well as the mean "propagator"[34] $g \doteq \langle \widetilde{g}\rangle$, and we shall be able to see how $g(\tau)$ relates the correlation function $C(t,t') \doteq \langle \delta\psi(t)\delta\psi^*(t')\rangle$ to the external forcing. Green's function for Eq. (2.24) obeys

$$[\partial_t + \nu + \mathrm{i}\widetilde{\omega}(t)]\widetilde{g}(t;t') = \delta(t-t') \tag{2.27}$$

with the boundary condition $\widetilde{g}(-\infty;t') = 0$. Upon ignoring an initial condition at $t = -\infty$, one finds that the exact driven solution of Eq. (2.24) is

$$\widetilde{\psi}(t) = \int_{-\infty}^{t} \mathrm{d}\overline{t}\, \widetilde{g}(t;\overline{t})\widetilde{f}^{\mathrm{ext}}(\overline{t}). \tag{2.28}$$

[The upper limit can be replaced by ∞ because $\widetilde{g}(t;\overline{t}) \propto H(t-\overline{t})$.] The mean field follows as

$$\langle\psi\rangle(t) = \int_{-\infty}^{t} \mathrm{d}\overline{t}\, g(t;\overline{t})\langle\widetilde{f}^{\mathrm{ext}}\rangle(\overline{t}), \tag{2.29}$$

where again $g \doteq \langle\widetilde{g}\rangle$. (Here I used the fact that $\widetilde{f}^{\mathrm{ext}}$ and $\widetilde{\omega}$ are uncorrelated, which for Gaussian variables implies statistical independence, in order to factor the average.)

It is straightforward to find that

$$\widetilde{g}(t;t') = H(t-t')\exp\left(-\nu(t-t') - \mathrm{i}\int_{t'}^{t}\mathrm{d}\overline{t}\,\widetilde{\omega}(\overline{t})\right). \tag{2.30}$$

Because $\widetilde{\omega}$ is Gaussian, the cumulant expansion of the average of Eq. (2.30) truncates at second order:

$$g(\tau) = H(\tau)\mathrm{e}^{-\mathrm{i}\overline{\omega}\tau-\nu\tau}\exp\left(-\frac{1}{2}\int_0^\tau\mathrm{d}\overline{t}\int_0^\tau\mathrm{d}\overline{t}'\,W(\overline{t}-\overline{t}')\right) \tag{2.31a}$$

$$= H(\tau)\mathrm{e}^{-\mathrm{i}\overline{\omega}\tau-\nu\tau}\begin{cases} \mathrm{e}^{-k^2D\tau} & \text{if } \tau_{\mathrm{ac}}^{(W)} \to 0, \\ \mathrm{e}^{-\frac{1}{2}\beta^2\tau^2} & \text{if } \tau_{\mathrm{ac}}^{(W)} \to \infty. \end{cases} \tag{2.31b}$$

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[32]One can bridge these limits with a formula motivated by Doob's theorem for a Markov process (Wang & Uhlenbeck 1945): $W(\tau) = \beta^2\exp(-|\tau|/\tau_{\mathrm{ac}}^{(W)})$, where $\beta \doteq k\overline{V}$ for some $\overline{V}$. This shows that a natural dimensionless variable is the *Kubo number* $\mathcal{K} \doteq \beta\tau_{\mathrm{ac}}^{(W)}$. For $\mathcal{K} \to 0$ ($\tau_{\mathrm{ac}}^{(W)} \to 0$), $W(\tau)$ approaches a delta function with $\int_0^\infty\mathrm{d}\tau\,W(\tau) = \beta^2\tau_{\mathrm{ac}}^{(W)} = k^2D$, where $D \doteq \overline{V}^2\tau_{\mathrm{ac}}^{(W)}$ (assumed to remain nonzero as $\tau_{\mathrm{ac}}^{(W)} \to 0$).

[33]The semicolon reminds one that the function is causal; i.e., it is proportional to $H(t-t')$.

[34]Strictly speaking, it is best to apply the nomenclature *propagator* to the function that multiplies the causality constraint $H(t-t')$; thus write $\widetilde{g}(t;t') = \widetilde{\chi}(t,t')H(t-t')$ and call $\widetilde{\chi}(t,t')$ the propagator. Although $\widetilde{\chi}$ is an actual propagator—it obeys the semigroup property $\widetilde{\chi}(t,t') = \widetilde{\chi}(t;\overline{t})\widetilde{\chi}(\overline{t};t')$—the average $\chi \doteq \langle\widetilde{\chi}\rangle$ does not necessarily obey that property but is still sometimes called a "propagator." The differing behaviour of χ arises because the average of a product need not equal the product of the averages. That fact is ultimately responsible for the difficulty that Eulerian theories have with satisfying random Galilean invariance (discussed in Sec. 5.1).

Both $\widetilde{g}$ and g contain an $\mathrm{e}^{-\nu\tau}$ damping, a consequence of ordinary linear dissipation. However, $g(\tau)$ decays even for $\nu = 0$ though $\widetilde{g}$ does not. That random fluctuations lead to decay of the mean response is one instance of *phase mixing*.[35] In the context of equations like Eq. (2.21), one can say that the effect is due to *random Doppler shifts*, an effect emphasized by Dupree.

Another perspective is obtained by looking in frequency space. The Fourier transform of Eq. (2.31b) is

$$\widehat{g}(\omega) = \begin{cases} \dfrac{1}{-\mathrm{i}[\omega - \overline{\omega} + \mathrm{i}(\nu + k^2 D)]} & \text{if } \tau_{\mathrm{ac}}^{(W)} \to 0, \\[2ex] \sqrt{\dfrac{\pi}{2}} \dfrac{1}{\beta} w[(\omega - \overline{\omega} + \mathrm{i}\nu)/(\sqrt{2}\beta)] & \text{if } \tau_{\mathrm{ac}}^{(W)} \to \infty, \end{cases} \tag{2.32}$$

where[36]

$$w(z) \doteq \mathrm{e}^{-z^2} \operatorname{erfc}(-\mathrm{i}z) = \mathrm{e}^{-z^2} \left(1 + \frac{2\mathrm{i}}{\sqrt{\pi}} \int_0^z \mathrm{d}t \, \mathrm{e}^{t^2} \right). \tag{2.33}$$

The small-$\tau_{\mathrm{ac}}^{(W)}$ result demonstrates that the effect of the random Doppler shifts is to enhance the dissipative broadening of the Lorentzian-shaped resonance:

$$\operatorname{Re}\widehat{g}(\omega) = \frac{\nu + k^2 D}{(\omega - \overline{\omega})^2 + (\nu + k^2 D)^2}. \tag{2.34}$$

Whereas for $\nu \to 0$ and $D \to 0$ one would have $\operatorname{Re}\widehat{g}(\omega) \approx \pi\delta(\omega - \overline{\omega})$, the random Doppler shifts provide additional line broadening, distributing power to all frequencies $0 \leqslant |\omega - \overline{\omega}| \lesssim \nu + k^2 D$. In Vlasov theory, this amounts to a broadening of the Landau wave–particle resonance, which is the effect discussed in the original paper of Dupree (1966). This justifies the nomenclature *resonance-broadening theory* for Dupree's early line of attack (which with regard to the nonlinearity was a theory of passive advection, though that was not clarified by Dupree[37]). For infinite τ_{ac} the resonance is not Lorentzian, but it maintains the same general features. The function $w(z)$ is graphed in Fig. 1. Also plotted there is the equivalent single-pole approximation or Lorentzian shape for which the functions agree at $z = 0$. An interpretation of this choice is given in footnote 40 on page 19.

Now let us turn to the fluctuation spectrum; for simplicity, I now consider $\langle f^{\mathrm{ext}} \rangle = 0$.

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[35]Mathematically, decay due to phase mixing is a consequence of the *Riemann–Lebesque lemma*. That is discussed, for example, by van Kampen & Felderhof (1967, Chapter XII.5).

[36]The standard function $w(z)$ is related to the widely used plasma dispersion function $Z(z)$ by $w(z) = Z(z)/(\mathrm{i}\sqrt{\pi})$.

[37]Some discussion that I gave in an earlier review article (Krommes 2002) is worth repeating here: "Dupree (1966), in describing his formal theory of test waves [not defined here], stated, 'The method we employ for solving the Vlasov–Maxwell equation consists of two distinct pieces. First, we assume knowledge of the electric field $\boldsymbol{E}$ . . . . As a second step, we must . . . require that the f so determined does . . . produce the assumed $\boldsymbol{E}$ [via Poisson's equation].' Later he asserted, 'The fact that the initial phases of the background waves in the subsidiary [test wave] problem are uncorrelated . . . does not prevent the [Fourier coefficients] so calculated from being used to describe an actual system in which all the initial phases have some precise relation to each other and to f.' However, freezing $\boldsymbol{E}$ in step 1 is the definition of a passive problem. Statistical correlations are lost at that point and cannot be recovered with the basic test-wave theory." However, this does not mean that theories of passive advection are useless. An example of an experimental situation to which resonance-broadening theory properly applies is described by Hershcovitch & Politzer (1979).
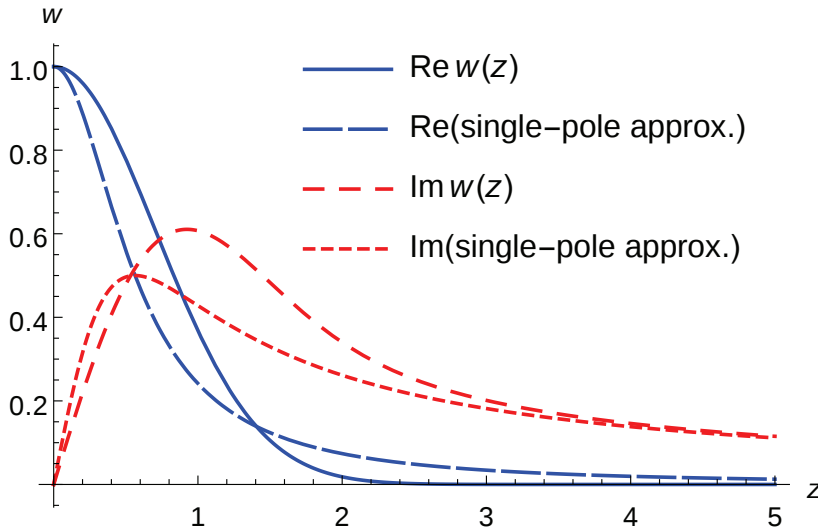
FIGURE 1. (Colour online.) The function $w(z)$ [Eq. (2.33)] and the equivalent single-pole approximation.

From Eq. (2.28), the two-time correlation function then follows as

$$C(t,t') = \int_{-\infty}^{t} \mathrm{d}\overline{t} \int_{-\infty}^{t'} \mathrm{d}\overline{t}' \, \langle \widetilde{g}(t;\overline{t}) F^{\mathrm{ext}}(\overline{t};\overline{t}') \widetilde{g}^*(t';\overline{t}') \rangle \qquad (2.35a)$$

$$= 2\varepsilon \int_{-\infty}^{\min(t,t')} \mathrm{d}\overline{t} \, \langle \widetilde{g}(t;\overline{t}) \widetilde{g}^*(t';\overline{t}) \rangle. \qquad (2.35b)$$

The upper limit of the integral in Eq. (2.35b) ensures that one of the integration variables in Eq. (2.35a) encounters the argument of the delta function in F^{ext} so that it can be integrated to 1, and it guarantees that $C(t,t')$ is properly symmetric.

Upon recalling Eq. (2.30), one can easily work out Eq. (2.35b):[38]

$$C(\tau) = \left(\frac{\varepsilon}{\nu}\right) \mathrm{e}^{-\mathrm{i}\overline{\omega}\tau - \nu|\tau|} \begin{cases} \mathrm{e}^{-k^2 D |\tau|} & \text{if } \tau_{\mathrm{ac}}^{(W)} \to 0, \\ \mathrm{e}^{-\frac{1}{2}\beta^2 \tau^2} & \text{if } \tau_{\mathrm{ac}}^{(W)} \to \infty; \end{cases} \qquad (2.36a)$$

$$\widehat{C}(\omega) = 2\left(\frac{\varepsilon}{\nu}\right) \mathrm{Re}\, \widehat{g}(\omega) \quad \text{(for arbitrary } \tau_{\mathrm{ac}}^{(W)}) \qquad (2.36b)$$

[cf. Eq. (2.14)]. Note that from either Eq. (2.36a) or Eq. (2.36b) follows the steady-state fluctuation level[39]

$$C(0) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \widehat{C}(\omega) = \frac{\varepsilon}{\nu}, \qquad (2.37)$$

which has the same form as Eq. (2.12).

2.3.3. Autocorrelation time

By contemplating the inverse Fourier transform of Eq. (2.36b), one learns that the detailed time dependence of the correlation function depends on all of the Fourier

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[38]Equation (2.36b) follows because $-\mathrm{i}\int_{\overline{t}}^{t} \mathrm{d}s\, \widetilde{\omega}(s) + \mathrm{i}\int_{\overline{t}}^{t'} \mathrm{d}s\, \widetilde{\omega}(s) = -\mathrm{i}\int_{t'}^{t} \mathrm{d}s\, \widetilde{\omega}(s)$. This integral is independent of $\overline{t}$, so the $\overline{t}$ integral can be done explicitly.

[39]$(2\pi)^{-1} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \widehat{g}(\omega) = g(0) = H(0) = \frac{1}{2}$.

components of $\widehat{g}(\omega)$. As a special case, one can define an autocorrelation time as[40]

$$\tau_{\rm ac} = [C(0)]^{-1} \int_0^\infty {\rm d}\tau\, C(\tau) = \frac{1}{2}[C(0)]^{-1} \int_{-\infty}^\infty {\rm d}\tau\, C(\tau) = \frac{1}{2}[C(0)]^{-1}\widehat{C}(0) = {\rm Re}\,\widehat{g}(0). \tag{2.38}$$

In general, this is built from the four frequencies or rates $\overline{\varpi}$, ν, β, and $(\tau_{\rm ac}^{(W)})^{-1}$. These four dimensional quantities determine three dimensionless parameters. I have already introduced the Kubo number $\mathcal{K} \doteq \beta\tau_{\rm ac}^{(W)}$; a second important parameter is $\mathcal{R} \doteq \beta/\nu$, which plays the role of a Reynolds number.[41] The third dimensionless ratio is $\nu/\overline{\varpi}$. Various regimes of the $\mathcal{K}$–$\mathcal{R}$ space for $\overline{\varpi} = 0$ were discussed by Krommes & Smith (1987).

To further understand the significance of $\tau_{\rm ac}$ or $\widehat{g}(0)$, let us write *in general*

$$\widehat{g}(\omega) = \frac{1}{-{\rm i}[\omega - \overline{\varpi} + {\rm i}(\nu + \widehat{\Sigma}_\omega)]} \tag{2.39}$$

for some $\widehat{\Sigma}_\omega$ [sometimes called the *mass operator*[42] because of deep connections to quantum field theory (Martin *et al.* 1973; Krommes 2002)]. It is technically convenient to introduce Σ because g has a resonance form that is difficult to handle in approximate theories; approximating the inverse of the propagator (i.e., changing the focus to Σ), is much easier to do. Upon writing $\widehat{\Sigma} = \widehat{\Sigma}^r + {\rm i}\widehat{\Sigma}^i$, then defining $\widehat{\varpi}_\omega \doteq \overline{\varpi} + \widehat{\Sigma}_\omega^i$ and $\widehat{\eta}_\omega \doteq \nu + \widehat{\Sigma}_\omega^r$, one finds

$$ {\rm Re}\,\widehat{g}(\omega) = \frac{\widehat{\eta}_\omega}{(\omega - \widehat{\varpi}_\omega)^2 + \widehat{\eta}_\omega^2}. \tag{2.40}$$

We see that $\widehat{\Sigma}^r$ introduces additional resonance broadening,[43] whereas $\widehat{\Sigma}^i$ provides a nonlinear frequency shift. If one defines $\varpi \doteq \widehat{\varpi}_0$ and $\eta \doteq \widehat{\eta}_0$, then one has

$$ \tau_{\rm ac} = {\rm Re}\,\widehat{g}(0) = \frac{\eta}{\varpi^2 + \eta^2} \approx \begin{cases} (\eta/\varpi)\varpi^{-1} & \text{if } \eta/\varpi \ll 1, \\ \eta^{-1} & \text{if } \eta/\varpi \gg 1. \end{cases} \tag{2.41}$$

Thus the size of $\tau_{\rm ac}$ depends on whether the broadening is weak or strong. For weak broadening, one has a weakly damped oscillator. For that limit, one has $\tau_{\rm ac} \approx (\eta/\varpi)\varpi^{-1} \ll \varpi^{-1}$. It may at first seem counterintuitive that $\tau_{\rm ac}$ is smaller than the oscillator period, but that is easily explained with the aid of Fig. 2; only a small amount of decorrelation or dissipation occurs per cycle.

Equation (2.41) makes sense only if $\eta \geqslant 0$. I shall demonstrate that at the end of the next section.

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[40]The result $\tau_{\rm ac} = {\rm Re}\,\widehat{g}(0)$ shows that the single-pole approximation plotted in Fig. 1 was chosen to maintain the equivalence of $\tau_{\rm ac}$; this is sometimes described as "preserving the adiabatic response." That is the same choice made by Hammett & Perkins (1990) and Hammett *et al.* (1992) in their discussions of fluid models of phase mixing. One can view the family of Markovian fluid closures developed by those authors as n-pole generalizations of the single-pole approximation mentioned here.

[41]If one writes $\beta = k\overline{V} \sim \overline{V}/L$ for some scale L, then $\mathcal{R} \sim \overline{V}L/\mu$ with $\mu \doteq L^2\nu$ playing the role of a viscosity.

[42]Distinguish the symbol Σ for the mass operator from the summation sign $\sum$. Also, for conciseness of notation in the following discussion, some symbols such as $\widehat{\Sigma}_\omega$ use frequency subscripts instead of arguments even though the continuous Fourier transform in time is used.

[43]The form of Eq. (2.40) is not strictly Lorentzian because $\widehat{\varpi}_\omega$ and $\widehat{\eta}_\omega$ depend on frequency.
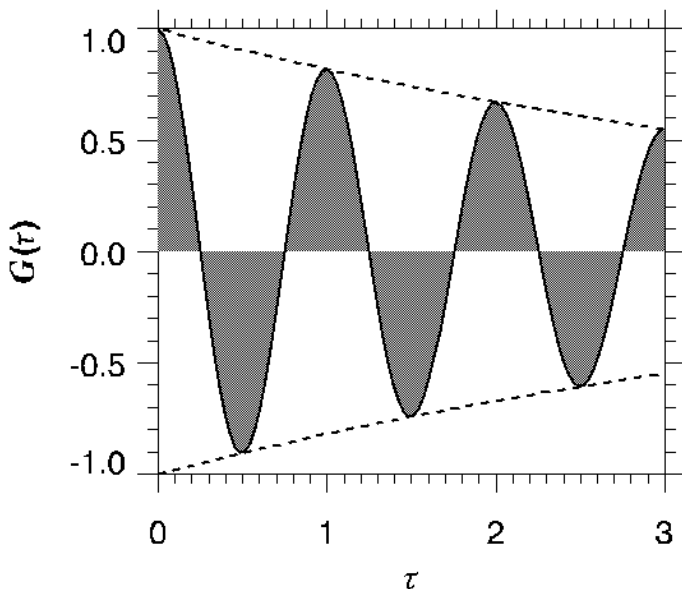
FIGURE 2. Example of a correlation function for weak broadening: $C(\tau) = \mathrm{Re}\exp(-\mathrm{i}\varpi\tau - \eta|\tau|)$. The autocorrelation time is the area under the curve. For a weakly damped oscillator, the positive and negative areas almost cancel over one cycle. Reprinted from Fig. 13 of Krommes (2002), copyright 2002, with permission from Elsevier.

2.3.4. Internal or incoherent noise

Note that Eq. (2.40) can be written as

$$\mathrm{Re}\,\widehat{g}(\omega) = |\widehat{g}(\omega)|^2(\nu + \widehat{\Sigma}_\omega^r). \tag{2.42}$$

Substitution of Eq. (2.42) into Eq. (2.36b) gives

$$\widehat{C}(\omega) = \widehat{g}(\omega)\widehat{F}(\omega)\widehat{g}^*(\omega), \tag{2.43}$$

where

$$\widehat{F}(\omega) \doteq 2\varepsilon[(\widehat{\Sigma}_\omega^r/\nu) + 1]. \tag{2.44}$$

Equation (2.43) should be compared with Eq. (2.35a), which involves $\widetilde{g}$ rather than g. In Eq. (2.44), the 1 arises from the external forcing. For the purpose of calculating second-order statistics, the interpretation of Eqs. (2.43) and (2.44) is that instead of propagating δf^{ext} with the random propagator $\widetilde{g}$ [Eq. (2.28)], then averaging at the end, one is entitled to propagate noise with the *mean* propagator g provided that one enhances the external noise with a certain effective *internal* noise [called *incoherent noise* by Dupree (1972)]. Thus one can write

$$\widehat{F}(\omega) = \widehat{F}^{\mathrm{int}}(\omega) + \widehat{F}^{\mathrm{ext}}(\omega), \tag{2.45}$$

where (for white-noise external forcing)

$$\widehat{F}^{\mathrm{int}}(\omega) = 2\varepsilon(\widehat{\Sigma}_\omega^r/\nu), \quad \widehat{F}^{\mathrm{ext}}(\omega) = 2\varepsilon. \tag{2.46}$$

The concept of internal noise is central to modern theories of statistical dynamics.

It is a consequence of Parseval's theorem that $\widehat{C}(\omega)$ is real and non-negative. From

Eq. (2.43), $\widehat{F}(\omega)$ inherits the same properties. From Eq. (2.44), this implies that $\widehat{\eta}_\omega \geqslant 0$, which is an important constraint on the turbulent broadening.

2.3.5. Generalized Langevin equation

The representation (2.39) together with Eqs. (2.45) and (2.46) lead one to the *generalized Langevin equation*[44]

$$(\partial_t + i\overline{\omega} + \nu)\widetilde{\psi}(t) + \int_{-\infty}^{t} d\overline{t}\, \Sigma(t;\overline{t})\widetilde{\psi}(\overline{t}) = \widetilde{f}^{\text{int}}(t) + \widetilde{f}^{\text{ext}}(t) \doteq \widetilde{f}^{\text{tot}}(t). \qquad (2.47)$$

Here $\Sigma(\tau)$ is the inverse Fourier transform of $\widehat{\Sigma}(\omega)$, and $\widetilde{f}^{\text{int}}$ is an auxiliary random variable (assumed to be statistically independent of $\widetilde{f}^{\text{ext}}$) whose covariance is

$$F^{\text{int}}(\tau) = 2\varepsilon\Sigma'(\tau)/\nu, \qquad (2.48)$$

where $\Sigma'(\tau)$ is the inverse Fourier transform of $\widehat{\Sigma}_\omega^r$. *Generalized* means that the equation is nonlocal in time due to the time-history integral. Such generalized Langevin representations arise frequently in modern turbulence theory.

The solution of Eq. (2.47) is

$$\widetilde{\psi}(t) = \int_{-\infty}^{t} d\overline{t}\, g(t;\overline{t})\widetilde{f}^{\text{tot}}(\overline{t}), \qquad (2.49)$$

which should be compared with Eq. (2.28). It must be emphasized that Eq. (2.49) is not an exact solution for a single realization since all realizations are coupled through Σ, which determines both g and the covariance[45] of $\widetilde{f}^{\text{int}}$. However, it is equivalent to that solution at the level of second-order statistics. This remark holds as well for the original Langevin equation (2.10).

2.4. *Statistical closure and the stochastic oscillator*

It is instructive to try to recover the essence of these exact results from statistical closure approximations because exact solutions are not available for realistic, nonlinear PDEs. This topic has been treated extensively in the literature. An important early paper (the first part of which is quite pedagogical) is by Kraichnan (1961), who considered aspects of the infinite-τ_{ac} problem; further discussion and references are given by Krommes (1984) and, more thoroughly, by Krommes (2002). Here I shall just give a brief introduction.

Consider the restricted problem of finding an approximation to the mean response function $g(\tau)$ for Eq. (2.24). The random response function $\widetilde{g}(t;t')$ obeys Eq. (2.27). Instead of writing a formally exact solution (which would not be useful if the primitive amplitude equation were really a PDE), let us write the evolution equation for g by averaging Eq. (2.27). One gets

$$\partial_t g(t;t') + (i\overline{\omega} + \nu)g + \langle i\delta\widetilde{\omega}(t)\delta\widetilde{g}(t;t')\rangle = \delta(t - t'). \qquad (2.50)$$

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[44]Generalized Langevin equations were originally discussed by Zwanzig (1961*a,b*) and Mori (1965). A modern reference is by Zwanzig (2001).

[45]In this version of the generalized Langevin formalism, the complete statistics of $\widetilde{f}^{\text{int}}$ are not specified. They could be chosen to be Gaussian. Then Eq. (2.47) would predict that $\widetilde{\psi}$ is Gaussian, which of course is not true for the actual problem.

To be consistent with Eq. (2.47), I shall write this as

$$\partial_t g(t;t') + (\mathrm{i}\overline{\omega} + \nu)g + \int_{t'}^{t} \mathrm{d}\overline{t}\, \Sigma(t;\overline{t})g(\overline{t};t') = \delta(t - t') \tag{2.51}$$

or

$$\partial_t g(\tau) + (\mathrm{i}\overline{\omega} + \nu)g + \int_{0}^{\tau} \mathrm{d}\overline{\tau}\, \Sigma(\overline{\tau})g(\tau - \overline{\tau}) = \delta(\tau). \tag{2.52}$$

Either of these forms is called a *Dyson equation*[46] after the famous unification by Dyson (1949) of various approaches to quantum electrodynamics.

The goal is to find an approximate formula for Σ. One has exactly

$$\partial_t \delta g(t;t') + (\mathrm{i}\overline{\omega} + \nu)\delta g + \mathrm{i}\delta\omega(t)g(t;t') + \mathrm{i}[\delta\omega(t)\delta g(t;t') - \langle\delta\omega\,\delta g\rangle] = 0. \tag{2.53}$$

To proceed, one must assess the importance of the (stochastically) nonlinear term $\delta\omega\,\delta g$, and that depends on the values of the dimensionless parameters. For present purposes, if either $\mathcal{K}$ or $\mathcal{R}$ is small it is appropriate to drop the nonlinear term, in which case the solution of Eq. (2.53) is

$$\delta g(t;t') = -\int_{t'}^{t} \mathrm{d}\overline{t}\, g_0(t;\overline{t})\mathrm{i}\delta\omega(\overline{t})g(\overline{t};t'), \tag{2.54}$$

where $g_0(\tau) \doteq H(\tau)\exp(-\mathrm{i}\overline{\omega}\tau - \nu\tau)$ is the lowest-order Green's function. Thus

$$\langle\mathrm{i}\delta\omega(t)\delta g(t;t')\rangle = \int_{t'}^{t} \mathrm{d}\overline{t}\, \Sigma(t;\overline{t})g(\overline{t};t') = \int_{0}^{\tau} \mathrm{d}\overline{\tau}\, \Sigma(\overline{\tau})g(\tau - \overline{\tau}), \tag{2.55}$$

where

$$\Sigma(\tau) \approx \Sigma_0(\tau) \doteq g_0(\tau)W(\tau). \tag{2.56}$$

This has the form of Eq. (2.52) with a particular approximation for Σ. Since in the present case Σ has been constructed with the unperturbed propagator g_0 but describes the nonlinearity, one may call this the *quasilinear approximation*.[47]

[As a consistency check, for $\mathcal{K} \to 0$ one has $\Sigma(\tau) \to g_0(\tau)2k^2D\delta(\tau) = k^2D\delta(\tau)$, in agreement with Eq. (2.32).]

For the quasilinear approximation to be valid, the nonlinear term in the equation for the fluctuations must be smaller than the linear term. For simplicity, assume that $\overline{\omega} = 0$. Then the ratio of nonlinear to linear rates is $\beta^2\tau_{\mathrm{ac}}^{(\Sigma_0)}/\nu$, which becomes large as $\nu \to 0$ ($\mathcal{R} \to \infty$). In this situation, quasilinear theory is unjustifiable and advanced techniques must be used to effect a proper closure. As discussed by Kraichnan (1961), a plausible approximation is the DIA. For the oscillator model, that amounts to replacing the g_0 in Eq. (2.56) by the as-yet-unknown g: $\Sigma(\tau) \approx g(\tau)W(\tau)$. The exact Dyson equation (2.52) is then used to solve for g self-consistently.[48] For $\mathcal{K} \to \infty$ one has $\eta = \beta^2\tau_{\mathrm{ac}}$, and the

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[46]Although in principle Dyson equations are nonlocal in time, it is sometimes possible to approximately eliminate the time convolution. Define a new autocorrelation time $\tau_{\mathrm{ac}}^{(\Sigma)} \doteq \beta^{-2}\int_0^\infty \mathrm{d}\tau\, \Sigma(\tau)$. If one is content to consider a coarse graining of the time axis into increments proportional to $\tau_{\mathrm{ac}}^{(\Sigma)}$, then for $\tau > \tau_{\mathrm{ac}}^{(\Sigma)}$ one has $\int_0^\infty \mathrm{d}\overline{\tau}\, \Sigma(\overline{\tau})g(\tau - \overline{\tau}) \approx [\int_0^\infty \mathrm{d}\overline{\tau}\, \Sigma(\overline{\tau})]g(\tau)$ provided that $\Sigma(\tau)$ decays more rapidly than $g(\tau)$; this is a *Markovian approximation*. One has $\eta \doteq \int_0^\infty \mathrm{d}\overline{\tau}\, \Sigma(\overline{\tau}) = \beta^2\tau_{\mathrm{ac}}^{(\Sigma)} = k^2D$, where $D \doteq \overline{V}^2\tau_{\mathrm{ac}}^{(\Sigma)}$. Compare this result to Eq. (2.20).

[47]A distinction between a non-Markovian *Bourret approximation* and a Markovian quasilinear approximation is sometimes made; see Krommes (2002, Sec. 3.9.2).

[48]For $\mathcal{K} \to \infty$ and $\mathcal{R} \to \infty$ the DIA for the response function of the stochastic oscillator, $\partial_\tau g(\tau) + \int_0^\tau \mathrm{d}\overline{\tau}\,[g(\overline{\tau})\beta^2]g(\tau - \overline{\tau}) = \delta(\tau)$, where the term in square brackets is $\Sigma(\overline{\tau})$, can be

balance $\partial_\tau \sim \eta$ self-consistently determines $\tau_{\rm ac}$ to be $O(\beta^{-1})$. This is consistent with the exact solution; see Eq. (2.36a). Note that in this strong-turbulence limit the Markovian approximation breaks down because $\Sigma(\tau)$ and $g(\tau)$ vary on the same time scale; the physics is intrinsically nonlocal in time. I shall give a derivation of the DIA in Sec. 4.1.

2.5. *Renormalization*

The replacement of g_0 by g is an example of *renormalization*. A thorough discussion of the technical meaning of renormalization and of all of its ramifications is unfortunately beyond the scope of this article. Heuristically, it implies the broadening of resonance functions due to random nonlinear effects; in perturbation theory, that broadening is represented as a sum of certain terms through all orders of the nonlinear coupling. In general, the broadening must be determined self-consistently. An elucidating collection of articles with historical perspectives was edited by Brown (1993), a beginner's guide to methodology is by McComb (2004), and an advanced book is by Zinn-Justin (1996). Some of the connections to quantum field theory are reviewed by Krommes (2002). Renormalization has a deep connection to so-called *anomalous scaling*, frequently discussed in the theory of phase transitions (Goldenfeld 1992). A brief discussion of anomalous scaling that uses the stochastic oscillator as an example is given in Sec. 6.1.2 of Krommes (2002). The basic point is that the naive quasilinear result for Σ scales with β^2, but a self-consistent calculation of the nonlinear damping rate for large $\mathcal{K}$ and $\mathcal{R}$ leads to a scaling with β^1; here the power 1 is called an *anomalous exponent*. Introductory pedagogical discussion of renormalization and anomalous scaling can also be found in the article by Krommes (2009), which provides some useful background for the present tutorial.

3. Self-consistency, polarization, and dielectric shielding

Passive systems are (relatively) simple because the advecting velocity is not modified by the system response. When instead that velocity is self-consistently determined, additional effects come into play. In particular, once a fluctuation arises it can polarize the medium (that backreaction being a manifestation of self-consistency); the resulting polarization field shields the fluctuation and modifies its effect. This is dielectric response.

3.1. *Prelude: Dielectric polarization in electromagnetism*

Ultimately I shall consider turbulent systems that are not necessarily electromagnetic in nature, such as the Navier–Stokes fluid. However, it is useful to begin on familiar footing by discussing electromagnetic systems, fully ionized plasmas being a prominent example.

3.1.1. Introductory concepts from electromagnetism

In electromagnetism, the dielectric function $\mathcal{D}$ is introduced as a way of describing the macroscopic polarizability properties of a microscopically random medium. Specifically, in elementary discussions the total charge density $\rho^{\rm tot}$ is divided into free charge and bound charge, and Maxwell's equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho^{\rm tot} = 4\pi(\rho^{\rm free} + \rho^{\rm bound}) \tag{3.1}$$

is rewritten as $\boldsymbol{\nabla} \cdot \boldsymbol{D} = 4\pi\rho^{\rm free}$, where the electrical displacement is $\boldsymbol{D} \doteq \boldsymbol{E} + 4\pi\boldsymbol{P}$ and the polarization vector $\boldsymbol{P}$ obeys $\boldsymbol{\nabla} \cdot \boldsymbol{P} = -\rho^{\rm bound}$. If $\boldsymbol{P}$ is assumed to be linear

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solved exactly by Fourier transformation (Kraichnan 1959, 1961). See also the article by Frisch & Bourret (1970).

in the macroscopic field $\boldsymbol{E}$, then one can introduce a susceptibility tensor $\boldsymbol{X}$ by $\boldsymbol{P} = (4\pi)^{-1}\boldsymbol{X} \cdot \boldsymbol{E}$ and one has $\boldsymbol{D} = \boldsymbol{D} \cdot \boldsymbol{E}$ with $\boldsymbol{D} \doteq \boldsymbol{I} + \boldsymbol{X}$ being the dielectric tensor. A standard generalization is to introduce the wave-number- and frequency-dependent dielectric tensor $\boldsymbol{D}(\boldsymbol{k},\omega)$, appropriate for a medium that is on the average invariant under translations in space (statistically homogeneous) and time (statistically stationary). For a homogeneous, isotropic, and stationary plasma in the electrostatic approximation $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$, it is sufficient to work with the longitudinal dielectric scalar $\mathcal{D}(\boldsymbol{k},\omega) \doteq \widehat{\boldsymbol{k}} \cdot \boldsymbol{D}(\boldsymbol{k},\omega) \cdot \widehat{\boldsymbol{k}}$, so $\boldsymbol{D}(\boldsymbol{k},\omega) = \mathcal{D}(\boldsymbol{k},\omega)\boldsymbol{E}(\boldsymbol{k},\omega)$. For simplicity, I shall consider only the longitudinal dielectric function in this article.

The quantity $\mathcal{D}^{-1}(\boldsymbol{k},\omega)$ describes the first-order response to the addition of free charge or external potential; it is a so-called *linear response function*. If a turbulent plasma is perturbed by a potential of the form $\phi^{\mathrm{ext}}(t) = \Delta\mathrm{e}^{\epsilon t}H(-t)$ (i.e., by adiabatically raising ϕ^{ext} from 0 at $t = -\infty$ to a value Δ at $t = 0_-$), it can be shown (Martin 1968) that the Fourier transform of the subsequent one-sided response of the internal or induced potential, $\phi_+^{\mathrm{int}}(t) \doteq H(t)\phi^{\mathrm{int}}(t)$, is

$$\widehat{\phi}_+^{\mathrm{int}}(\boldsymbol{k},\omega) = \frac{\Delta}{\mathrm{i}\omega\mathcal{D}(\boldsymbol{k},\omega)}\left(\frac{\chi(\boldsymbol{k},\omega)}{\chi(\boldsymbol{k},0)} - 1\right). \tag{3.2}$$

Note that this expression has no pole at $\omega = 0$. If $\mathcal{D}$ were to vanish at a real frequency $\omega = \Omega_{\boldsymbol{k}}$, the system would oscillate even in the absence of external forcing; this is called a *normal mode*. That does not usually happen for real ω; however, the analytical continuation of $\mathcal{D}(\boldsymbol{k},\omega)$ into the complex ω plane can possess a complex zero $\omega_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}} + \mathrm{i}\gamma_{\boldsymbol{k}}$. One continues to speak of a normal mode provided that $|\gamma_{\boldsymbol{k}}/\Omega_{\boldsymbol{k}}| \ll 1$ (i.e., that the oscillation is underdamped). For the sign of $\gamma_{\boldsymbol{k}}$, see the discussion at the end of this section.

It is important to understand that all of $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{P}$, and ρ are statistical averages over the random microscopic configuration; for example, $\rho \doteq \langle\widetilde{\rho}\rangle$, where $\widetilde{\rho}(\boldsymbol{x},t) \doteq \sum_{\overline{s}}(\overline{n}q)_{\overline{s}}\int\mathrm{d}\overline{\boldsymbol{v}}\,\widetilde{N}_{\overline{s}}(\boldsymbol{x},\overline{\boldsymbol{v}},t)$ is the microscopic charge density. Thus $\mathcal{D}$ describes the *mean* response of a random system of charged particles to infinitesimal perturbations. But although $\mathcal{D}^{-1}$ is a linear response function[49] (see Fig. 3), the nature of that response inherits properties of the fluctuations in the unperturbed system; $\mathcal{D}$ is a *nonlinear*[50] functional of those fluctuations through all orders. In thermal equilibrium one has the famous fluctuation–dissipation theorem,[51] which relates the two-time correlation function C to $\mathcal{D}^{-1}$. For nonequilibrium turbulence, we shall see that generalized fluctuation–dissipation relations can be written. Indeed, a spontaneous fluctuation in the random medium effectively behaves as another internal source of free charge that is then shielded by the dielectric properties of the medium, so we shall find that $\mathcal{D}$ figures importantly in the spectral balance equation that describes the non-Gaussian final state achieved by the turbulent system.

When $\mathcal{D}$ is calculated in the linear approximation, one can often find unstable eigenvalues with linear growth rate $\gamma_{\boldsymbol{k}}^{\mathrm{lin}} > 0$. But in a turbulent steady state, which is assumed

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[49]Linear response functions enjoy important analyticity properties as consequences of causality; see Martin (1968) for the details.

[50]It must be stressed that although $\mathcal{D}$ is a nonlinear functional it describes first-order response. It is also possible to define nth-order response functions for $n \geqslant 2$. Those have important applications in physics, but they are not discussed in this article.

[51]A review article about the general fluctuation–dissipation theorem is by Kubo (1966). A good discussion of the plasma fluctuation–dissipation theorem is by Ecker (1972); see also Martin (1968).
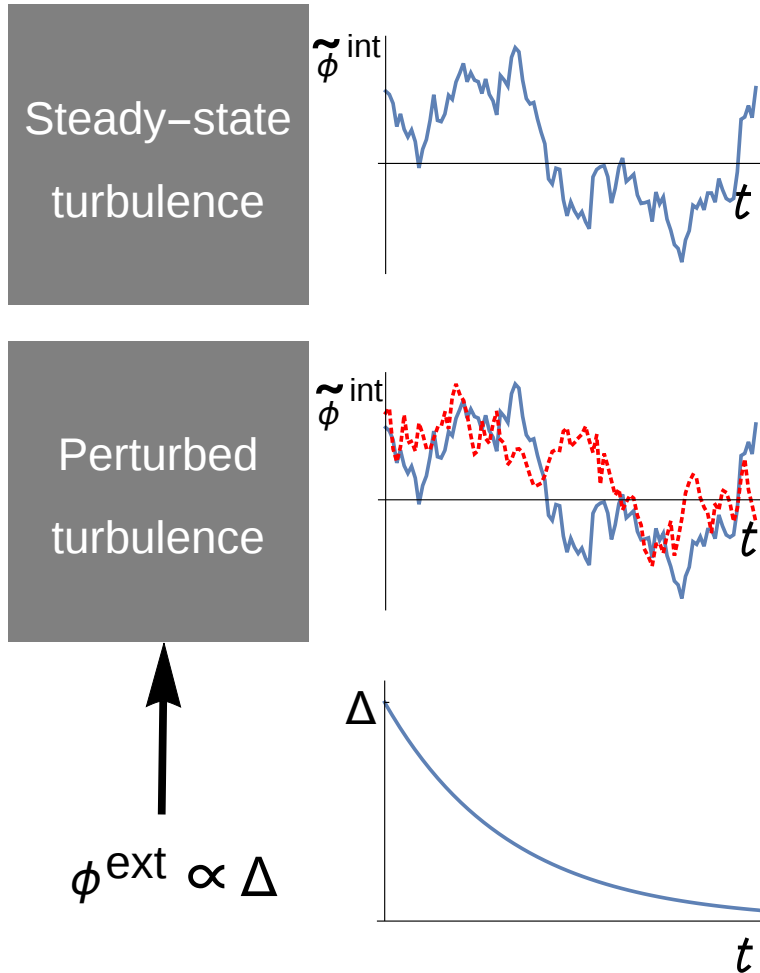
FIGURE 3. (Colour online.) Illustration of the linear response of a turbulent system. The top sketch shows that the internal potential measured in the steady state is a random function of time $\widetilde{\phi}^{\mathrm{int}}(t)$. In the middle sketch, the system is perturbed by an external potential ϕ^{ext} of size Δ. Because of nonlinearity, the measured output (red, dotted curve) is a random signal containing terms of all orders in Δ: $\widetilde{\phi}^{\mathrm{int}} = \widetilde{\phi} + \sum_{n=1}^{\infty} O(\Delta^n)$. The bottom plot shows the first-order part of the perturbed output after ensemble averaging. If the turbulence level is C, nonlinear scrambling should make the mean perturbation decay faster for larger C—for example, like $\exp(-Ct)$. Thus the mean linear response will depend on the fluctuation level of the background turbulence through all orders in C. The inverse of the dielectric function, $\mathcal{D}^{-1}$, is a first-order response function. Terms of $O(\Delta^n)$ for $n > 1$ define higher-order response functions, which are not discussed in this article.

to be stable against small perturbations, Eq. (3.2) guarantees that any complex zeros of $\mathcal{D}(\boldsymbol{k}, \omega)$ lie in the lower (stable) half of the ω plane; i.e., the true "growth" rate satisfies $\gamma_{\boldsymbol{k}} < 0$. *Thus in steady-state ("saturated") turbulence, linear growth must be overcompensated by nonlinear damping effects.* Further discussion of this point can be found in Krommes (2007).

3.1.2. Digression: Gyrokinetics and dielectric response

Before delving into the difficult subject of dielectric response in the presence of nonlinearity and turbulence, I shall illustrate some basic concepts about polarization with the aid of the linear approximation to the gyrokinetic dielectric function.[52]

Low-frequency gyrokinetics is an analytical reduction of the description of magnetized plasma in which the rapid gyrospiraling of a particle around a magnetic field line is eliminated in favour of particle drifts and effective (gyroaveraged) potentials. It is appropriate for the treatment of fluctuations whose frequencies are much smaller than the ion gyrofrequency: $\omega/\omega_{ci} \ll 1$. A revolution in analytical formalism occurred with the development of nonlinear gyrokinetics by Frieman & Chen (1982), Lee (1983), Dubin *et al.* (1983), and many subsequent researchers. Krommes (2012b) provides an introductory review and also cites previous reviews [e.g., Garbet *et al.* (2010)] and fundamental papers, including ones on linear gyrokinetics [e.g., Catto (1978b)]. The gyrokinetic formalism is now the major tool for both analytical descriptions and numerical simulations of low-frequency fluctuations in turbulent magnetized plasmas. The theory of gyrokinetic transport equations is discussed in a comprehensive review of multiscale gyrokinetics by Abel *et al.* (2013).

The works of Lee (1983) and Dubin *et al.* (1983) showed that gyrocentres can be treated in the same sense as the free charge introduced in the theory of dielectric media (Sec. 3.1.1). Gyrocentres can be said to move in an effective *gyrokinetic vacuum* (Krommes 1993, 2012b) endowed with a large dielectric permittivity[53] $\mathcal{D}_\perp$ that takes account of the polarization drift that particles undergo when exposed to a slowly varying potential. Thus gyrocentres do *not* move with the polarization drift; instead, polarization is taken into account in the gyrokinetic Poisson equation. This interpretation is satisfyingly intuitive, and it is given a rigorous justification by modern mathematical methods (Brizard & Hahm 2007).

Let us illustrate the procedure for obtaining a dielectric function by deriving the simplest (linear) gyrokinetic $\mathcal{D}$, then using that to recover the drift-wave dispersion relation. (The symbols used in this discussion are defined in Appendix A.) For uniform magnetic field, uniform background density, and cold ions, the gyrokinetic equation reduces (Dubin *et al.* 1983) to

$$\partial_t F + v_\| \nabla_\| F + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} F + E_\| \partial_\| F = 0 \qquad (3.3)$$

(describing the facts that gyrocentres are electrically accelerated along the field lines and move across the field with the $\boldsymbol{E} \times \boldsymbol{B}$ drift), and the gyrokinetic Poisson equation (in the approximation of quasineutrality) becomes

$$-\mathcal{D}_\perp \nabla_\perp^2 \phi = 4\pi\rho^{\mathrm{G}} \quad \text{or} \quad -\rho_s^2 \nabla_\perp^2 \varphi = n_i^{\mathrm{G}}/\overline{n}_i - n_e^{\mathrm{G}}/\overline{n}_e, \qquad (3.4)$$

where $\mathcal{D}_\perp \doteq \rho_s^2/\lambda_{\mathrm{D}e}^2$ and $n_s^{\mathrm{G}} \doteq \overline{n}_s \int \mathrm{d}\boldsymbol{v} F_s$. The right-hand side of this equation describes gyrocentre (free) charge; the left-hand side is the negative of the polarization charge ρ^{pol}, which plays the role of bound charge in the general theory. This can be seen by considering the linearized continuity equation $\partial_t \delta n_i^{\mathrm{pol}} + \boldsymbol{\nabla} \cdot (\delta\boldsymbol{V}^{\mathrm{pol}} n_i) = 0$, where

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[52]It is not necessary to introduce gyrokinetics at this point; one can discuss polarization and dielectric response for the unmagnetized Vlasov equation. However, gyrokinetics is important in its own right, and the drift-wave dispersion relation (3.7) that I shall derive will be referred to later in the article.

[53]The symbols $\perp$ and $\|$ refer to the directions perpendicular to and parallel to the magnetic field.

$\boldsymbol{V}^{\mathrm{pol}} \doteq \omega_{\mathrm{ci}}^{-1}\partial_t(c\boldsymbol{E}_\perp/B)$ is the ion polarization drift. Upon removing the ∂_t's, one is led to the expression[54] $\delta n_i^{\mathrm{pol}}/n_i = \rho_{\mathrm{s}}^2\nabla_\perp^2\varphi$.

Thus the coupled gyrokinetic–Poisson equations build in the particle polarization that arises from slowly varying fields in the magnetized plasma. This new dynamical system possesses interesting properties. It contains some, though not all, of the normal modes of the full Vlasov–Poisson system. To learn about the gyrokinetic normal modes, one can straightforwardly linearize Eq. (3.3) around a Maxwellian, add an external potential φ^{ext} (that must be done in both of the $\boldsymbol{V}_E$ and $E_\parallel$ terms), and find that the total potential (the induced or polarization part proportional to δF, plus φ^{ext}) obeys $\widehat{\varphi}^{\mathrm{tot}}(\boldsymbol{k},\omega) = (\mathcal{D}_{\mathrm{G}}^{(0)})^{-1}(\boldsymbol{k},\omega)\widehat{\varphi}^{\mathrm{ext}}(\boldsymbol{k},\omega)$, where

$$\mathcal{D}_{\mathrm{G}}^{(0)}(\boldsymbol{k},\omega) \doteq 1 + \frac{1}{\mathcal{D}_\perp}\sum_{\overline{s}}\frac{k_{\mathrm{D}\overline{s}}^2}{k_\perp^2}\int \mathrm{d}\overline{\boldsymbol{v}}\left(\frac{\omega_{*\overline{s}} - k_\parallel\overline{v}_\parallel}{\omega - k_\parallel\overline{v}_\parallel + \mathrm{i}\epsilon}\right)F_{\mathrm{M}\overline{s}}(\overline{\boldsymbol{v}}). \tag{3.5}$$

The (0) superscript indicates that nonlinear corrections have been ignored. This form closely parallels the one that follows from unmagnetized linearized Vlasov theory, the crucial difference being the appearance of the large $\mathcal{D}_\perp$—that being a consequence of ion polarization and a fundamental property of the gyrokinetic vacuum. In the limit $k_\parallel v_{\mathrm{ti}} \ll \omega$, $\omega_{*s} \ll k_\parallel v_{\mathrm{te}}$, $\mathcal{D}_{\mathrm{G}}^{(0)}$ reduces to

$$\mathcal{D}_{\mathrm{G}}^{(0)}(\boldsymbol{k},\omega) \approx 1 + \frac{1}{\mathcal{D}_\perp}\left[\left(\frac{k_{\mathrm{D}e}^2}{k_\perp^2}\right) + \left(\frac{k_{\mathrm{D}i}^2}{k_\perp^2}\right)\left(\frac{\omega_{*i}}{\omega}\right)\right] \tag{3.6a}$$

$$= 1 + \frac{1}{k_\perp^2\rho_{\mathrm{s}}^2} - \frac{1}{k_\perp^2\rho_{\mathrm{s}}^2}\left(\frac{\omega_{*e}}{\omega}\right). \tag{3.6b}$$

This leads one to the drift-wave dispersion relation

$$\omega_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}} \doteq \frac{\omega_{*\boldsymbol{k}}}{1 + k_\perp^2\rho_{\mathrm{s}}^2}. \tag{3.7}$$

The dispersive term in the denominator arises from ion polarization; it persists in the limit of cold ions, so it is not fundamentally due to a finite (nonzero) Larmor radius. The nonlinear description of this wave will be discussed in Sec. 5.2.

More refined treatments predict the existence of various drift instabilities driven by gradients in the background density or temperature profiles. Thus fluctuations will grow and ultimately saturate due to nonlinear effects; in general, steady-state turbulence results. That turbulence can respond self-consistently to perturbations, so it behaves like a polarizable medium and possess a *nonlinear* dielectric response function $\mathcal{D}_{\mathrm{G}}$ that generalizes $\mathcal{D}_{\mathrm{G}}^{(0)}$ to the nonlinear regime. The spectral balance describes the fluctuations that arise from a renormalized gyrocentre that moves in the turbulent background and is shielded by nonlinear dielectric response. $\mathcal{D}_{\mathrm{G}}$ is the permittivity of the turbulent background state in the same sense that $\mathcal{D}_\perp$ is the permittivity of the gyrokinetic vacuum. As a generalization of the previous algorithm for the calculation of $\mathcal{D}_{\mathrm{G}}^{(0)}$, $\mathcal{D}_{\mathrm{G}}$ will be calculated from the relation between total *averaged* potential and external potential. Indeed, the procedure is not restricted to gyrokinetics. We shall see that turbulent systems quite generally possess a dielectric function $\mathcal{D}$ that describes *turbulent*

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[54]If one calculates the vorticity of $\boldsymbol{E} \times \boldsymbol{B}$ motion in a constant magnetic field $\boldsymbol{B} = B\widehat{\boldsymbol{b}}$, one finds $\boldsymbol{\nabla} \times (c\boldsymbol{E}/B) = \omega_{\mathrm{ci}}(\rho_{\mathrm{s}}^2\nabla_\perp^2\varphi)\widehat{\boldsymbol{b}}$. Thus the ion polarization density and the plasma vorticity are essentially the same thing, which explains why a vorticity equation is a central ingredient of virtually all drift-wave theories.

polarization effects analogous to the particle polarization discussed above. This is true even for nonelectromagnetic systems such as the Navier–Stokes fluid.

3.2. *Self-consistency and the dielectric function of a turbulent plasma*

I shall show in Sec. 4.2 how dielectric response can be defined for an arbitrary quadratically nonlinear PDE Eq. (2.5). However, for initial discussion it is best to be more concrete, so I shall first discuss the Klimontovich equation (2.1) as a particular, very important example. Here $\widetilde{E}$ is linearly related to $\widetilde{N}$ through Poisson's equation, so one has $\widetilde{E} = \mathbb{E}\widetilde{N}$ for a particular linear (vector) operator $\mathbb{E}$. In general, it is necessary to introduce a dielectric tensor that describes the response to external currents. However, it is simplest for pedagogical purposes to consider the electrostatic limit $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$, in which case one can work with a dielectric scalar that describes the response to external charge or electrostatic potential. In electrostatics, Green's-function solution of Poisson's equation leads to[55]

$$\mathbb{E}(1,\overline{1}) \doteq -\boldsymbol{\nabla}_1 |\boldsymbol{x}_1 - \boldsymbol{x}_{\overline{1}}|^{-1} 4\pi(\overline{n}q)_{s_{\overline{1}}} \delta(t_1 - t_{\overline{1}}). \tag{3.8}$$

Thus the Klimontovich equation can be written as

$$\partial_t \widetilde{N}_s(\boldsymbol{x},\boldsymbol{v},t) + \boldsymbol{v}\cdot\boldsymbol{\nabla}\widetilde{N} + (\mathbb{E}\widetilde{N})\cdot\boldsymbol{\partial}\widetilde{N} = 0. \tag{3.9}$$

This is a self-consistent ($\widetilde{E}$ is determined by $\widetilde{N}$, not specified externally), multiplicatively stochastic PDE. It has the form of Eq. (2.5) with coupling coefficients

$$U_2(1,2) \doteq -\boldsymbol{v}_1\cdot\boldsymbol{\nabla}_1\delta(1-2), \tag{3.10a}$$

$$U_3(1,2,3) \doteq -\mathbb{E}(1,2)\cdot\boldsymbol{\partial}_1\delta(1-3) + (2\leftrightarrow 3). \tag{3.10b}$$

[The symmetrization in Eq. (3.10b) is a manifestation of the self-consistency.] To inquire about linear response to an external or free charge, introduce an external (statistically sharp) electric field $\Delta\boldsymbol{E}^{\text{ext}}$ ($\widetilde{E} \to \mathbb{E}\widetilde{N} + \Delta\boldsymbol{E}^{\text{ext}}$) and consider the equation for first-order response $\Delta\widetilde{N}$:

$$\partial_t\Delta\widetilde{N} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\Delta\widetilde{N} + \widetilde{E}\cdot\boldsymbol{\partial}\Delta\widetilde{N} + \underline{(\mathbb{E}\Delta\widetilde{N})}\cdot\boldsymbol{\partial}\widetilde{N} = -\boldsymbol{\partial}\widetilde{N}\cdot\Delta\boldsymbol{E}^{\text{ext}}. \tag{3.11}$$

The underlining here and subsequently identifies the self-consistent response term that would be absent for passive advection. Note that although $\Delta\boldsymbol{E}^{\text{ext}}$ is not random, $\Delta\widetilde{N}$ is random because it evolves in the random background of $\widetilde{N}$. That is, there are random coefficients $\widetilde{E}$ and $\widetilde{N}$ on the left-hand side of Eq. (3.11), and $\Delta\boldsymbol{E}^{\text{ext}}$ couples to the random $\widetilde{N}$ on the right-hand side. Ultimately we must calculate the mean response $\langle\Delta\widetilde{N}\rangle$, which will involve us with a closure problem.

The left-hand side of Eq. (3.11) is just the linearization of the Klimontovich equation (2.1). It can be solved by introducing a *random infinitesimal response function* $\widetilde{R}$, which is just Green's function[56] for the left-hand side of Eq. (3.11). It obeys

$$\left(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \widetilde{E}\cdot\boldsymbol{\partial} + \boldsymbol{\partial}\widetilde{N}\cdot\underline{\mathbb{E}}\right)\widetilde{R}_{ss'}(\boldsymbol{x},\boldsymbol{v},t;\boldsymbol{x}',\boldsymbol{v}',t') = \delta_{ss'}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\boldsymbol{v}-\boldsymbol{v}')\delta(t-t').$$
$$\tag{3.12}$$

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[55]The Fourier transform of this function is perhaps more familiar: $\mathbb{E}_{s\overline{s}}(\boldsymbol{k},\omega) = \boldsymbol{\epsilon}_{\boldsymbol{k}}(\overline{n}q)_{\overline{s}}$, where $\boldsymbol{\epsilon}_{\boldsymbol{k}} \doteq -4\pi\mathrm{i}\boldsymbol{k}/k^2$ describes the electric field due to a unit point charge.

[56]$\widetilde{R}$ describes the response to an infinitesimal additive source on the right-hand side of the dynamical equation. It is thus a different kind of object than the ones studied in linear response theory (Martin 1968), which describe the response to a multiplicative perturbation (an additive term in the Hamiltonian, if such exists).

Thus, upon omitting all variables except the time for brevity, one finds

$$\Delta \widetilde{N}(t) = - \int_{-\infty}^{t} d\overline{t}\, \widetilde{R}(t;\overline{t}) \boldsymbol{\partial} \widetilde{N}(\overline{t}) \cdot \Delta \boldsymbol{E}^{\text{ext}}(\overline{t}). \tag{3.13}$$

This result is analogous to Eq. (2.28); the right-hand side of Eq. (3.11) can be recognized as a particular kind of forcing. However, there are two crucial differences. First, the forcing is not purely external because it involves the background quantity $\widetilde{N}$; consequently, it is correlated to $\widetilde{R}$ and the neat factorizations involved in Eqs. (2.29) and (2.35a) no longer hold. Second, $\widetilde{R}$ is distinct from $\widetilde{g}$ because of the presence of the $\mathbb{E}$ term on the left-hand side of Eq. (3.12). That describes the self-consistent response to a small perturbation; it is absent in a passive problem.

Given $\widetilde{R}$, one can calculate the mean internally produced field by averaging Eq. (3.13) and evaluating $\mathbb{E}\langle \Delta \widetilde{N}\rangle$. One can then add $\Delta \boldsymbol{E}^{\text{ext}}$ to obtain the total mean field in the medium. Finally, $\mathcal{D}$ is defined from the relation

$$\Delta \boldsymbol{E}^{\text{tot}} = \mathcal{D}^{-1} \Delta \boldsymbol{E}^{\text{ext}}; \tag{3.14}$$

one finds[57]

$$\mathcal{D}^{-1} = 1 - \mathbb{E} \cdot \langle \widetilde{R}\, \boldsymbol{\partial} \widetilde{N}\rangle. \tag{3.15}$$

(This symbolic notation is a bit tricky because $\mathcal{D}$ is a function of only spatial coordinates whereas $\mathbb{E}$ and $\widetilde{R}$ operate in phase space.[58]) This is as far as one can go without facing up to statistical closure, i.e., an approximate calculation of $\langle \widetilde{R}\,\widetilde{N}\rangle$.

One way of proceeding, ultimately not fruitful, is to first express $\widetilde{R}$ in terms of $\widetilde{g}$, then attempt to average. The Klimontovich $\widetilde{g}$ obeys Eq. (3.12) with the last term on the left-hand side omitted. Thus Eq. (3.12) can be written schematically as

$$\widetilde{g}^{-1}\widetilde{R} + \boldsymbol{\partial} \widetilde{N} \cdot \underline{\mathbb{E}\widetilde{R}} = 1. \tag{3.16}$$

Given $\mathbb{E}\widetilde{R}$, the formal solution of Eq. (3.16) is

$$\widetilde{R} = \widetilde{g} - \widetilde{g}\, \boldsymbol{\partial} \widetilde{N} \cdot \underline{\mathbb{E}\widetilde{R}}, \tag{3.17}$$

An equation for $\mathbb{E}\widetilde{R}$ follows by applying $\mathbb{E}$ to both sides of Eq. (3.17), then bringing the $\mathbb{E}\widetilde{R}$ term on the right-hand side to the left:

$$(\boldsymbol{I} + \mathbb{E}\widetilde{g}\boldsymbol{\partial}\widetilde{N}) \cdot \mathbb{E}\widetilde{R} = \mathbb{E}\widetilde{g}. \tag{3.18}$$

In the electrostatic approximation, this can be solved to find

$$\mathbb{E}\widetilde{R} = \widetilde{\mathcal{D}}^{-1}\mathbb{E}\widetilde{g}, \tag{3.19}$$

where

$$\widetilde{\mathcal{D}} \doteq 1 + \underline{\mathbb{E} \cdot \widetilde{g}\, \boldsymbol{\partial}\widetilde{N}}. \tag{3.20}$$

Thus

$$\widetilde{R} = \widetilde{g} - \widetilde{g}\, \boldsymbol{\partial}\widetilde{N} \cdot \underline{\widetilde{\mathcal{D}}^{-1}\mathbb{E}\widetilde{g}}. \tag{3.21}$$

The quantity $\widetilde{\mathcal{D}}$ is a kind of "random dielectric function" (although that concept is not defined), and the form (3.21) emphasizes, through its second term, that self-consistency

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[57]The scalar product between $\mathbb{E}$ and $\boldsymbol{\partial}$ is a consequence of the electrostatic approximation.

[58]In detail, the operator notation means $\mathcal{D}^{-1}(\underline{1}, \underline{1}') = \delta(\underline{1} - \underline{1}') - \sum_{s'} \int d\boldsymbol{v}'\, \mathbb{E}(\underline{1}, \overline{1}) \cdot \langle \widetilde{R}(\overline{1}; 1') \times \boldsymbol{\partial}_{1'}\widetilde{N}(1')\rangle$, where (in this footnote only) the underlining denotes the set of just spatial and temporal variables, excluding velocity and species.

is crucial. However, how to calculate the correlation between $\widetilde{R}$ and $\widetilde{N}$ that is required for formula (3.15) is not apparent because of the wildly nonlinear way in which random variables are mixed throughout the last term of Eq. (3.21). Thus a different approach must be taken, as I now describe.

Instead of following the route of first solving for $\Delta\widetilde{N}$, then averaging, in traditional closures one averages Eq. (3.11) directly. Upon recalling that $f = \langle\widetilde{N}\rangle$, one finds

$$\partial_t \Delta f + \boldsymbol{v} \cdot \boldsymbol{\nabla} \Delta f + \boldsymbol{E} \cdot \boldsymbol{\partial} \Delta f + \underline{\underline{\Delta\langle\delta\boldsymbol{E}\,\delta N\rangle}} = -\boldsymbol{\partial} f \cdot \Delta\boldsymbol{E}^{\text{ext}}. \qquad (3.22)$$

The last term on the left-hand side must be evaluated by statistical closure. (A doubly underlined term indicates that at least part of the term contains self-consistent response that would be absent for passive advection.) The problem is essentially the same one that arises in closing the equation for R itself. The most general way of proceeding is to use the *method of sources* introduced in the classical context by Martin *et al.* (1973, henceforth MSR), motivated by prior developments in quantum field theory. I shall not give a full description of the MSR formalism here; see Krommes (2002) for extensive discussion. Briefly, the procedure is to inquire about the response to an external, nonrandom source. By looking at infinitesimal variations of that source, one can obtain functional relations between the various statistical observables that can be usefully closed. For example, add an external source[59] $\widehat{\eta}$ to the right-hand side of the Klimontovich equation (3.9). The random response function $\widetilde{R}$ is defined by the functional derivative (Beran 1968)

$$\widetilde{R}(t;t') = \left.\frac{\delta\widetilde{N}(t)}{\delta\widehat{\eta}(t')}\right|_{\widehat{\eta}=0}, \qquad (3.23)$$

and the mean response is

$$R(t;t') = \langle\widetilde{R}(t;t')\rangle = \left.\frac{\delta f(t)}{\delta\widehat{\eta}(t')}\right|_{\widehat{\eta}=0}. \qquad (3.24)$$

Thus one can obtain an equation for R by functionally differentiating the mean equation,

$$\partial_t f + \boldsymbol{v}\cdot\boldsymbol{\nabla}f + \boldsymbol{E}\cdot\boldsymbol{\partial}f + \boldsymbol{\partial}\cdot\langle\delta\boldsymbol{E}\,\delta N\rangle = \widehat{\eta}(t), \qquad (3.25)$$

to find

$$\partial_t R(t;t') + \boldsymbol{v}\cdot\boldsymbol{\nabla}R + \boldsymbol{E}\cdot\boldsymbol{\partial}R + \boldsymbol{\partial}f\cdot\underline{\underline{\boldsymbol{E}R}} + \underline{\underline{\left.\frac{\delta[\boldsymbol{\partial}\cdot\langle\delta\boldsymbol{E}\,\delta N\rangle(t)]}{\delta\widehat{\eta}(t')}\right|_{\widehat{\eta}=0}}} = \delta(t-t'). \qquad (3.26)$$

The key to usefully reducing the last term on the left-hand side is to recognize that as $\widehat{\eta}$ changes $f[\widehat{\eta}]$ changes in concert. One can therefore use a functional chain rule:

$$\frac{\delta}{\delta\widehat{\eta}(t')} = \int d\overline{t}\,\frac{\delta f(\overline{t})}{\delta\widehat{\eta}(t')}\frac{\delta}{\delta f(\overline{t})} = \int d\overline{t}\,R(\overline{t};t')\frac{\delta}{\delta f(\overline{t})}, \qquad (3.27)$$

where Eq. (3.24) was used in the last step. This formally closes the equation for R in the form of a Dyson equation (Dyson 1949; Krommes 2002, 2009):

$$(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{E}\cdot\boldsymbol{\partial}R + \boldsymbol{\partial}f\cdot\underline{\underline{\boldsymbol{E}R}} + \underline{\underline{\Sigma}})R(t;t') = \delta(t-t'), \qquad (3.28)$$

where[60]

$$\Sigma(t;\overline{t}) \doteq \frac{\delta[\boldsymbol{\partial}\cdot\langle\delta\boldsymbol{E}\,\delta N\rangle(t)]}{\delta f(\overline{t})} \qquad (3.29)$$

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[59] Here the hat does not denote a Fourier transform; instead, it denotes a particular kind of source. The nomenclature is that used by Martin *et al.* (1973) and Krommes (2002).

[60] Multiplication of two-time functions implies convolution; i.e., $(\Sigma R)(t;t') \equiv \int d\overline{t}\,\Sigma(t;\overline{t})R(\overline{t};t')$.

is still unknown at this point. It can be evaluated in various statistical closures such as the DIA; see Sec. 4.

In a seminal paper, DuBois & Espedal (1978) showed that one further manipulation puts the R equation into a very suggestive form that is an obvious generalization of linear theory. Note that Σ, whatever it is, will contain two kinds of terms: ones that operate directly on R, such as $\ldots \boldsymbol{\partial} R$; and ones that apply the electric-field operator to R ($\ldots \mathbb{E} R$). Call those terms $\Sigma^g R$ and $\Sigma^f R$, respectively, and group the Σ^f terms with the explicit $\boldsymbol{\partial} f \cdot \mathbb{E} R$ term in Eq. (3.28), thereby changing $\boldsymbol{\partial} f$ to a fluctuation-dependent quantity $\boldsymbol{\partial} \overline{f} \doteq \boldsymbol{\partial} f + \boldsymbol{\partial} \delta \overline{f}$ with $\boldsymbol{\partial} \delta \overline{f} \cdot \mathbb{E} \equiv \Sigma^f$. Equation (3.28) then becomes

$$g^{-1} R + \underline{\boldsymbol{\partial} \overline{f} \cdot \mathbb{E}} R = 1, \tag{3.30}$$

where the *renormalized particle propagator* g obeys

$$(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\nabla} + \boldsymbol{E} \cdot \boldsymbol{\partial} + \Sigma^g) g = 1. \tag{3.31}$$

Equation (3.30) now has the same form as Eq. (3.16), but importantly it contains only statistically averaged quantities. The method of solution is the same, however, so one obtains [cf. Eqs. (3.20) and (3.21)]

$$R = g - g \, \underline{\boldsymbol{\partial} \overline{f} \cdot \mathcal{D}^{-1} \mathbb{E}} g, \tag{3.32a}$$

$$\mathcal{D} \doteq 1 + \underline{\mathbb{E} \cdot g \, \boldsymbol{\partial} \overline{f}}, \tag{3.32b}$$

where $\mathcal{D}$ is the required dielectric function. The first term of Eq. (3.32a) describes the propagation of a renormalized *test particle*; the second one represents the shielding cloud that arises as the test particle polarizes the medium. The formula (3.32b) has the same form as the linearized Vlasov dielectric, but it contains a renormalized particle propagator g and a fluctuation-dependent function $\boldsymbol{\partial} \overline{f}$ that formally replaces the derivative of the background distribution f. In plasmas, where there is inevitably a nonvanishing mean field (namely f), formula (3.32b) has a nontrivial limit in which all nonlinear terms are neglected, i.e., the linearized Vlasov $\mathcal{D}^{(0)}$ discussed in the textbooks and illustrated by formula (3.5). In other situations with no mean field, $\mathcal{D}$ is still nontrivial, but the corrections to the vacuum limit $\mathcal{D} = 1$ are entirely nonlinear, being proportional to $\delta \overline{f}$.[61]

The somewhat mysterious correction $\delta \overline{f}$ is necessary for one to recover the proper form of the matrix elements in a reduction to weak-turbulence theory (Krommes 2002). It is also required for the satisfaction of constraints such as energy conservation. A special case was considered by Dupree & Tetreault (1978) and also discussed by Diamond *et al.* (2010, Sec. 4.4.4).

It follows from Eq. (3.32a) [and was an intermediate step of the solution methodology; cf. Eq. (3.19)] that

$$\mathbb{E} R = \mathcal{D}^{-1} \mathbb{E} g. \tag{3.33}$$

This crucial result is a statement of dielectric shielding. Its interpretation is that the field due to a particle streaming through the random medium (that streaming being appropriately renormalized by the effects contained in Σ^g, such as turbulent diffusion) is shielded by the self-consistent response of that medium, which behaves on the average as a macroscopic dielectric.

One can generalize these results to define the dielectric function for an arbitrary turbulent system. I shall show how to do this in Sec. 4.2 after I have given further background on closure by deriving the DIA in Sec. 4.1.

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[61] A good illustration of this case is the guiding-centre plasma model (Taylor 1974; Krommes & Similon 1980); see further discussion after Eq. (4.41).

3.3. *Spectral balance and dielectric response*

In Sec. 2.3 I showed that for a particular problem of passive advection the two-time correlation function $C(\tau)$ or the fluctuation spectrum $\widehat{C}(\omega)$ can be written schematically as [cf. Eq. (2.43)]

$$C = gFg^{\mathrm{T}}, \tag{3.34}$$

where g is the mean Green's function, T denotes transpose $[g^{\mathrm{T}}(1,2) \doteq g(2,1)]$, and F is the sum of the internal and external covariances of the incoherent noise. Although I shall not prove it here, it is plausible that a similar formula holds for self-consistent systems:

$$C(1,1') = R(1;\overline{1})F(\overline{1},\overline{1}')R(1';\overline{1}') \tag{3.35}$$

for some noise covariance $F = F^{\mathrm{int}} + F^{\mathrm{ext}}$, where F^{int} is determined by closure. [It is intimately related to Σ by a generalization of Eq. (2.48).] The form $R^{-1}C = FR^{\mathrm{T}}$ can also be viewed as a Dyson equation.[62] This equation follows in general from the MSR formalism.[63] Closures such as the DIA discussed in Sec. 4 provide specific formulas for Σ and F^{int}. Note that each of the functions in Eq. (3.35) depends on all of the independent variables of the problem. For the immediately following discussion, it is useful to imagine that one is dealing with a kinetic plasma description, for which the independent variables are space, velocity, species, and time. Electromagnetic fields are calculated by performing weighted integrations over velocity and summations over species. By doing so, we shall see how dielectric response is contained in the balance equation (3.35).

Because R comprises two terms according to Eq. (3.32a), there are various pieces to the formula (3.35), but those collapse when one calculates the electric-field fluctuation spectrum by applying $\mathbb{E}$ to each of the arguments 1 and 1' of Eq. (3.35) and using Eq. (3.33):

$$\langle \delta\boldsymbol{E}\,\delta\boldsymbol{E}\rangle(\boldsymbol{k},\omega) \doteq \frac{[\mathbb{E}(gFg^{\mathrm{T}})\mathbb{E}^{\mathrm{T}}](\boldsymbol{k},\omega)}{|\mathcal{D}(\boldsymbol{k},\omega)|^2}. \tag{3.36}$$

Thus the appropriately propagated incoherent noise is shielded by the coherent dielectric polarization. The beauty and elegance of this representation, which is one form of a *spectral balance equation*, are hopefully apparent. This balance was already known to Kadomtsev & Petviashvili (1963) in the context of weak turbulence theory. Some physical interpretations of the balance equation for steady-state, isotropic, neutral-fluid turbulence were given by Kraichnan (1964a), although he did not explicitly introduce the concept of a dielectric function.

Equations (3.35) and (3.36) are generalized fluctuation–dissipation relations. They determine the fluctuation spectrum implicitly since it occurs in all of g, $\mathcal{D}$, and F (or in R and F).

The fact that the fluctuation balance can be represented as $C = RFR^{\mathrm{T}}$ [Eq. (3.35)] implies that it is unnecessary to explicitly introduce either the renormalized propagator g or the dielectric function $\mathcal{D}$; one only requires approximate formulas for Σ and F^{int}. Indeed, initial-value solutions of two-time statistical closures such as the DIA merely advance $R(t;t')$ and $C(t,t')$. The purposes of stressing the decomposition (3.32a) for R and the formula (3.32b) for $\mathcal{D}$ are to provide insight into the physical content of R and

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[62]It is actually one component of a 2×2 matrix Dyson equation discussed by Martin *et al.* (1973).

[63]All such statements about the MSR formalism are subject to the caveat that the external forcing and initial conditions (which can be treated as a special case of external forcing) must be Gaussian. Non-Gaussian external statistics lead to significant additional complications (Rose 1979).

		Eq.
The Dyson equations:	$(g_0^{-1} + \boldsymbol{\partial} f \cdot \mathbb{E} + \Sigma)R = 1,$	(3.28)
	$C = RFR^{\mathrm{T}}$	(3.35)
Particle propagation:	$(g_0^{-1} + \Sigma^g)g = 1$	(3.31)
Self-consistent response:	$R = g - g\,\overline{\partial f} \cdot \mathcal{D}^{-1}\mathbb{E}g$	(3.32a)
Dielectric function:	$\mathcal{D} = 1 + \mathbb{E} \cdot g\,\boldsymbol{\partial}\overline{f}$	(3.32b)
Shielding identity:	$\mathbb{E}R = \mathcal{D}^{-1}\mathbb{E}g$	(3.33)
Spectral balance:	$\langle \delta\boldsymbol{E}\,\delta\boldsymbol{E} \rangle = \dfrac{\mathbb{E}(gFg^{\mathrm{T}})\mathbb{E}^{\mathrm{T}}}{|\mathcal{D}|^2}$	(3.36)

TABLE 1. Key formulas of statistical turbulence theory. For a turbulent plasma, $g_0^{-1} \doteq \partial_t + \boldsymbol{v} \cdot \boldsymbol{\nabla} + \boldsymbol{E} \cdot \boldsymbol{\partial}$. The effective distribution function is $\overline{f} \doteq f + \delta f$. The quantities Σ (or equivalently Σ^g and $\boldsymbol{\partial}\overline{\delta f}$) and F^{int} must be approximated by a statistical closure. The interpretation of the spectral balance [Eq. (3.36)] is that incoherent electric-field fluctuations (with covariance F) behave like moving test particles (g) that are shielded by the dielectric or polarization properties of the turbulent medium ($\mathcal{D}$).

to show that the effects of turbulence can be incorporated as a natural generalization of the familiar linear theory. Awareness of the relationships between R, g, and $\mathcal{D}$ is useful in understanding the content and limitations of various approximations, beginning with the approach taken in Dupree's early work, discussed in the next section.

The principal results of this section are summarized in Table 1.

3.4. *The $D_\perp = \gamma/k_\perp^2$ formula*

In the last section I argued that one representation of the steady-state fluctuation spectrum, $\mathcal{E}(\boldsymbol{k}, \omega) \doteq \langle \delta\boldsymbol{E} \cdot \delta\boldsymbol{E} \rangle(\boldsymbol{k}, \omega)$, is given by

$$\mathcal{E}(\boldsymbol{k}, \omega) = \frac{\mathcal{N}(\boldsymbol{k}, \omega)}{|\mathcal{D}(\boldsymbol{k}, \omega)|^2}, \qquad (3.37)$$

where $\mathcal{N}$ is the trace of the numerator of Eq. (3.36). In general, explicitly solving this equation for $\mathcal{E}$ is quite complicated because $\mathcal{E}$ is buried in both $\mathcal{N}$ and $\mathcal{D}$. A straightforward though nontrivial method of attack would be to solve the kinetic DIA directly as an initial-value problem; if that were done, spectral functions would be calculated as subsidiary quantities. However, one can use Eq. (3.37) directly to interpret the approach taken by Dupree, which was very popular for a time and is still instructive. Let us inquire about the wave-number spectrum

$$\mathcal{E}(\boldsymbol{k}) \doteq \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\mathcal{E}(\boldsymbol{k}, \omega) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\frac{\mathcal{N}(\boldsymbol{k}, \omega)}{|\mathcal{D}(\boldsymbol{k}, \omega)|^2}. \qquad (3.38)$$

The ω dependence of both $\mathcal{N}$ and $\mathcal{D}$ contribute to this integral, so it cannot be done in general.[64] But assume that the system supports a weakly damped normal mode; i.e., $\mathcal{D}(\boldsymbol{k}, \omega_{\boldsymbol{k}}) = 0$ for complex frequency $\omega_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}} + \mathrm{i}\gamma_{\boldsymbol{k}}$ with $|\gamma_{\boldsymbol{k}}/\Omega_{\boldsymbol{k}}| \ll 1$. Here $\gamma_{\boldsymbol{k}}$ is the total growth rate, including all nonlinear corrections; crucially, as I remarked previously, it must be negative in order that a steady state can be achieved. (Therefore nonlinear contributions to $\gamma_{\boldsymbol{k}}$ are essential in order to overcompensate a positive $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$ and balance

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[64] A well-known special case is the thermal equilibrium of the discrete many-body system, for which the fluctuation–dissipation theorem relates $\mathcal{N}$ to $\mathcal{D}$ in a simple way; then the frequency integral can be done with the aid of the Kramers–Kronig relations (Ichimaru 1973).

the positive-definite nonlinear noise.) To inquire about the contribution of a given normal mode to the spectrum, one can expand

$$\mathcal{D}(\omega) \approx \mathcal{D}(\omega_{\boldsymbol{k}}) + (\omega - \omega_{\boldsymbol{k}}) \left.\frac{\partial \mathcal{D}}{\partial \omega}\right|_{\omega_{\boldsymbol{k}}} \tag{3.39a}$$

$$\equiv (\omega - \Omega_{\boldsymbol{k}} - \mathrm{i}\gamma_{\boldsymbol{k}})\mathcal{D}'_{\boldsymbol{k}}. \tag{3.39b}$$

The right-hand side of Eq. (3.38) then involves

$$\frac{1}{(\omega - \Omega_{\boldsymbol{k}})^2 + \gamma_{\boldsymbol{k}}^2} \approx \frac{1}{|\gamma_{\boldsymbol{k}}|}\pi\delta(\omega - \Omega_{\boldsymbol{k}}). \tag{3.40}$$

Let us assume that only this mode contributes to the fluctuation level. (This cannot really be correct since the modes are coupled nonlinearly.) Then the ω integration can be performed:

$$-2\gamma_{\boldsymbol{k}}\mathcal{E}(\boldsymbol{k}) = \mathcal{N}(\boldsymbol{k}, \Omega_{\boldsymbol{k}})/|\mathcal{D}'_{\boldsymbol{k}}|^2, \tag{3.41}$$

where I used the fact that $\gamma_{\boldsymbol{k}} < 0$. This is still an implicit equation for $\mathcal{E}(\boldsymbol{k})$, which enters in both $\mathcal{N}$ and $\mathcal{D}'$. However, if the nonlinear noise is neglected altogether (see further discussion at the end of this section), one is led to the saturation condition

$$\gamma_{\boldsymbol{k}} = 0. \tag{3.42}$$

If one further writes

$$\gamma_{\boldsymbol{k}} = \gamma_{\boldsymbol{k}}^{\mathrm{lin}} - \eta_{\boldsymbol{k}}, \tag{3.43}$$

where $\eta_{\boldsymbol{k}}$ describes[65] the nonlinear effects, then the saturation condition becomes

$$\eta_{\boldsymbol{k}} = \gamma_{\boldsymbol{k}}^{\mathrm{lin}}, \tag{3.44}$$

describing a balance between the nonlinear coherent damping and the linear drive. For the case of magnetized plasma, for which fluid elements move cross field with $\boldsymbol{E} \times \boldsymbol{B}$ drifts, the approximations of Dupree (1967) and Weinstock (1968, 1969, 1970) for $\eta_{\boldsymbol{k}}$ amounted to saying

$$\eta_{\boldsymbol{k}} \approx k_\perp^2 D_\perp \tag{3.45}$$

for some $D_\perp$ that is a measure of cross-field transport. Not accidentally, this is the same formula for $\eta_{\boldsymbol{k}}$ that we found in the small-τ_{ac} passive oscillator discussed in Sec. 2.3.

The virtue of Eq. (3.45) is that in conjunction with Eq. (3.44) it seems to directly determine the diffusion coefficient to be $D_\perp = \gamma_{\boldsymbol{k}}^{\mathrm{lin}}/k_\perp^2$. An immediate objection is that this cannot literally be so because $D_\perp$ was assumed to be $\boldsymbol{k}$ independent but $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$ is not proportional to $k_\perp^2$ in general. (Furthermore, not all $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$'s need be positive.) The more general Eq. (3.44) does not necessarily have this problem because $\eta_{\boldsymbol{k}}$ is a functional of the wave-number spectrum, which could in principle adjust so that Eq. (3.44) holds for all $\boldsymbol{k}$. However, the diffusive approximation is valid only at small $\boldsymbol{k}$,[66] so it is pointless to worry about detailed $\boldsymbol{k}$ dependence. The saturation criterion is best evaluated at some

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[65]I have reverted here to a Markovian description. Markovian approximations are further discussed in Sec. 5.2.3.

[66]Classical diffusion of long-wavelength disturbances (small k) arises from random kicks due to short-scale fluctuations (large k). In turbulence, there is no clean scale separation, and the nonlinear interactions of fluctuations of comparable scale cannot be argued to be diffusive. Worse, in self-consistent problems $\eta_{\boldsymbol{k}}$ need not even be positive; witness the inverse energy cascade in 2D turbulence. In such situations the nonlinear noise cannot be neglected. Some further discussion can be found in Krommes (2002, Sec. 4.3.5).

typical[67] wave vector $\overline{\boldsymbol{k}}$ (one must have $\gamma_{\overline{\boldsymbol{k}}}^{\mathrm{lin}} > 0$); then a gross estimate is

$$D_\perp = \gamma_{\overline{\boldsymbol{k}}}^{\mathrm{lin}}/\overline{k}_\perp^2. \tag{3.46}$$

This formula, essentially also given by Kadomtsev (p. 107),[68] is crude but effective. It states that the fluctuation and transport levels scale with a typical linear growth rate, which can be tested. It is often reasonable.

This is not the place for a thorough review of the subsequent developments of resonance-broadening theory, which addressed various refinements and applications of the basic theme. Regarding its validity, there are really two issues: how does one adequately approximate the coherent damping $\eta_{\boldsymbol{k}}$, and how important is the nonlinear noise $\mathcal{N}_{\boldsymbol{k}}$? Note that there are important situations in which $\mathcal{N}$ can be neglected. In weak-turbulence theory, where the spectrum comprises weakly coupled waves, the lowest-order approximation to $\mathcal{N}$ vanishes unless the resonance condition $\Omega_{\boldsymbol{k}} + \Omega_{\boldsymbol{p}} + \Omega_{\boldsymbol{q}} = 0$ is satisfied for $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{0}$. Linear dispersion relations of "nondecay" type (Kadomtsev 1965; Sagdeev & Galeev 1969) do not (by definition) obey that condition. In that case, nontrivial steady states require that Eq. (3.42) be satisfied (unless higher-order, e.g., four-wave interactions are considered). The nonlinear corrections to $\mathcal{D}$ in this case describe wave–wave–particle (and higher-order) interactions and can be systematically calculated by perturbation theory.[69] Kadomtsev and later Sagdeev & Galeev (1969) and Sagdeev (1979) discussed some instances of this situation; an example of a greatly refined analysis of such effects is the work of Horton & Choi (1979).

In later research, Dupree attempted to take the nonlinear noise into consideration. A difficulty is that at the kinetic level the incoherent noise is associated with several distinct kinds of physical processes: renormalized free-streaming particles, which enter into discussions by Dupree (1972) and later authors (DuBois & Espedal 1978; Boutros-Ghali & Dupree 1981) of *phase-space granulations*[70] in plasmas; and fluid mode-coupling effects that are not sensitive to the details of the velocity space. Those effects must not be neglected; failure to recognize this can easily lead to paradoxes or misconceptions (Krommes & Kim 1988). The mode coupling is exposed more transparently in statistical closures such as the DIA applied directly to fluid models (Weinstock & Williams 1971), for which there has been much work (some of which is described below).

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[67]What does "typical" mean in this context? Elementary discussions often refer to the wave vector of the fastest growing linear mode. However, that is usually not the mode in which the fluctuation energy concentrates in the nonlinearly saturated steady state. At the level of the present argument, which is essentially dimensional, one merely needs to postulate the existence of a continuous band of nonlinearly coupled, physically similar, excited wave numbers. The estimate (3.45) captures the fact that the turbulence is driven by some source of free energy (like a profile gradient), typified by a linear instability. Then, by a mean-value theorem, Eq. (3.45) can be made exact by an appropriate choice of $\overline{\boldsymbol{k}}$. This argument needs to be refined for cases of subcritical turbulence, which by definition exists even when all $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$'s are negative; however, that is beyond the scope of the present discussion.

[68]Kadomtsev actually wrote $\langle \gamma/k_\perp^2 \rangle$, where the averaging operation was not defined. Presumably an average over the steady-state wave-number spectrum was intended; if one models that spectrum as being concentrated at a typical wave number $\overline{k}$, then one recovers Eq. (3.46).

[69]When one does this, one finds that Dupree's diffusive resonance-broadening approximation is an asystematic approximation to *part* of the wave–wave–particle effects (Krommes 2002, Secs. 4.3.5 and 6.5.4).

[70]As discussed by Dupree (1972) and later authors, one of the roles of the incoherent noise term is to ensure that the relative motion of closely separated trajectories is represented correctly.

4. The direct-interaction approximation

The general equations (3.32*b*) for $\mathcal{D}$, (3.28) for R, and (3.35) for C are not useful until the various terms Σ^g, $\delta\overline{f}$ ($\Sigma = \Sigma^g + \boldsymbol{\partial}\delta\overline{f} \cdot \mathbf{E}$), and F^{int} are calculated from statistical closure. That is a vast subject that cannot be fully treated here. The most famous closure is the direct-interaction approximation of Kraichnan (1959), and the literature is full of many discussions, developments, and offshoots of that theory. [Some perspectives on Kraichnan's seminal contributions are given by Eyink & Frisch (2011).] However, a tutorial article of this nature would not be complete without some words about the DIA. Its derivation has been variously discussed in terms of the perturbative effect of one Fourier amplitude out of many (Kraichnan 1959), random-coupling models (Kraichnan 1961; Orszag & Kraichnan 1967), renormalized perturbation theory (Kraichnan 1961), Langevin representations (Kraichnan 1970; Leith 1971), systems with infinite dimensionality (Mou & Weichman 1993), and more; discussion and many references can be found in Krommes (2002). The book of Leslie (1973) explains in great detail Kraichnan's work up to that time; McComb (1990) also treats the DIA extensively.

Sections 4.1 and 4.2 contain more technical details than does the rest of the article. They can be skipped on a first reading—but do take the time to absorb the basic message stated in the next paragraph.

4.1. *Heuristic derivation of the DIA*

In the present section, I shall present a route to the DIA that seems to be modestly new. It is designed to emphasize the principal intuitive points that all beginners must grasp, which are that the statistics of turbulence are non-Gaussian and the goal of closure is to calculate those non-Gaussian corrections self-consistently. The method essentially provides an $\boldsymbol{x}$-space version of Kraichnan's original approach (Kraichnan 1959), which was anchored in $\boldsymbol{k}$ space.

If one were dealing with Gaussian statistics, the Furutsu–Novikov theorem (Novikov's theorem for brevity) would be useful. That theorem (also known as *Gaussian integration by parts*) states that for a centred Gaussian random variable $\widetilde{\psi}'$ (with covariance C) and a functional $\mathcal{F}$ of that variable, one has

$$\langle \mathcal{F}[\widetilde{\psi}']\widetilde{\psi}'(1)\rangle = \left\langle \frac{\delta\mathcal{F}[\widetilde{\psi}']}{\delta\widetilde{\psi}'(\overline{1})}\right\rangle C(\overline{1},1), \tag{4.1}$$

where $\delta/\delta\widetilde{\psi}'$ denotes functional differentiation and I use $\widetilde{\psi}'$ in this section as an alternate notation for the fluctuation $\widetilde{\psi} - \langle\widetilde{\psi}\rangle$. See Appendix B for the proof. Cook (1978) discussed the use of Novikov's theorem to obtain closures for passive advection; the present work focuses on self-consistent problems. Thus the goal is to approximate the triplet correlation function T introduced in Sec. 2; see Eqs. (2.8) and (2.9). One can cast T into the form of the left-hand side of Novikov's theorem:

$$T(2,3,1') = \langle \mathcal{F}[\widetilde{\psi}']\widetilde{\psi}'(1')\rangle, \tag{4.2}$$

where $\mathcal{F}[\widetilde{\psi}'] = \widetilde{C}(2,3)$ with $\widetilde{C}(2,3) \doteq \widetilde{\psi}'(2)\widetilde{\psi}'(3)$. However, one cannot trivially employ Novikov's theorem to evaluate T because that theorem applies only to Gaussian statistics and, as I discussed in Sec. 2.1, $\widetilde{\psi}'$ becomes non-Gaussian already after the first time step. If one attempts to assert that $\widetilde{\psi}'$ is Gaussian, then one finds the trivial result $T = 0$ (third- and higher-order cumulants of a Gaussian vanish), consistent with the direct use

of Novikov's theorem:

$$\langle \mathcal{F}[\widetilde{\psi}']\widetilde{\psi}'(1')\rangle = \left\langle \frac{\delta[\widetilde{\psi}'(2)\widetilde{\psi}'(3)]}{\delta\widetilde{\psi}'(\overline{1})}\right\rangle C(\overline{1},1') \tag{4.3a}$$

$$= \langle \delta(2-\overline{1})\widetilde{\psi}'(3) + \widetilde{\psi}'(2)\delta(3-\overline{1})\rangle C(\overline{1},1') \tag{4.3b}$$

$$= 0 \tag{4.3c}$$

because $\langle\widetilde{\psi}'\rangle \equiv 0$.

Non-Gaussian statistics are induced by the last U_3 term in Eq. (2.7). Let us write

$$\widetilde{\psi}' = \widetilde{\psi}'_{\mathrm{G}} + \Delta\widetilde{\psi}', \tag{4.4}$$

where $\widetilde{\psi}'_{\mathrm{G}}$ is a Gaussian fluctuation with covariance $\langle\widetilde{\psi}'_{\mathrm{G}}(1)\widetilde{\psi}'_{\mathrm{G}}(1')\rangle = C(1,1')$ (this will later be determined self-consistently) and $\Delta\widetilde{\psi}'$ is a non-Gaussian correction. One therefore has

$$\begin{aligned} T(2,3,1') = &\langle\widetilde{\psi}'_{\mathrm{G}}(2)\widetilde{\psi}'_{\mathrm{G}}(3)\widetilde{\psi}'_{\mathrm{G}}(1')\rangle \\ &+ \langle[\Delta\widetilde{\psi}'(2)\widetilde{\psi}'_{\mathrm{G}}(3)]\widetilde{\psi}'_{\mathrm{G}}(1')\rangle + \langle[\Delta\widetilde{\psi}'(3)\widetilde{\psi}'_{\mathrm{G}}(1')]\widetilde{\psi}'_{\mathrm{G}}(2)\rangle + \langle[\Delta\widetilde{\psi}'(1')\widetilde{\psi}'_{\mathrm{G}}(2)]\widetilde{\psi}'_{\mathrm{G}}(3)\rangle \\ &+ O(\Delta\widetilde{\psi}'^2). \end{aligned} \tag{4.5}$$

The first term vanishes because it is the third-order cumulant of a Gaussian process. The remaining three explicit terms have been written in a form suitable for application of Novikov's theorem; the terms in square brackets define $\mathcal{F}$. [I shall actually evaluate the second explicit term as $\langle[\Delta\widetilde{\psi}'(3)\widetilde{\psi}'_{\mathrm{G}}(2)]\widetilde{\psi}'_{\mathrm{G}}(1')\rangle$.] The DIA is defined by the neglect of the terms of $O(\Delta\widetilde{\psi}'^2)$. (Unfortunately, $\Delta\widetilde{\psi}'$ is not small in general, which leads to subtle deficiencies in the DIA; see Sec. 5.1 for further discussion.) An equation for $\Delta\widetilde{\psi}'$ follows from Eq. (2.7); correct to first order in $\Delta\widetilde{\psi}'$, it is

$$\begin{aligned} \partial_t \Delta\widetilde{\psi}'(1) = &U_2(1,2)\Delta\widetilde{\psi}'(2) + U_3(1,2,3)[\langle\psi\rangle(2) + \widetilde{\psi}'_{\mathrm{G}}(2)]\Delta\widetilde{\psi}'(3) \\ &+ \frac{1}{2}U_3(1,2,3)[\widetilde{\psi}'_{\mathrm{G}}(2)\widetilde{\psi}'_{\mathrm{G}}(3) - C(2,3)]. \end{aligned} \tag{4.6}$$

This equation can be represented more compactly by introducing the random infinitesimal response function

$$\widetilde{R}(1;1') \doteq \frac{\delta\widetilde{\psi}(1)}{\delta\widetilde{f}(1')}, \tag{4.7}$$

which according to Eq. (2.5) obeys

$$\partial_{t_1}\widetilde{R}(1;1') - U_2(1,2)\widetilde{R}(2;1') - U_3(1,2,3)\widetilde{\psi}(2)\widetilde{R}(3;1') = \delta(1-1'). \tag{4.8}$$

$\widetilde{R}$ is a functional of $\widetilde{\psi}$: $\widetilde{R} = \widetilde{R}[\widetilde{\psi}]$. The quantity $\widetilde{R}[\langle\psi\rangle + \widetilde{\psi}'_{\mathrm{G}}] \equiv \widetilde{R}_{\mathrm{G}}$ is Green's function for the terms explicitly involving $\Delta\widetilde{\psi}'$ in Eq. (4.6); thus Eq. (4.6) has the formal solution

$$\Delta\widetilde{\psi}'(1) = \widetilde{R}_{\mathrm{G}}(1;\overline{4})\frac{1}{2}U_3(\overline{4},\overline{2},\overline{3})[\widetilde{\psi}'_{\mathrm{G}}(\overline{2})\widetilde{\psi}'_{\mathrm{G}}(\overline{3}) - C(\overline{2},\overline{3})]. \tag{4.9}$$

This describes how non-Gaussian statistics are generated by the nonlinear coupling of Gaussian fluctuations.

One can now work out all of the $O(\Delta\widetilde{\psi}')$ terms in Eq. (4.5) by approximating the consequences of Novikov's theorem, assuming that all terms are functionals of $\widetilde{\psi}'_{\mathrm{G}}$. One

has

$$\langle [\Delta\widetilde{\psi}'(2)\widetilde{\psi}'_{\mathrm{G}}(3)]\widetilde{\psi}'_{\mathrm{G}}(1')\rangle = \left\langle \frac{\delta[\Delta\widetilde{\psi}'(2)\widetilde{\psi}'_{\mathrm{G}}(3)]}{\delta\widetilde{\psi}'_{\mathrm{G}}(\overline{1})} \right\rangle C(\overline{1},1'). \tag{4.10}$$

From Eq. (4.9), the functional derivative is approximately

$$\frac{\delta[\Delta\widetilde{\psi}'(2)\widetilde{\psi}'_{\mathrm{G}}(3)]}{\delta\widetilde{\psi}'_{\mathrm{G}}(\overline{1})} = \widetilde{R}_{\mathrm{G}}(2;\overline{4})\{U_3(\overline{4},\overline{2},\overline{1})\widetilde{\psi}'_{\mathrm{G}}(\overline{2})\widetilde{\psi}'_{\mathrm{G}}(3)$$

$$+ \frac{1}{2}U_3(\overline{4},\overline{2},\overline{3})[\widetilde{\psi}'_{\mathrm{G}}(\overline{2})\widetilde{\psi}'_{\mathrm{G}}(\overline{3}) - C(\overline{2},\overline{3})]\delta(3-\overline{1})\}$$

$$+ \frac{\delta\widetilde{R}_{\mathrm{G}}(1;\overline{4})}{\delta\widetilde{\psi}'_{\mathrm{G}}(\overline{1})}\frac{1}{2}U_3(\overline{4},\overline{2},\overline{3})[\widetilde{\psi}'_{\mathrm{G}}(\overline{2})\widetilde{\psi}'_{\mathrm{G}}(\overline{3}) - C(\overline{2},\overline{3})]. \tag{4.11}$$

Upon differentiating the identity

$$\widetilde{R}(1;2)\widetilde{R}^{-1}(2;1') = \delta(1-1'), \tag{4.12}$$

which leads to $\Delta\widetilde{R} = -\widetilde{R}\Delta(\widetilde{R}^{-1})\widetilde{R}$, and noting from Eq. (4.8) that

$$\widetilde{R}^{-1}(1;1') = \partial_{t_1}\delta(1-1') - U_2(1,1') - U_3(1,2,1')\widetilde{\psi}(2), \tag{4.13}$$

one finds

$$\frac{\delta\widetilde{R}_{\mathrm{G}}(1;\overline{4})}{\delta\widetilde{\psi}'_{\mathrm{G}}(\overline{1})} = -\widetilde{R}_{\mathrm{G}}(1;5)\frac{\delta\widetilde{R}^{-1}_{\mathrm{G}}(5;\overline{5})}{\delta\widetilde{\psi}'_{\mathrm{G}}(\overline{1})}\widetilde{R}_{\mathrm{G}}(\overline{5};\overline{4}) \tag{4.14a}$$

$$= \widetilde{R}_{\mathrm{G}}(1;5)U_3(5,\overline{1},\overline{5})\widetilde{R}_{\mathrm{G}}(\overline{5};\overline{4}). \tag{4.14b}$$

For use in Eq. (4.10), one must average (4.11) using the result (4.14b). Since all quantities have been expressed in terms of $\widetilde{\psi}'_{\mathrm{G}}$, which is Gaussian by assumption, the averages can be performed by using the functional generalization of results like

$$\langle \mathcal{F}(\widetilde{\psi}')\widetilde{\psi}'\widetilde{\psi}'\rangle = \langle [\mathcal{F}(\widetilde{\psi}')\,\widetilde{\psi}']\widetilde{\psi}'\rangle = \left\langle \frac{\mathrm{d}(\mathcal{F}\,\widetilde{\psi}')}{\mathrm{d}\widetilde{\psi}'} \right\rangle \sigma^2$$

$$= \langle\mathcal{F}\rangle\sigma^2 + \left\langle \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\widetilde{\psi}'}\,\widetilde{\psi}' \right\rangle\sigma^2 = \langle\mathcal{F}\rangle\sigma^2 + \left\langle \frac{\mathrm{d}^2\mathcal{F}}{\mathrm{d}\widetilde{\psi}'^2} \right\rangle\sigma^4. \tag{4.15}$$

(Here $\sigma^2 \doteq \langle\widetilde{\psi}'\widetilde{\psi}'\rangle$; see Appendix B.) However, terms that bring in extra factors of U_3 (any of the terms that arise from differentiating $\widetilde{R}^{-1}$) should be neglected because the role of U_3 is to induce allegedly small non-Gaussian statistics and we have already (for better or worse) neglected terms of higher than first order in that correction. This amounts to saying that one should retain only the first term of Eq. (4.15) and should approximate $\widetilde{R}_{\mathrm{G}} \approx \langle\widetilde{R}\rangle \equiv R$. Only the first line of Eq. (4.11) then survives the average, and one finds a contribution to $\frac{1}{2}U_3T$ of the form $-\frac{1}{2}\Sigma(1;\overline{1})C(\overline{1},1')$, where

$$\Sigma(1;\overline{1}) \doteq -U_3(1,2,3)R(2;\overline{2})C(3,\overline{3})U_3(\overline{2},\overline{3},\overline{1}). \tag{4.16}$$

The second and third terms of $O(\Delta\widetilde{\psi}')$ in Eq. (4.5) can be worked out in a similar fashion. By symmetry, the second term is equal to the first. The third term gives a distinct contribution to the term $\frac{1}{2}U_3T$ in Eq. (2.8) of the form $F^{\mathrm{int}}(1,\overline{1})R(1';\overline{1})$, where

$$F^{\mathrm{int}}(1,\overline{1}) \doteq \frac{1}{2}U_3(1,2,3)C(2,\overline{2})C(3,\overline{3})U_3(\overline{1},\overline{2},\overline{3}). \tag{4.17}$$

This function can be interpreted as the covariance of an internal emission or noise term arising from the nonlinearity.

Next, one must evaluate the cross-correlation $\langle \widetilde{f}^{\mathrm{ext}\prime}(1)\widetilde{\psi}'(1')\rangle$ required in Eq. (2.8). This can be done exactly if one assumes that $\widetilde{\psi}$ evolves from Gaussian initial conditions $\widetilde{\psi}_0$ that are uncorrelated with the Gaussian $\widetilde{f}^{\mathrm{ext}}$. From Novikov's theorem, one has

$$\langle \widetilde{f}^{\mathrm{ext}\prime}(1)\widetilde{\psi}'(1')\rangle = \langle \widetilde{\psi}(1')\widetilde{f}^{\mathrm{ext}\prime}(1)\rangle \tag{4.18a}$$

$$= \left\langle \frac{\delta\widetilde{\psi}(1')}{\delta\widetilde{\psi}_0(\underline{1})}\right\rangle \langle \widetilde{\psi}'_0(\underline{1})\widetilde{f}^{\mathrm{ext}\prime}(1)\rangle + \left\langle \frac{\delta\widetilde{\psi}(1')}{\delta\widetilde{f}^{\mathrm{ext}}(\overline{1})}\right\rangle \langle \widetilde{f}^{\mathrm{ext}\prime}(\overline{1})\widetilde{f}^{\mathrm{ext}\prime}(1)\rangle. \tag{4.18b}$$

(Here the notation $\underline{1}$ denotes the set of all independent variables except time.) By assumption one has $\langle \widetilde{\psi}_0\,\widetilde{f}^{\mathrm{ext}\prime}\rangle = 0$, and by definition one has $\langle \delta\widetilde{\psi}(1')/\delta\widetilde{f}^{\mathrm{ext}}(\overline{1})\rangle = R(1';\overline{1})$. Thus[71]

$$\langle \widetilde{f}^{\mathrm{ext}\prime}(1)\widetilde{\psi}'(1')\rangle = R(1';\overline{1})F^{\mathrm{ext}}(1,\overline{1}). \tag{4.19}$$

This leads to the covariance equation

$$\partial_{t_1}C(1,1') - U_2(1,\overline{1})C(\overline{1},1') - U_3(1,2,\overline{1})\langle\psi\rangle(2)C(\overline{1},1') + \Sigma(1;\overline{1})C(\overline{1},1')$$
$$= [F^{\mathrm{int}}(1,\overline{1}) + F^{\mathrm{ext}}(1,\overline{1})]R(1';\overline{1}). \tag{4.20}$$

Finally, one can obtain a closed equation for the mean response function R by noting from Eq. (4.8) that R obeys

$$\partial_t R(1;1') - U_2(1,2)R(2;1') - U_3(1,2,3)\langle\psi\rangle(2)R(3;1')$$
$$- U_3(1,2,3)\langle\widetilde{R}(2;1')\widetilde{\psi}'(3)\rangle = \delta(1-1'). \tag{4.21}$$

If one approximates $\widetilde{\psi}' \approx \widetilde{\psi}'_{\mathrm{G}}$, the last term can be evaluated by Novikov's theorem:

$$\langle \widetilde{R}_{\mathrm{G}}(2;1')\widetilde{\psi}'_{\mathrm{G}}(3)\rangle = \left\langle \frac{\delta\widetilde{R}_{\mathrm{G}}(2;1')}{\delta\widetilde{\psi}'_{\mathrm{G}}(\overline{3})}\right\rangle C(\overline{3},3) \tag{4.22a}$$

$$= -\left\langle \widetilde{R}_{\mathrm{G}}(2;\overline{2})\frac{\delta\widetilde{R}_{\mathrm{G}}^{-1}(\overline{2},\overline{1})}{\delta\widetilde{\psi}'_{\mathrm{G}}(\overline{3})}\widetilde{R}(\overline{1};1')\right\rangle C(\overline{3},3) \tag{4.22b}$$

$$\approx R(2;\overline{2})C(\overline{3},3)U_3(\overline{2},\overline{3},\overline{1})R(\overline{1};1'). \tag{4.22c}$$

Thus, upon recalling Eq. (4.16), one finds that the last term on the left-hand side of Eq. (4.21) reduces to $\Sigma(1;\overline{1})R(\overline{1};1')$; therefore

$$\partial_{t_1}R(1;1') - U_2(1,\overline{1})R(\overline{1};1') - U_3(1,2,\overline{1})\langle\psi\rangle(2)R(\overline{1};1') + \Sigma(1;\overline{1})R(\overline{1};1') = \delta(1-1'). \tag{4.23}$$

This result shows that the operator on the left-hand side of Eq. (4.20) is just R^{-1}. Upon formally solving Eq. (4.20), one is then led to the symmetrical form

$$C(1,1') = R(1;\overline{1})[F^{\mathrm{int}}(\overline{1},\overline{1}') + F^{\mathrm{ext}}(\overline{1},\overline{1}')]R(1';\overline{1}') \tag{4.24}$$

or concisely $C = RFR^{\mathrm{T}}$.

This closure, consisting of Eq. (2.6) for the mean field, Eqs. (4.20) or (4.24) for the covariance, and Eq. (4.23) for the response function, along with the definitions (4.16) for Σ and (4.17) for F^{int}, is precisely the DIA. A thorough discussion and alternate derivation of the DIA for Vlasov turbulence was given by DuBois & Pesme (1985).

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[71] This result was known to Martin, Siggia, and Rose, who used the words "it is not difficult to show that . . ." but did not actually describe the proof.

For homogeneous turbulence, the DIA equations are often written in $\boldsymbol{k}$ space. If the quadratic PDE for a fluid description (no velocity space) is written as[72]

$$\partial_t \widetilde{\psi}_{\boldsymbol{k}} = L_{\boldsymbol{k}} \widetilde{\psi}_{\boldsymbol{k}} + \frac{1}{2} \sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},\boldsymbol{0}} M_{\boldsymbol{kpq}} \widetilde{\psi}_{\boldsymbol{p}}^* \widetilde{\psi}_{\boldsymbol{q}}^* + \widetilde{f}_{\boldsymbol{k}}^{\text{ext}}, \tag{4.25}$$

then in the DIA

$$\widehat{\Sigma}_{\boldsymbol{k}}(t;\overline{t}) = - \sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},\boldsymbol{0}} M_{\boldsymbol{kpq}} M_{\boldsymbol{pqk}}^* \widehat{R}_{\boldsymbol{p}}^*(t;\overline{t}) \widehat{C}_{\boldsymbol{q}}^*(t;\overline{t}) \tag{4.26}$$

and

$$\widehat{F}_{\boldsymbol{k}}^{\text{int}}(t,\overline{t}) = \frac{1}{2} \sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 \widehat{C}_{\boldsymbol{p}}^*(t,\overline{t}) \widehat{C}_{\boldsymbol{q}}^*(t,\overline{t}), \tag{4.27}$$

where $M \equiv U_3$. If $\psi_{\boldsymbol{k}}$ has been normalized in such a way that $M_{\boldsymbol{kpq}} + M_{\boldsymbol{pqk}} + M_{\boldsymbol{qkp}} = 0$, then one can verify that

$$\sum_{\boldsymbol{k}} [-\widehat{\Sigma}_{\boldsymbol{k}}(t;\overline{t})\widehat{C}_{\boldsymbol{k}}^*(t,\overline{t}) + \widehat{F}_{\boldsymbol{k}}(t;\overline{t})\widehat{R}_{\boldsymbol{k}}^*(t;\overline{t})] = 0; \tag{4.28}$$

this is a statement of "energy" conservation by the nonlinear terms: $\sum_{\boldsymbol{k}} C_{\boldsymbol{k}}(t,t)|_{\text{n.l.}} = 0$. (The proof can as easily be conducted in $\boldsymbol{x}$ space.) This result is inherited from the primitive amplitude equation; it is an essential property that any credible closure should satisfy.

The pros and cons of the DIA have been discussed at great length (Krommes 2002, and references therein). Significantly, it is known that the DIA is realizable; it provides the exact description of the second-order statistics for certain stochastic models. Some general discussion and original references are given by Kraichnan (1991). Although the first demonstration of realizability involved a so-called *random-coupling model* (Kraichnan 1961), a more intuitive generalized Langevin representation was later displayed by Kraichnan (1970) and Leith (1971). For homogeneous statistics, that model is

$$\partial_t \widetilde{\psi}_{\boldsymbol{k}}(t) - L_{\boldsymbol{k}} \widetilde{\psi}_{\boldsymbol{k}} + \int_0^t d\overline{t}\, \widehat{\Sigma}_{\boldsymbol{k}}(t;\overline{t}) \widetilde{\psi}_{\boldsymbol{k}}(\overline{t}) = \widetilde{f}_{\boldsymbol{k}}^{\text{int}}(t) + \widetilde{f}_{\boldsymbol{k}}^{\text{ext}}(t), \tag{4.29}$$

where $\widehat{\Sigma}_{\boldsymbol{k}}$ has the DIA form (4.26),

$$\widetilde{f}_{\boldsymbol{k}}^{\text{int}}(t) \doteq \frac{1}{2} \sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},\boldsymbol{0}} M_{\boldsymbol{kpq}} \widetilde{\xi}_{\boldsymbol{p}}^*(t) \widetilde{\xi}_{\boldsymbol{q}}^*(t), \tag{4.30}$$

and $\widetilde{\xi}_{\boldsymbol{p}}$ is an auxiliary random variable having covariance $C_{\boldsymbol{p}}$. It is left as an instructive exercise for the reader to verify that the equations for the response function and covariance that follow from Eq. (4.29) reproduce the DIA.

In addition to realizability (which guarantees a positive-semidefinite covariance), the DIA properly reduces to weak-turbulence theory at the level of wave–wave–particle or three-wave coupling (but omits some terms associated with higher-order processes[73]). Its

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[72]In terms of the previous general notation involving the coupling coefficients U_n, then upon Fourier transformation $U_2(\boldsymbol{x}_1 - \boldsymbol{x}_2) \to L_{\boldsymbol{k}}$ and $U_3(\boldsymbol{x}_2 - \boldsymbol{x}_1, \boldsymbol{x}_3 - \boldsymbol{x}_1) \to M_{\boldsymbol{kpq}}$. The introduction of conplex conjugates in the nonlinear term is done so that all of the wave vectors enter symmetrically in the triad constraint $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{0}$.

[73]In the language of Martin, Siggia, and Rose, the missing terms are contained in *vertex corrections*. Thus, for example, the DIA does not include all of the four-wave coupling effects that are contained in weak-turbulence theory.

self-consistent, nonlinearly energy-conserving determination of R and C is a definite plus, and it has enjoyed considerable successes (Kraichnan 1964a; Krommes 2002). However, the approximation described in the present section, and others like it, merely define algorithms; for arbitrarily strong turbulence, they cannot be systematically defended because one does not have a controlled estimate of the size of the non-Gaussian effects. A well-known qualitative issue related to the lack of random Galilean invariance is discussed in Sec. 5.1. The DIA cannot deal properly with mixtures of turbulent motions and coherent structures. And even for regimes in which the DIA is expected to be good, it is quite complicated. Alternate, simpler approximations are mentioned later in the article.

4.2. *Perturbative algorithm for the dielectric function of a general turbulent medium in the direct-interaction approximation*

As I discussed after Eq. (3.29), the essence of the R–g–$\mathcal{D}$ factorization obtained by DuBois and Espedal was the grouping of the nonlinear terms in the equation for the response function into two parts. In their words, they "temporarily consider $[f]$ and $[\boldsymbol{E}]$ as independent functional variables" and replace $\delta/\delta f$ in Eq. (3.29) by

$$\frac{\delta}{\delta f}\bigg|_{\boldsymbol{E}} + \frac{\delta \boldsymbol{E}}{\delta f} \cdot \frac{\delta}{\delta \boldsymbol{E}}\bigg|_f = \frac{\delta}{\delta f}\bigg|_{\boldsymbol{E}} + \frac{\delta}{\delta \boldsymbol{E}}\bigg|_f \cdot \mathbb{E}. \tag{4.31}$$

However, if one tries the same approach for the Navier–Stokes equation, the nonlinear term of which is essentially (modulo the constraint of incompressibility) $\langle \delta \boldsymbol{u} \cdot \boldsymbol{\nabla} \delta \boldsymbol{u} \rangle$, confusion arises because the two $\boldsymbol{u}$'s are obviously not functionally independent. In fact, the assumption of functional independence is problematical even for the plasma case because $\boldsymbol{E}$ and f are related by $\boldsymbol{E} = \mathbb{E}f$. Operationally, what is really being done is keeping track of which term is responsible for the advection, then organizing the terms in such a way that the self-consistent response is split off. This leads one to a formula for the dielectric of a general (quadratically nonlinear) system. I shall illustrate the procedure in the DIA.

The approach will be to calculate nonlinear corrections to R perturbatively, then to heuristically renormalize by replacing certain zeroth- or second-order quantities by their exact counterparts. This is one of the standard elementary procedures for obtaining the DIA itself (Kraichnan 1977; DuBois & Pesme 1985); where the methodology differs is that the symmetrized form of the mode-coupling coefficient will not be used here. Working with the unsymmetrized form allows one to distinguish terms associated with advection and with self-consistent response; this leads to a procedure that generalizes the approach of DuBois and Espedal.

For conciseness, I shall use a subscript notation that looks like a Fourier representation. In fact, however, the notation works in either $\boldsymbol{x}$ space or $\boldsymbol{k}$ space. A subscript k could be a combination of discrete index (say for a Cartesian velocity component) and continuous variable (either $\boldsymbol{k}$ or $\boldsymbol{x}$, and t). Again, sums over repeated indices are assumed. Let the dynamical equation be written

$$(R_k^{(0)})^{-1}\widetilde{\psi}_k - \widehat{M}_{k\,\widehat{p}\,q}\widetilde{\psi}_{\widehat{p}}\widetilde{\psi}_q = \widehat{\eta}_k. \tag{4.32}$$

The hatted index identifies the advection term. Here $\widehat{M}$ is not symmetric in its last two indices. Let me again underline terms that do not exist for passive advection. Then the random response function obeys

$$(R^{(0)})^{-1}\widetilde{R}_{k\,k'} - \widehat{M}_{k\,\widehat{p}\,q}(\widetilde{\psi}_{\widehat{p}}\widetilde{R}_{q\,k'} + \underline{\widetilde{R}_{\widehat{p}\,k'}\widetilde{\psi}_q}) = \delta_{k\,k'}. \tag{4.33}$$

At first order, one has

$$\widetilde{R}^{(1)}_{k\,k'} = R^{(0)}_{k\,\overline{k}}\widehat{M}_{\overline{k}\,\overline{p}\,\overline{q}}(\widetilde{\psi}^{(1)}_{\widehat{\overline{p}}}\,\widetilde{R}^{(0)}_{\overline{q}\,k'} + \underline{\widetilde{R}^{(0)}_{\widehat{\overline{p}}\,k'}\,\widetilde{\psi}^{(1)}_{\overline{q}}}). \tag{4.34}$$

Upon averaging Eq. (4.33), one finds that through second order R obeys

$$(R^{(0)})^{-1}R_{k\,k'} - \widehat{M}_{k\,p\,q}(\langle\widetilde{\psi}^{(1)}_{\widehat{p}}\,\widetilde{R}^{(1)}_{q\,k'}\rangle + \underline{\langle\widetilde{R}^{(1)}_{\widehat{p}\,k'}\,\widetilde{\psi}^{(1)}_{q}\rangle}) = \delta_{k\,k'}. \tag{4.35}$$

Substitution of Eq. (4.34) gives

$$(R^{(0)})^{-1}R_{k\,k'} - \widehat{M}_{k\,\widehat{p}\,q}R^{(0)}_{q\,\overline{k}}\widehat{M}_{\overline{k}\,\widehat{\overline{p}}\,\overline{q}}(C^{(2)}_{\widehat{p}\,\widehat{\overline{p}}}R^{(0)}_{\overline{q}\,k'} + \underline{C^{(2)}_{\widehat{p}\,\overline{q}}R^{(0)}_{\widehat{\overline{p}}\,k'}})$$
$$- \widehat{M}_{k\,\widehat{p}\,q}R^{(0)}_{\widehat{\overline{p}}\,\overline{k}}\widehat{M}_{\overline{k}\,\widehat{p}\,\overline{q}}(C^{(2)}_{\widehat{\overline{p}}\,q}R^{(0)}_{\overline{q}\,k'} + \underline{R^{(0)}_{\widehat{\overline{p}}\,k'}C^{(2)}_{\overline{q}\,q}}) = \delta_{k\,k'}. \tag{4.36}$$

In the last expression, the underlined quantities identify terms in which a hatted index is associated with R. Applying a hatted index to R is the generalization of applying the $\mathbf{E}$ operator in the original manipulations of DuBois and Espedal; i.e., it is related to the last term in Eq. (4.31). I shall group terms involving a hatted index on R separately, and I shall also interchange indices so that the names of the indices on the C's and R's match. Furthermore, one can heuristically renormalize by dropping the superscripts on all quantities. Then one obtains

$$g^{-1}_{k\,\overline{k}}R_{\overline{k}\,k'} + \Sigma^{f}_{k\,\widehat{\overline{k}}}R_{\widehat{\overline{k}}\,k'} = \delta_{k\,k'}, \tag{4.37}$$

where

$$(g^{-1}_0 + \Sigma^{g}_{k\,\overline{k}})g_{\overline{k}\,k'} = \delta_{k\,k'} \tag{4.38}$$

and

$$\Sigma^{g}_{k\,\overline{k}} \doteq -\widehat{M}_{k\,\widehat{p}\,q}(R_{q\,\overline{q}}C_{\widehat{p}\,\widehat{\overline{p}}}\widehat{M}_{\overline{q}\,\widehat{\overline{p}}\,\overline{k}} + R_{\widehat{p}\,\overline{p}}C_{\widehat{\overline{q}}q}\widehat{M}_{\overline{p}\,\overline{q}\,\overline{k}}) \tag{4.39a}$$

$$= -M_{k\,p\,q}R_{p\,\overline{p}}C_{q\,\overline{q}}\widehat{M}_{\overline{p}\,\overline{q}\,\overline{k}}, \tag{4.39b}$$

$$\Sigma^{f}_{k\,\widehat{\overline{k}}} \doteq -\widehat{M}_{k\,\widehat{p}\,q}(R_{q\,\overline{q}}C_{\widehat{p}\,\overline{p}}\widehat{M}_{\overline{q}\,\widehat{\overline{k}}\,\overline{p}} + R_{\widehat{p}\,\overline{p}}C_{\overline{q}\,q}\widehat{M}_{\overline{p}\,\widehat{\overline{k}}\,\overline{q}}) \tag{4.40a}$$

$$= -M_{k\,p\,q}R_{p\,\overline{p}}C_{q\,\overline{q}}\widehat{M}_{\overline{p}\,\widehat{\overline{k}}\,\overline{q}}. \tag{4.40b}$$

(Here $M_{k\,p\,q} \doteq \widehat{M}_{k\,p\,q} + \widehat{M}_{k\,q\,p}$ and redundant hats were dropped in each of the final expressions.) The quantity g is a generalized "particle" or passive fluid-element propagator, while Σ^f generalizes the plasma $\partial\delta\overline{f}\cdot\mathbf{E}$.

These general formulas work for both kinetic problems, where the indices are six dimensional (plus time), and fluid problems, where there is no velocity variable. When the latter case is applied to a scalar field, the operations become purely multiplicative; no summations or integrations occur. Then one can define a dielectric function simply by the ratio of particle to total response:

$$\mathcal{D}(\boldsymbol{k},\omega) = \frac{g(\boldsymbol{k},\omega)}{R(\boldsymbol{k},\omega)} = \frac{(g^{-1}_0 + \Sigma)(\boldsymbol{k},\omega)}{(g^{-1}_0 + \Sigma^g)(\boldsymbol{k},\omega)} = 1 + \frac{\Sigma^f(\boldsymbol{k},\omega)}{(g^{-1}_0 + \Sigma^g)(\boldsymbol{k},\omega)}, \tag{4.41}$$

where $\Sigma \doteq \Sigma^g + \Sigma^f$ and formula (4.41) holds for the case of zero mean field. An important example of this situation is the guiding-centre plasma model (a system of charged rods moving across a very large magnetic field with $\boldsymbol{E}\times\boldsymbol{B}$ motion). In a most elegant paper, Taylor (1974) found the proper form of $\mathcal{D}$ for the guiding-centre model ($g^{-1}_0 = -\mathrm{i}\omega$) in thermal equilibrium. He used a linear response formalism that applied only to Gibbsian

			Eq.
Mass operator:	$\Sigma(1;\overline{1})$	$= -U_3(1,2,3)R(2;\overline{2})C(3,\overline{3})U_3(\overline{2},\overline{3},\overline{1}) = \Sigma^g + \Sigma^f$	(4.16)
Particle renormalization:	$\Sigma^g(1;\overline{1})$	$= -U_3(1,2,3)R(2;\overline{2})C(3,\overline{3})\widehat{U}_3(\overline{2},\overline{3},\overline{1})$	(4.39b)
Effective mean field:	$\Sigma^f(1,\overline{1})$	$= -U_3(1,2,3)R(2;\overline{2})C(3,\overline{3})\widehat{U}_3(\overline{2},\overline{1},\overline{3}) = \boldsymbol{\partial}\delta\overline{f}\cdot\mathbb{E}$	(4.40b)
Nonlinear noise:	$F^{\mathrm{int}}(1,\overline{1})$	$= \frac{1}{2}U_3(1,2,3)C(2,\overline{2})C(3,\overline{3})U_3(\overline{1},\overline{2},\overline{3})$	(4.17)

TABLE 2. Statistical closure in the DIA. See Table 1 for the general form of the Dyson equations. The symmetrized mode-coupling coefficient is $U_3(1,2,3) = \widehat{U}_3(1,2,3) + \widehat{U}_3(1,3,2)$, where advection is associated with the second argument of $\widehat{U}_3$.

thermal equilibrium; subsequently Krommes & Similon (1980) showed how Taylor's result follows from the DIA. The present formalism recovers those results.

It should be noted that self-consistency is not only responsible for dielectric shielding, it also affects the form of the renormalized particle propagator g. Whereas passive advection leads to a generalized diffusion operator for Σ^g, polarization modifies that operator, producing among other things a smaller effective diffusion coefficient. This was discussed by Krommes and Similon for the guiding-centre model; the general polarization effect is described by the second term in Eq. (4.39a). (The first term in that equation would be the sole one arising in passive advection because it involves a C with two hatted indices, i.e., the covariance of the advecting velocity.)

The method of construction guarantees that the formulas (3.32b) and (3.33) for the dielectric function and shielded response function hold provided that one can identify in $\widehat{M}$ the analogue of the $\mathbb{E}$ and $\boldsymbol{\partial}$ operators; cf. Eq. (3.10b). This appears to be most useful for cases with a two-level description—for example, involving a velocity space that with the $\widehat{p}$ index in Eq. (4.32) must be integrated over, as in Klimontovich theory. For the plasma case, the results are recorded in Table 2. For fluid cases without a phase-space variable (such as the guiding-centre plasma or the Navier–Stokes equation), $\mathcal{D}$ can still be defined although the implications of this are not fully understood. It is noteworthy that dielectric response is mentioned nowhere in the most recent book on homogeneous turbulence (McComb 2014). Indeed, it is unnecessary to explicitly introduce $\mathcal{D}$ since the response function R carries the relevant information. Nevertheless, it is clear that nonlinear dielectric response operates in Navier–Stokes turbulence. It is not excluded that new perspectives on certain nonlinear processes in fluid turbulence may be obtained by expressing them in terms of the nonlinear $\mathcal{D}$.

The key DIA formulas obtained in this section are summarized in Table 2.

5. Some key points about statistical closures, especially for plasma physicists

We have now gained some intuition about the DIA, and we have seen how to obtain the dielectric function for turbulence in the DIA. I shall say nothing explicit about practical calculations involving $\mathcal{D}_{\mathrm{DIA}}$. But the discussion in Secs. 3, 4.1, and 4.2 shows that $\mathcal{D}_{\mathrm{DIA}}$ is included in the full DIA formalism. Any calculation that obtains statistics in the DIA (or in any similar closure) is *ipso facto* calculating the turbulent dielectric function along the way.

5.1. *Kadomtsev and the DIA*

The most general mathematical formulation of the last statement was presumably not known to Kadomtsev, although it is probable that he appreciated it at least intuitively. What is clear is that Kadomtsev understood an important deficiency of the DIA,[74] which I shall now describe.

It was already known to Kraichnan in 1959 that the DIA did not predict the famous $k^{-5/3}$ inertial-range spectrum of Navier–Stokes turbulence that follows from the $\boldsymbol{x}$-space self-similarity considerations of Kolmogorov (1941). [For some careful discussion, see Frisch (1995).] Instead, the DIA predicts $k^{-3/2}$, and for a time Kraichnan entertained the possibility that Kolmogorov was incorrect. However, the tidal-basin measurements of Grant *et al.* (1962) confirmed Kolmogorov's prediction, so Kraichnan was led to reconsider the structure of the DIA. In 1964 he explained (Kraichnan 1964*c*) that the difficulty is that the DIA is not invariant to *random Galilean transformations*. Ordinary Galilean invariance implies that under the addition of an infinite-wavelength flow short-scale eddies should be carried along unchanged. In fact, the DIA is Galilean invariant in that respect (McComb 2014). But if one considers an ensemble of such flows with random direction and strength, phase-mixing effects arise as a consequence of ensemble averaging. Those are captured correctly by the two-time R, as in the stochastic-oscillator model. However, because of the intimate way in which the DIA couples $R(t; t')$ and $C(t, t')$ to $C(t, t)$ [discussed in mathematical detail by Kraichnan (1964*c*); see also footnote 33 on page 16], that phase mixing introduces spurious distortion into the small-scale eddies (see Fig. 4), which shows up as an incorrect inertial-range spectrum.

Also in 1964, Kadomtsev's review appeared in Russian. It contains, after some discussion about the differences between resonant and adiabatic wave interactions and the role of wave packets, the following observation (p. 55):

> "[The weak coupling approximation, *a.k.a.* DIA] over-estimates the part played by the large-scale fluctuations, which is in fact no more than the convection of higher modes which are deformed adiabatically in the process."

Kadomtsev's discussion was not as crisply technical as Kraichnan's, but certainly he understood the intuitive essence of the issue. Because of the publication dates, writing styles, and lack of cross-references, it seems clear that the analyses were independent.

The issue of random Galilean invariance has received much attention and fostered many attempts at a cure. In general, it has been understood that the problem cannot be cured within a systematically renormalized Eulerian formulation (which includes the MSR formalism). A heuristic Eulerian approximation that does respect random Galilean invariance is the *eddy-damped quasi-normal Markovian* closure (Orszag 1970), and various more or less successful Lagrangian schemes have also been discussed (Kraichnan 1965; McComb 2014). But it should be emphasized that in many problems of practical interest the issue is only of secondary importance. In particular, turbulent transport is usually dominated by the long-wavelength, energy-containing fluctuations, in which case it is insensitive to the precise form of the inertial range. Although there are situations (for example, the intermittent mixing of contaminants) in which one cares about detailed inertial-range structure, they are well beyond the scope of this article; see Falkovich *et al.* (2001). Fusion plasma physicists should not be overly concerned with inertial-

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[74]Kadomtsev called the DIA the *weak-coupling approximation*. The equivalence between the weak-coupling approximation (formulated by Kadomtsev in the frequency domain) and the DIA (formulated by Kraichnan in the time domain) was demonstrated by Sudan & Pfirsch (1985). Some errors in Kadomtsev's formulation for Vlasov turbulence were corrected by DuBois & Pesme (1985).
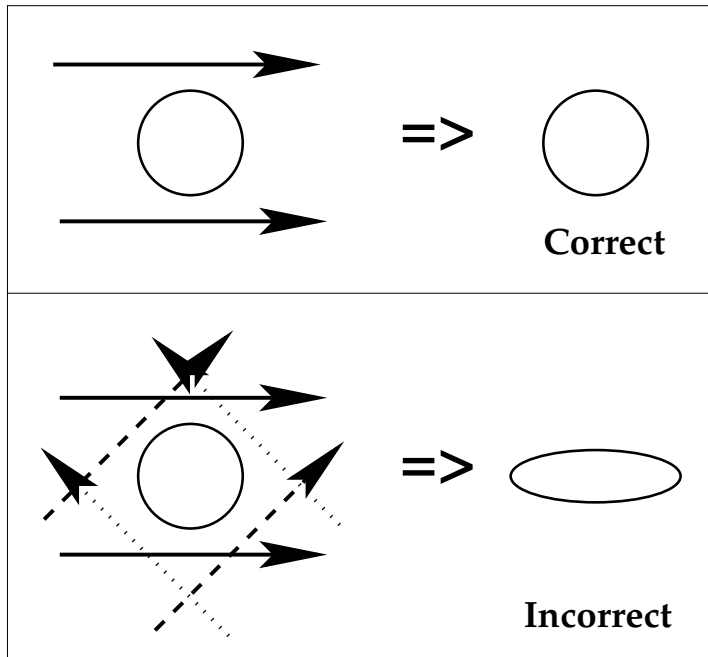
FIGURE 4. Illustration of difficulties with random Galilean invariance. In the top panel, an infinite-wavelength flow (no shear) translates an eddy unchanged. In the bottom panel, the average over an ensemble of such flows produces in the DIA a spurious distortion of a typical eddy, violating random Galilean invariance.

range spectra, although the shapes of those spectra sometimes figure in discussions of closures for large-eddy simulations.[9]

5.2. *Drift-wave saturation and the mixing-length formula*

The DIA for Vlasov turbulence was first discussed by Orszag & Kraichnan (1967); further contributions were made by DuBois & Espedal (1978), Krommes (1978), and DuBois & Pesme (1985). However, kinetic DIA calculations proved mostly too hard to perform at the time. Plasma physicists slowly latched on to the use of the DIA for fluid models (Krommes 1982; Waltz 1983; Sudan & Pfirsch 1985), though even there it is rather complicated for anisotropic turbulence. That difficulty fostered the development of DIA-based Markovian closures for plasma turbulence (Waltz 1983; Bowman 1992; Bowman *et al.* 1993; Bowman & Krommes 1997), and those can be used to quantify some important heuristic drift-wave scalings discussed by Kadomtsev. Here I shall give a brief introduction. Definitions of the symbols used in the following discussion are given in Appendix A.

5.2.1. The Hasegawa–Mima equation

More than ten years after Kadomtsev's monograph, Hasegawa & Mima (1978) proposed a nonlinear PDE, now known as the *Hasegawa–Mima equation* (HME), that serves as a very useful paradigm for understanding the basic physics of drift-wave turbulence. The

46

equation is[75]

$$\left(\underbrace{1}_{(i)} - \underbrace{\rho_s^2 \nabla_\perp^2}_{(ii)}\right)\partial_t \varphi + \underbrace{V_* \partial_y \varphi}_{(iii)} + \underbrace{\mathbf{V}_E \cdot \boldsymbol{\nabla}(-\rho_s^2 \nabla_\perp^2 \varphi)}_{(iv)} = 0; \tag{5.1}$$

it has been studied extensively. Indeed, the HME has the same form as the equation of Charney & Stern (1962) for Rossby-wave turbulence with finite deformation radius[76] L_d. (In the HME, ρ_s plays the role of L_d.) For infinite L_d, the equation reduces to the barotropic vorticity equation that is a subject of intense interest in geophysics and related areas; see various articles in Galperin & Read (2015).

In plasma physics, the interpretations of the various terms of the HME are as follows: (i) adiabatic response of electrons, streaming along magnetic field lines, to the slowly changing electrostatic potential of the drift wave; (ii) wave dispersion due to ion polarization drift; (iii) $\mathbf{E} \times \mathbf{B}$ advection of the background density profile; and (iv) $\mathbf{E} \times \mathbf{B}$ advection of vorticity (frequently called the *polarization-drift nonlinearity*). The linear dispersion relation of the HME reproduces Eq. (3.7).

The drift-wave dispersion relation (3.7) contains no linear growth or damping. That is consistent with the fact that the basic HME is time reversible. That property stems from the approximate derivation of the HME from kinetic theory, in which the wave–particle resonance is neglected in the determination of the electron response. That effect can be restored by hand by calculating the effects of the resonance directly from linear kinetic theory [see formula (3.5)] and expressing the result for the electron response as $\delta n_e/n_e = (1 - \mathrm{i}\delta)\varphi$, where δ is a wave-number-dependent operator. This changes $1 - \rho_s^2 \nabla_\perp^2$ to $1 - \rho_s^2 \nabla_\perp^2 - \mathrm{i}\delta$ and leads to the modified linear dispersion relation $\omega_{\boldsymbol{k}} = \omega_{*\boldsymbol{k}}/(1 + k_\perp^2 \rho_s^2 - \mathrm{i}\delta_{\boldsymbol{k}}) - \nu_{i\boldsymbol{k}}$, where an ion damping ν_i was also added. (Thus $\delta_{\boldsymbol{k}}$ is proportional to the ratio of the electron growth rate and the mode frequency.) When the $\mathrm{i}\delta$ is included in both the linear and nonlinear terms, the result is called the *Terry–Horton equation* after the original research of Terry & Horton (1982). For tutorial purposes, I shall instead assume that $\mathrm{i}\delta$ is neglected in the nonlinear term and thus just introduces a linear growth rate, leading one to another example of a quadratically nonlinear PDE, called *the* $\mathrm{i}\delta$ *model* (Waltz 1983), that takes the form (4.25) with $L_{\boldsymbol{k}} = -\mathrm{i}\omega_{\boldsymbol{k}}$ and

$$M_{\boldsymbol{kpq}} = \left(\frac{c_s}{\rho_s}\right)\frac{\widehat{\boldsymbol{b}} \cdot (\boldsymbol{p} \times \boldsymbol{q})(q^2 - p^2)}{1 + k^2} \tag{5.2}$$

(upon dropping the $\perp$ and normalizing lengths to ρ_s). Although the details of these various nonlinear equations definitely matter, the basic point is that irreversible effects can lead to instability and forced, dissipative turbulence. This was already well known to Kadomtsev.

5.2.2. Drift-wave saturation at the mixing-length level

Kadomtsev did not have the HME available to him, but he did understand the basic physics of the drift wave, which involves radial cross-field advection of a parcel of background ion density profile by the $\mathbf{E} \times \mathbf{B}$ velocity according to

$$\partial_t \delta n_i = -\delta \mathbf{V}_E \cdot \boldsymbol{\nabla} n_i. \tag{5.3}$$

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[75]The clearest and most succinct way of deriving the HME is via the cold-ion limit of the gyrokinetic formalism. Pedagogical discussions can be found in Krommes (2006) and Krommes (2012b).

[76]The Rossby radius of deformation is the horizontal scale at which Coriolis forces come into (geostrophic) balance with horizontal pressure forces. It is a characteristic mesoscale important, for example, in the theory of ocean circulation at large scales.

Let Δt be a characteristic mode period. Then the ion density fluctuation generated under coherent advection in a time Δt is

$$\frac{\delta n_i}{n_i} \sim \frac{\Delta \ell}{L_n}, \tag{5.4}$$

where $\Delta \ell$ is the distance $\delta V_{E,x} \Delta t$ moved by the fluid element and $L_n^{-1} \doteq -\partial_x \ln n_i$. By quasineutrality, the electron density fluctuation must equal this. Since the electrons are free to stream along the field lines,[77] they rapidly adjust to the ambient potential according to $\delta n_e / n_e = e \delta \phi / T_e$. If one balances this with the right-hand side of Eq. (5.4) and uses $\delta V_E = -(c/B) k_y \delta \phi$, one is readily led to the linear mode frequency $\Omega = \omega_*$. A more refined argument (Stoltzfus-Dueck *et al.* 2013) can take account of the dispersive polarization-drift correction as well and can also explain how dissipation leads to instability. As the modes grow, the density fluctuations continue to obey Eq. (5.4), while $\Delta \ell$ increases with δV_E. However, $\Delta \ell$ cannot increase indefinitely. When the turbulence saturates, it will possess a characteristic correlation length $L_{\rm ac}$. The coherency of the advection postulated above will be destroyed on that length scale; thus at saturation $\Delta \ell \sim L_{\rm ac}$. Since one has $L_{\rm ac} \sim \overline{k}_\perp^{-1}$, where $\overline{k}_\perp$ is a characteristic wave number,[78] one can rewrite the steady-state balance as $\overline{k}_\perp \delta n \sim n/L_n$, which means that at saturation the gradient of the typical density fluctuation becomes of the order of the background profile gradient. This criterion was asserted (with less discussion) by Kadomtsev (pp. 106–107),[79] and it is often quoted. Note that the argument does not by itself determine $L_{\rm ac}$. But dimensional analysis of the HME with periodic boundary conditions shows that the sole perpendicular scale in the problem is the sound radius $\rho_{\rm s}$, so $L_{\rm ac}$ must scale with $\rho_{\rm s}$. This leads to the saturation level $\delta n/n \sim \rho_{\rm s}/L_n$, which corresponds to fluctuating velocities of the order of the diamagnetic speed V_*.

An alternate term for correlation length in this context is *mixing length*, after the famous work of Prandtl (1925) on turbulent jets, and the drift-wave saturation level estimated[80] above has come to be called the *mixing-length level*. Various authors such as Connor & Pogutse (2001) cite Kadomtsev (1965) in this context. In fact, however, Kadomtsev did not actually use the phrase "mixing length" in his Sec. IV.4 where he discusses turbulent diffusion due to drift waves, though he had introduced the concept earlier in his review in connection with Reynolds stress, a topic that I shall discuss in more detail in Sec. 6.1.

Originally Kadomtsev (1962) believed that the cross-field diffusion coefficient due to saturated drift waves should be the one that Bohm (1949) obtained experimentally and that is usually quoted as $D_\perp = (1/16)D_{\rm B}$, where $D_{\rm B} \doteq cT_e/eB = \rho_{\rm s} c_{\rm s}$. Taylor (1961) employed Langevin equations to argue that the diffusion of a test ion can never

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[77]This is true provided the parallel wave number does not vanish. The argument must be modified for $k_\parallel = 0$ (zonal) modes. That leads one to a modified HME (Sec. 5.3) that describes the interactions between drift-wave turbulence and zonal flows.

[78]This is a simple consequence of dimensional analysis.

[79]Kadomtsev stated, "For $\gamma \sim \omega$, when strong turbulence develops, the oscillation amplitude increases to such an extent that the perturbation of the density gradient $k_\perp n'$ becomes of the order of the mean gradient, i.e. $k_\perp n' \sim \kappa n$ [where $\kappa \doteq L_n^{-1}$]."

[80]The argument is essentially dimensional in nature. As such, it is not a universal result. It is not possible to define a unique mixing length for physics models involving multiple length scales, as cautioned by Tennekes & Lumley (1972, p. 57) and Diamond & Carreras (1987). An extended excerpt from Tennekes and Lumley is reproduced by Krommes (2002, footnote 96, p. 149).

exceed the Bohm level[81] (with some uncertainty in the numerical coefficient). In his book, Kadomtsev referred to Bohm diffusion but presciently observed (p. 4),

> "It has now become evident, however, that the coefficient of turbulent diffusion cannot be obtained without a detailed investigation of the instability of an inhomogeneous plasma and in particular of its drift instability."

Indeed, the above arguments based on the HME imply that a local gyrokinetic description must lead to *gyro-Bohm diffusion*:

$$D_\perp \sim \left(\frac{\rho_s}{L_n}\right) D_B \equiv D_{gB},\qquad(5.5)$$

which is much smaller than the Bohm value and has a more favourable scaling with magnetic field. This scaling follows from the basic dimensional analysis of drift-wave physics given above and in Appendix A. It can also be recovered from the general estimate $D_\perp \sim \gamma^{\mathrm{lin}}/k_\perp^2$, obtained by both Kadomtsev (Chapter IV.4) and Dupree (1967), if the linear growth rate is taken to scale with the drift-wave frequency ω_* and fluctuation wavelengths are taken to scale with ρ_s [$k_\perp \rho_s = O(1)$]: $\omega_*/k_\perp^2 = (k_y \rho_s c_s/L_n)/k_\perp^2 = (k_y \rho_s)(\rho_s^2 c_s/L_n)/(k_\perp \rho_s)^2 \sim (\rho_s/L_n)D_B$. It took a surprisingly long time for this fact to be fully appreciated, especially in the face of experimental results that often showed Bohm scaling (Perkins *et al.* 1993). However, with the advent of modern supercomputers, it was ultimately possible to give numerical demonstrations (Lin *et al.* 2002, 2012) that turbulent gyrokinetic plasmas transition from Bohm to gyro-Bohm diffusion as $\rho_* \doteq \rho_i/a$ is reduced. Here a is the minor radius of a tokamak.

5.2.3. Mixing-length saturation and statistical closure

Given that all of these results are essentially dimensional in nature, one can pose the question, What does statistical closure have to add? One obvious answer is that it can make quantitative predictions; for example, dimensional analysis cannot pin down the numerical value of a diffusion coefficient. Closure also shows in detail how the nonlinear mode coupling works. That is of both conceptual and practical interest since predictions for the wave-number spectrum can be compared with numerical simulations or, in principal, with experiments. Early important work along these lines was by Sudan & Pfirsch (1985), who discussed the relationship between the mixing-length concept and DIA-type theories. However, since those authors did not consider drift waves, let us contemplate the statistical analysis of an iδ generalization of the HME.

Detailed solutions of statistical closures such as the DIA must be done numerically (LoDestro *et al.* 1991); that is entirely nontrivial in general because of both mode coupling and especially time-history integrations.[82] Therefore let us consider a Markovian approximation, which eliminates the time-history integrals at the price of a less faithful representation of the two-time correlations. A Markovianized version of the DIA

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[81]Kadomtsev (p. 29) criticized the calculations of Taylor (1961) on the grounds that the diffusion of a test ion (neither Taylor nor Kadomtsev actually used that nomenclature) differs from the diffusion of the plasma fluid because of considerations of ambipolarity. As remarked by Kadomtsev & Petviashvili (1963), "it is hazardous to extend to a total plasma conclusions that refer only to a single particle, neglecting the correlation of its motion with the motion of the other particles."

[82]The computationally challenging fact that in the original DIA the time-history integrations go all of the way back to $t = 0$ motivated Rose (1985) to develop the clever *cumulant-update DIA* (CUDIA), in which the integrations are periodically restarted from a time-evolved state involving non-Gaussian initial conditions. For further discussion and comparisons between various closures, see Frederiksen *et al.* (1994).

provides an evolution equation for the equal-time covariance[83] $C_{\boldsymbol{k}}(t)$ that takes the form (Kraichnan 1971; Bowman 1992; Bowman *et al.* 1993; Bowman & Krommes 1997)

$$\partial_t C_{\boldsymbol{k}} - 2\gamma_{\boldsymbol{k}}^{\text{lin}} C_{\boldsymbol{k}} + 2\operatorname{Re}\eta_{\boldsymbol{k}} C_{\boldsymbol{k}} = 2F_{\boldsymbol{k}}, \tag{5.6}$$

where

$$\eta_{\boldsymbol{k}} \doteq -\sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},\boldsymbol{0}} M_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}} M_{\boldsymbol{p}\boldsymbol{q}\boldsymbol{k}}^* \theta_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}}^* C_{\boldsymbol{q}}, \tag{5.7a}$$

$$F_{\boldsymbol{k}}^{\text{int}} \doteq \frac{1}{2}\sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},\boldsymbol{0}} |M_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}}|^2 \operatorname{Re}\theta_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}} C_{\boldsymbol{p}} C_{\boldsymbol{q}}. \tag{5.7b}$$

Here the triad interaction time $\theta_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}}$ is a measure[84] of the coherence time between the three interacting Fourier modes. The difficult coupled-mode nature of this system is apparent. But for pedagogical purposes, imagine that the spectrum consists of precisely three modes $\boldsymbol{K}$, $\boldsymbol{P}$, and $\boldsymbol{Q}$. Then it turns out that the steady-state Markovian closure can be solved analytically because the model contains just one triad interaction time, and that simple problem is sufficient to show how the theory works. The details are recorded in Appendix J of Krommes (2002). Of particular interest are the results for the coherent damping and nonlinear noise,

$$\frac{\operatorname{Re}\eta_{\boldsymbol{k}}}{\gamma_{\boldsymbol{k}}^{\text{lin}}} = 1 - \frac{\gamma_{\boldsymbol{k}}^{\text{lin}}}{\Delta\gamma} \quad\text{and}\quad \frac{F_{\boldsymbol{k}}^{\text{int}}/C_{\boldsymbol{k}}}{\gamma_{\boldsymbol{k}}^{\text{lin}}} = -\frac{\gamma_{\boldsymbol{k}}^{\text{lin}}}{\Delta\gamma}, \tag{5.8}$$

as well as the total saturation level:

$$\mathcal{E} = \left(\frac{\Delta(\gamma^2)}{(\Delta\gamma)^2}\right)\left(\frac{(\Delta\Omega)^2 + (\Delta\gamma)^2}{(\Delta\gamma)^2}\right)\left(\frac{\Gamma^2}{M^2}\right), \tag{5.9}$$

where $\mathcal{E} \doteq \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} C_{\boldsymbol{k}}$ is an energy-like quantity. The weight factors $\sigma_{\boldsymbol{k}}$ are assumed to be positive definite and to obey $\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} M_{\boldsymbol{k}} = 0$ ($M_{\boldsymbol{K}} \equiv M_{\boldsymbol{KPQ}}$). The characteristic mode-coupling coefficient M and growth rate Γ are defined by

$$\frac{1}{M^2} \doteq \Delta\left(\frac{1}{M_{\boldsymbol{P}}M_{\boldsymbol{Q}}}\right), \quad \Gamma^2 \doteq M^2\Delta\left[\sigma_{\boldsymbol{K}}\left(\frac{\gamma_{\boldsymbol{P}}^{\text{lin}}\gamma_{\boldsymbol{Q}}^{\text{lin}}}{M_{\boldsymbol{P}}M_{\boldsymbol{Q}}}\right)\right]. \tag{5.10}$$

In these expressions the Δ operator sums over all wave numbers (e.g., $\Delta\gamma \equiv \gamma_{\boldsymbol{K}}^{\text{lin}} + \gamma_{\boldsymbol{P}}^{\text{lin}} + \gamma_{\boldsymbol{Q}}^{\text{lin}}$). It can be shown that steady state requires $\Delta\gamma < 0$. First focus on Eq. (5.8). Basic RBT would correspond to $\operatorname{Re}\eta_{\boldsymbol{k}}/\gamma_{\boldsymbol{k}}^{\text{lin}} = 1$ and $F_{\boldsymbol{k}}^{\text{int}}/C_{\boldsymbol{k}} = 0$; the actual solution is clearly different, involving nonvanishing nonlinear noise. (This is not surprising since both $F_{\boldsymbol{k}}^{\text{int}}$ and $\eta_{\boldsymbol{k}}$ are measures of the same basic triadic mode coupling. Furthermore, as I noted in Sec. 4, nonvanishing noise is required in order that energy is conserved by the nonlinear terms.) Indeed, the fraction $|\gamma_{\boldsymbol{k}}/\Delta\gamma|$ can be larger, possibly much larger, than one because at least one $\gamma_{\boldsymbol{k}}^{\text{lin}}$ must be negative, which reduces $\Delta\gamma$. [This property was noted in the related numerical work by Hu *et al.* (1995, 1997) on the system of Hasegawa & Wakatani (1983); it was responsible for large differences in the measured flux from the quasilinear prediction.] Nevertheless, the total saturation level is in accord

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[83]In this section I shall drop the hats that indicate Fourier transforms.

[84]A common formula is $\theta_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}} = [(\mathrm{i}\Omega_{\boldsymbol{k}} + \eta_{\boldsymbol{k}}) + (\boldsymbol{k} \to \boldsymbol{p} \to \boldsymbol{q})]^{-1}$. However, the resulting theory cannot be shown to be realizable for $\Omega_{\boldsymbol{k}} \neq 0$ (Bowman 1992). Bowman *et al.* (1993) offered a pragmatic cure that was used in the numerical work cited in the text, but I shall not discuss that here.

with expectations. For the typical case[85] in which $|\Delta\gamma/\Delta\Omega| \ll 1$, formula (5.9) becomes

$$\mathcal{E} \sim \mathcal{E}^{\mathrm{ml}} \doteq (\Delta\Omega)^2/M^2. \qquad (5.11)$$

This is indeed the mixing-length level, as can be seen by taking $\Delta\Omega \sim \omega_* \sim c_{\mathrm{s}}/L_n$ and $M \sim c_{\mathrm{s}}/\rho_{\mathrm{s}}$ [see Eq. (5.2)]; then

$$\mathcal{E} \doteq \left\langle \left(\frac{e\delta\phi}{T_e} \right)^2 \right\rangle \sim \left(\frac{\rho_{\mathrm{s}}}{L_n} \right)^2. \qquad (5.12)$$

Obviously the details of the wave-number spectrum are complicated, but they are also predicted by the closure.

This simple example shows how closure can predict the intricacies of the nonlinear mode coupling and the steady-state balance between linear and nonlinear effects (in a way that is compatible with the constraints arising from scaling analysis). Details of the analysis of Markovian closures with many coupled modes can be found in the papers by Bowman *et al.* (1993) and Bowman & Krommes (1997) for general three-wave coupling and the Hasegawa–Mima equation, and by Hu *et al.* (1995, 1997) for the Hasegawa–Wakatani equations. A message is that technically one has come a long way from the early qualitative considerations. However, Kadomtsev's insights are seen to be upheld;[86] they capture the essence of the basic drift-wave paradigm.[87]

5.3. *The impact of zonal flows on the Hasegawa–Mima equation*

Zonal potentials are ones with $k_y = k_z = 0$ (in slab geometry) or with no variation within a flux surface (in toroidal geometry). The resulting $\boldsymbol{E} \times \boldsymbol{B}$ drifts (mostly poloidal)

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[85] In order that $|\Delta\gamma/\Delta\Omega|$ be small, it is necessary that the frequency mismatch is sufficiently large (i.e., that the waves be sufficiently dispersive). The drift wave is dispersive provided that the contribution due to the ion polarization drift—the $k_\perp^2 \rho_{\mathrm{s}}^2$ term in the denominator of Eq. (3.7)—is retained.

[86] Note that there is a distinction between $\eta_{\boldsymbol{k}}/k_\perp^2$ and the density diffusion coefficient $D_\perp$ for equations like the HME. $D_\perp$ vanishes for purely adiabatic electron response, whereas $\eta_{\boldsymbol{k}}$ does not. A consequence is that $D_\perp$ acquires an extra factor of $\gamma^{\mathrm{lin}}/\Omega$. This was recognized by Kadomtsev (p. 107).

In more detail, the quantity $\eta_{\boldsymbol{k}}/k_\perp^2$ is a generalization of the Taylor formula (2.17) for a test-particle diffusion coefficient; it describes a decorrelation or nonlinear scrambling effect due to the turbulence. An $\eta_{\boldsymbol{k}}$ exists even for passive advection. But the diffusion of the plasma density is subject to the self-consistency constraint that the $\boldsymbol{E} \times \boldsymbol{B}$ velocity and the fluid density are connected by the functional relationships between the densities n_s and the potential ϕ. For example, the continuity equation for electron gyrocentres, $\partial_t n_e + \boldsymbol{\nabla} \cdot (\boldsymbol{V}_E n_e) = 0$, leads after averaging to the electron gyrocentre flux $\Gamma_e = \langle \delta V_{E,x} \delta n_e \rangle$. In an $\mathrm{i}\delta$ model, one has $\delta n_{e,\boldsymbol{k}}/n_e = (1 - \mathrm{i}\delta_{\boldsymbol{k}})\varphi_{\boldsymbol{k}}$. Thus $\Gamma_e = (cT_e/eB) \sum_{\boldsymbol{k}} k_y \delta_{\boldsymbol{k}} \langle |\delta\varphi_{\boldsymbol{k}}|^2 \rangle n_e$. This is an exact formula for the flux, given the wave-number spectrum of the potential. An estimate using the mixing-length saturation level (5.12) leads to $\Gamma_e = \left[\underbrace{\left(\frac{\rho_{\mathrm{s}}}{L_n} \right) D_{\mathrm{B}}}_{D_{\mathrm{gB}}} \underbrace{(\overline{k}_y \rho_{\mathrm{s}})}_{O(1)} \underbrace{\delta_{\overline{\boldsymbol{k}}}}_{O(\gamma^{\mathrm{lin}}/\Omega)} \right] \left(\frac{n_e}{L_n} \right)$, where $\overline{\boldsymbol{k}}$ is a typical wave vector. The quantity in square brackets is the turbulent diffusion coefficient, which is seen to have the gyro-Bohm scaling reduced by a factor of $\gamma^{\mathrm{lin}}/\Omega$. (It is not hard to show, under mild assumptions about boundary conditions, that the ion diffusion coefficient is equal to the electron one; the polarization-charge contribution to Γ_i does not contribute.)

[87] But, as I have noted previously, one must not carry simple mixing-length arguments too far. For example, Diamond & Carreras (1987) have discussed the breakdown of such arguments for a model of resistive pressure-gradient-driven turbulence.

are called *zonal flows* (ZFs); they are important in multiple contexts.[88] In fusion plasmas, for example, they play an important role in the regulation of the levels of drift-wave turbulence and transport, either by shearing of turbulent eddies (Diamond *et al.* 2005) or by catalysing the coupling of unstable fluctuations to stable normal modes (Hatch *et al.* 2011*b*).

Zonal flows are strongly nonadiabatic: because they have $k_\parallel = 0$, the adiabatic ordering $\omega/k_\parallel v_{\text{t}e} \ll 1$ is violated. Dorland & Hammett (1993) emphasized that Poisson's equation must be modified as a consequence, giving rise to an enhanced response for the zonal modes. In the simplest model, the electron response is taken to vanish altogether for zonal modes. This can be implemented by writing $\delta n_{e,\boldsymbol{k}}/n_e = \alpha_{\boldsymbol{k}}\delta\phi_{\boldsymbol{k}}$, where $\alpha_{\boldsymbol{k}}$ vanishes for zonal modes and is equal to 1 otherwise. A consequence is that the derivation of the Hasegawa–Mima equation must be reconsidered; one finds instead the *modified Hasegawa–Mima equation*

$$(\alpha - \rho_{\text{s}}^2\nabla_\perp^2)\partial_t\varphi + \alpha V_*\partial_y\varphi + \boldsymbol{V}_E \cdot \boldsymbol{\nabla}(\alpha - \rho_{\text{s}}^2\nabla_\perp^2\varphi) = 0. \tag{5.13}$$

This equation has been used in various studies related to the physics of zonal flows (Krommes & Kim 2000; Parker & Krommes 2013); it is further discussed in Sec. 6.2.

5.4. *Further developments in closure theory*

For other facets of closure theory, I point the reader to the discussion and references in Krommes (2002) and McComb (2014). The latter focuses on homogeneous turbulence. In the next section, I shall describe some aspects of the theory of inhomogeneous turbulence.

6. Recent results on inhomogeneous turbulence

The easiest applications of closure approximations are to spatially homogeneous situations. However, while it is often possible to impose a homogeneous statistical ensemble, that may not reflect the physical reality. Kraichnan (1964*b*) pointed out that when inhomogeneous turbulence is of concern the choice of ensemble is crucial. For example, it seems counterproductive to describe the visibly banded Jovian atmosphere[88] by a homogeneous ensemble. In the present section, I shall discuss some basic results relating to inhomogeneous ensembles. They include fundamental ideas about Reynolds stress (Sec. 6.1), the symmetry-breaking bifurcation to inhomogeneous turbulence (Sec. 6.2), and some closures particularly suitable for states of inhomogeneous turbulence with significant mean fields (Sec. 6.3).

6.1. *Reynolds stress*

Kadomtsev (p. 59) wrote,

"We have shown . . . that the transition to strong turbulence leads to integral equations in the [DIA], in which the resonant and adiabatic interaction must be separated. At present we have no rigorous method of performing this separation and of reducing the integral equations, and the description given by this theory is inevitably only approximate. We have as yet no indication of the accuracy of this approximation.

"However, in describing strong turbulence in a plasma we can use an analogy with ordinary turbulence. Here the principal results have been obtained from a purely

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[88]The book edited by Galperin & Read (2015) is replete with observational data and theoretical analyses of zonal jets in many natural settings (such as planetary atmospheres) and laboratory experiments (such as tokamaks).

phenomenological approach. It is natural, therefore, to use this approach in plasma turbulence theory."

In the last paragraph, Kadomtsev was referring to the modelling of the so-called *Reynolds stress* by means of "the mixing-length concept introduced by Prandtl (1925)." In its original interpretation, Reynolds stress $\boldsymbol{\tau}$ described the contribution of fluctuations in the fluid velocity to the mean Navier–Stokes momentum equation:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\rho_{\mathrm{m}}^{-1} \boldsymbol{\nabla} P - \rho_{\mathrm{m}}^{-1} \boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \mu \nabla^2 \boldsymbol{u}, \tag{6.1}$$

where[89] $\boldsymbol{\tau} \doteq \rho_{\mathrm{m}} \langle \delta \boldsymbol{u} \, \delta \boldsymbol{u} \rangle$. Prandtl's estimate for the size of a fluctuating velocity component in a free jet was $\delta u \approx \ell |\partial \langle u \rangle / \partial y|$, where u is the longitudinal velocity, y is the transverse coordinate, and ℓ is the transverse mixing length. Kadomtsev (p. 59) reviewed a solution by Tollmein of the steady-state system of momentum and continuity equations that results when one takes $\ell = cx$, where c is a fitting constant; agreement with experiment is excellent when c is chosen appropriately.

Note that it is only the divergence of the stress tensor that contributes to the mean momentum equation. For homogeneous turbulence that divergence vanishes, so inhomogeneity is essential.[90] As an example, let us discuss the off-diagonal Reynolds stress due to $\boldsymbol{E} \times \boldsymbol{B}$ flows in a 2D model with constant magnetic field. One has $\tau_{xy} \propto \langle \delta V_{E,x} \delta V_{E,y} \rangle$ with $\boldsymbol{V}_E \propto \widehat{\boldsymbol{b}} \times \boldsymbol{\nabla} \widetilde{\phi}$ and $\widetilde{\phi}$ being the random electrostatic potential. Contours of constant $\widetilde{\phi}$ are the streamlines of the velocity. Consider an elliptical "eddy," of major axis a and minor axis b, inclined at random angle $\widetilde{\theta}$ from the x direction:

$$\widetilde{\phi}(x, y) = \frac{(\widetilde{c}x - \widetilde{s}y)^2}{a^2} + \frac{(\widetilde{s}x + \widetilde{c}y)^2}{b^2}, \tag{6.2}$$

where $\widetilde{c} \equiv \cos \widetilde{\theta}$ and $\widetilde{s} \equiv \sin \widetilde{\theta}$. It is straightforward to calculate $\widetilde{V}_{E,x} \propto -\partial_y \widetilde{\phi}$, $\widetilde{V}_{E,y} \propto \partial_x \widetilde{\phi}$, and the product $\widetilde{\tau}_{xy} \doteq \delta \widetilde{V}_{E,x} \delta \widetilde{V}_{E,y}$. I assume that y is a direction of symmetry.[91] The purely x-dependent part of the result (which after averaging gives the contribution to $\boldsymbol{\nabla} \cdot \boldsymbol{\tau}$) is

$$\widetilde{\tau}_{xy} \propto \widetilde{c}\,\widetilde{s} \left(\frac{\widetilde{c}^2}{a^2} + \frac{\widetilde{s}^2}{b^2} \right) \left(\frac{1}{a^2} - \frac{1}{b^2} \right) x^2. \tag{6.3}$$

For the case of circular eddies ($a^2 = b^2 = r^2$), this vanishes.[92] Even for elliptical eddies, $\tau_{xy} \doteq \langle \widetilde{\tau}_{xy} \rangle$ vanishes by symmetry when it is averaged over a symmetrical distribution

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[89]Kadomtsev and some others define Reynolds stress with a minus sign.

[90]In fact, even in statistically homogeneous turbulence a single realization is inhomogeneous, so one can argue that a kind of Reynolds stress operates microscopically even in homogeneous situations. This idea is behind the popular paradigm (Diamond *et al.* 2005; Fujisawa 2009) that zonal flows driven by turbulence back react on the turbulence and limit its magnitude. Detailed calculations of long-wavelength flow generation using a homogeneous Markovian closure applied to the modified Hasegawa–Mima equation were done by Krommes & Kim (2000), who were inspired by the earlier work of Diamond *et al.* (1998). However, the situation is subtle because such models assume random zonal flows with zero mean; they provide information only in a mean-square sense, as discussed by Krommes & Parker (2015). Do not confuse those homogeneous calculations with the inhomogeneous ones discussed in Secs. 6.2 and 6.3. See also challenge 6 in Sec. 8.

[91]One should really refer the y origin of the ellipse to a uniformly distributed random position. In a torus, one should further generalize the model to account for periodicity in the poloidal direction.

[92]This effect of symmetry is relevant to contemporary discussions of issues relating to momentum conservation and flow generation in gyrokinetics. Those are summarized in a lengthy technical report by Krommes & Hammett (2013), which contains a large pedagogical component.

in θ. But if $\widetilde{\theta}$ is distributed according to the PDF $f(\theta) = f_s(\theta) + \epsilon f_a(\theta)$, where s and a denote symmetric and antisymmetric parts, then the antisymmetric part contributes and one finds $\tau_{xy} \propto \epsilon \Delta x^2$, where $\Delta \doteq b^{-2} - a^{-2}$. This example shows the importance of asymmetry in producing nonvanishing Reynolds stress, and it suggests that toroidal devices with up–down asymmetry may, in the presence of microturbulence, have larger Reynolds stresses. Those may be important in limiting the size of the turbulence and in the generation of intrinsic rotation.[93]

It is well known that the deformation tensor $\boldsymbol{\nabla u}$ of a general flow field $\boldsymbol{u}(\boldsymbol{x})$ can be decomposed into a symmetric part (the rate-of-strain tensor $\boldsymbol{S}$, associated with both stretching at constant volume and possibly compression) and an antisymmetric part (associated with vorticity and leading to rigid-body rotation). Therefore tilted elliptical eddies, hence Reynolds stress, can arise by subjecting a circular eddy to an inhomogeneous velocity field. One source of inhomogeneity is the generation of *zonal flows* due to the (divergence of the) Reynolds stress. As an example, consider a slab zonal flow with constant shear:[94] $\boldsymbol{u} = \Omega_0 x \widehat{\boldsymbol{y}}$, with $\boldsymbol{\nabla} \times \boldsymbol{u} = \Omega_0 \widehat{\boldsymbol{z}}$ and $\boldsymbol{S} = \frac{1}{2}\Omega_0(\widehat{\boldsymbol{x}}\,\widehat{\boldsymbol{y}} + \widehat{\boldsymbol{y}}\,\widehat{\boldsymbol{x}})$ [the principal axes of which are $(1, \pm 1)^{\mathrm{T}}/\sqrt{2}$]. Tilt arises from the vorticity, and ellipticity (at constant area) arises from the straining. Thus one has a classic problem of the chicken and the egg. Inhomogeneous Reynolds stress drives zonal flows, and such stress can arise from the modulation of homogeneous turbulence by zonal flows; a feedback loop is indicated. But given homogeneous turbulence, there is no divergence of the Reynolds stress, hence no zonal-flow generation,[90] hence no inhomogeneity, hence no effect due to Reynolds stress. However, such homogeneous turbulence can be unstable to the generation of inhomogeneous flows, as I shall describe in the next section. Thus inhomogeneous states can result in which zonal flows and microturbulence are coupled self-consistently.

6.2. *Infinitesimal response, symmetry breaking, and zonostrophic instability*

A basic question is, Under what circumstances is a state of steady homogeneous turbulence unstable to the symmetry-breaking generation of inhomogeneity? This can be addressed by linearizing around the homogeneous state (allowing for inhomogeneous perturbations) and inquiring about the onset of a zero eigenvalue that solves a certain dispersion relation. I shall show, expanding on the original beautiful paper of Carnevale & Martin (1982), that that dispersion relation is intimately related to a certain approximation to the inverse of the infinitesimal response function R for the homogeneous turbulent state. In the theory of zonal-flow generation, the instability of homogeneous turbulence is called the *zonostrophic instability* (Srinivasan & Young 2012).

Mention of a zero eigenvalue in the context of homogeneous turbulence might be confusing. Recall that in earlier sections we saw how the nonlinear dielectric function participates in the spectral balance of the turbulence by shielding nonlinear noise, which I argued is important and cannot generally be neglected. Also recall that in steady state the zeros of $\mathcal{D}(\boldsymbol{k}, \omega)$ or $R^{-1}(\boldsymbol{k}, \omega)$ are in the lower half of the ω plane, not on the real line. Now the dielectric function (or inverse response function) referred to here relates internal

[93] I am grateful to G. Hammett (private communications, 2012, 2013) for emphasizing the possible roles of up–down asymmetries. For some discussion of symmetries in gyrokinetic turbulence, see Parra *et al.* (2011). For the importance of up–down asymmetries in flow generation, see Ball *et al.* (2014).

[94] A review of many effects associated with shearing of turbulent eddies, including the suppression of turbulence and the onset of transport barriers, is by Terry (2000). A region of constant shear might be reasonable for zonal flows arising from externally imposed radial electric fields. It is not a good approximation for self-consistently generated zonal flows; Kim & Diamond (2004) have studied the case of random shearing.

incoherent noise to the covariance of the fluctuations (for homogeneous turbulence, there is no mean flow). But if one admits the possibility of inhomogeneity, there will arise an effective *dielectric tensor* that operates on an extended vector containing not only fluctuations but also mean fields. At the symmetry-breaking bifurcation of homogeneous turbulence to the generation of mean fields, that tensor will develop a zero eigenvalue.

If one writes $Z \equiv \langle \psi \rangle$, Eq. (2.6) for zero forcing becomes

$$\partial_{t_1} Z(1) = U_2(1,2)Z(2) + \frac{1}{2}U_3(1,2,3)Z(2)Z(3) + \frac{1}{2}U_3(1,2,3)C(2,3). \qquad (6.4)$$

The linearization of this equation is

$$\partial_{t_1} \Delta Z(1) = U_2(1,2)\Delta Z(2) + U_3(1,2,3)\underline{Z(2)}\Delta Z(3) + \frac{1}{2}U_3(1,2,3)\Delta C(2,3). \qquad (6.5)$$

In particular, I shall linearize around a homogeneous state for which it is assumed that $Z = 0$; the terms that vanish with this assumption are underlined. The (generally inhomogeneous) fluctuation spectrum is assumed to obey the Dyson equation

$$\partial_{t_1} C(1,1') = U_2(1,2)C(2,1') + U_3(1,2,3)Z(2)C(3,1')$$
$$- \Sigma(1,\overline{1})C(\overline{1},1') + F(1,\overline{1})R(1';\overline{1}). \qquad (6.6)$$

Linear perturbations to the spectrum obey

$$\partial_{t_1} \Delta C(1,1') = U_2(1,2)\Delta C(2,1') + U_3(1,2,3)[\Delta Z(2)C(3,1') + \underline{Z(2)\Delta C(3,1')}]$$
$$- \underline{\Sigma(1,\overline{1})\Delta C(\overline{1},1')} - \underline{\Delta\Sigma(1;\overline{1})C(\overline{1},1')} + F(1,\overline{1})\Delta R(1';\overline{1}) + \underline{\Delta F(1,\overline{1})R(1';\overline{1})}. \qquad (6.7)$$

Doubly underlined terms are $O(C\,\Delta C)$; they describe fluctuation–fluctuation interactions. Let us neglect those and focus on the coupling between the mean fields and the turbulence. Because $\Delta R = -R\Delta(R^{-1})R$ and

$$R^{-1}(1;\overline{1}) = \partial_{t_1}\delta(1-\overline{1}) - U_2(1,\overline{1}) - U_3(1,2,\overline{1})Z(2) + \Sigma(1,\overline{1}), \qquad (6.8)$$

one has

$$\Delta(R^{-1})(1;\overline{1}) = -U_3(1,2,\overline{1})\Delta Z(2) + \underline{\Delta\Sigma(1;\overline{1})}. \qquad (6.9)$$

The doubly underlined term contains pieces of order $C\,\Delta R$ and $R\,\Delta C$; those can be ignored because when multiplied by F, as required for Eq. (6.7), they contribute effects of order C^2 or $C\,\Delta C$. The solution to Eq. (6.7) is then, after some permutation of indices,

$$\Delta C(2,3) \approx R_0(2;\overline{2})U_3(\overline{2},\overline{3},\overline{1})C(\overline{3},3)\Delta Z(\overline{1}) + (2 \leftrightarrow 3). \qquad (6.10)$$

[The last term, which ensures symmetry, comes from the first term on the right-hand side in Eq. (6.9) after using steady-state balance to replace the F in $F\Delta R$ with $R^{-1}C(R^{-1})^{\mathrm{T}}$.] When this solution is inserted into Eq. (6.5), one finds

$$[\partial_{t_1}\delta(1-\overline{1}) - U_2(1,\overline{1}) + \Sigma_0(1;\overline{1})]\Delta Z(\overline{1}) = 0, \qquad (6.11)$$

where

$$\Sigma_0(1;\overline{1}) \doteq -U_3(1,2,3)R^{(0)}(2;\overline{2})C(3,\overline{3})U_3(\overline{2},\overline{3},\overline{1}). \qquad (6.12)$$

A consistent (fluctuation–dissipation) approximation (Carnevale & Martin 1982) is to replace the two-time dependence of $C(3,\overline{3})$ with that of $R^{(0)}(3;\overline{3})$. Because the coefficients of ΔZ are all calculated in the homogeneous state, one can Fourier transform as usual. If one follows the common convention of using $\boldsymbol{q}$ to refer to the wave vector of the mean field, one then arrives at the dispersion relation

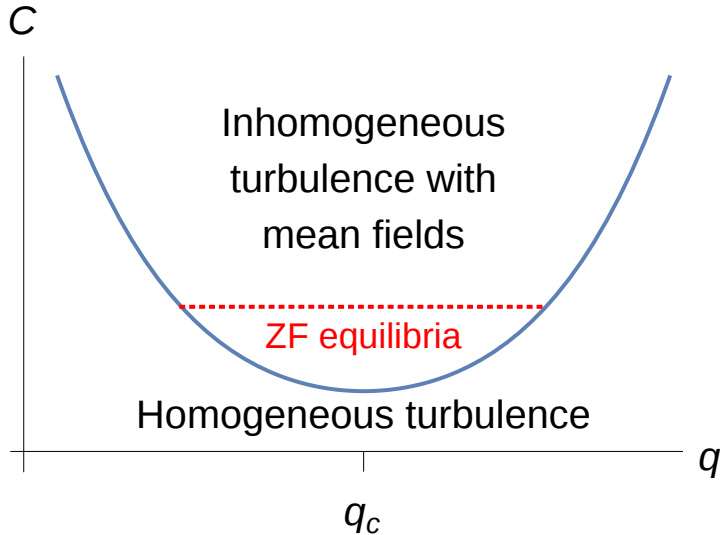$$R_1^{-1}(\boldsymbol{q},\omega) = 0, \qquad (6.13)$$

FIGURE 5. (Colour online.) Cartoon of a neutral curve $C = C(q)$ for zonostrophic instability. Below the neutral curve, the homogeneous turbulent state is stable. Above it, the homogeneous turbulence is unstable to the generation of inhomogeneous mean fields. Instability first sets in at a critical wave number q_c. For given C in the unstable region. zonal-flow equilibria with a continuum of q's are allowed. Further analysis must be done to examine the stability of those solutions (Parker & Krommes 2014).

where R_1 is the first renormalized approximation to R. [In Eq. (6.8), replace Σ by Σ_0 and R by R_1.] The solution determines the eigenvalue ω as a functional of the level C of the homogeneous turbulence (as well as $\boldsymbol{q}$ and the various parameters): $\omega = \omega_{\boldsymbol{q}}[C]$. Setting $\omega_{\boldsymbol{q}}[C] = 0$ determines the *neutral curve* $C = C(\boldsymbol{q})$; larger values of C are unstable to the generation of inhomogeneous mean fields (see Fig. 5). This process is analogous to the general phenomenon of pattern formation, as discussed by Parker & Krommes (2013).[95] In the context of the barotropic vorticity equation, the zonostrophic dispersion relation was analysed in detail in a beautiful paper by Srinivasan & Young (2012). Further results have been reported by Parker & Krommes (2014, 2015).

This zonostrophic instability is a generalization of the often-discussed *modulational instability* (Dewar & Abdullatif 2006; Connaughton *et al.* 2010), in which one considers the instability of a fixed "pump" wave at wave vector $\boldsymbol{K}$ to the generation of fluctuations at wave vector $\boldsymbol{Q}$ with sidebands at $\boldsymbol{P}_{\pm} = -\boldsymbol{K} \pm \boldsymbol{Q}$ (see Fig. 6). The relationship between the zonostrophic and modulational instabilities was first demonstrated by Carnevale & Martin (1982), though their result was not widely appreciated. It has recently been discussed in more detail by Parker & Krommes (2015), whose work was further generalized by Bakas *et al.* (2015, Appendix C).

6.3. *The CE2 and S3T closures*

The dispersion relation (6.13) also follows from a superficially distinct closure theory, the so-called CE2 (second-order cumulant) closure [Tobias *et al.* (2011), Srinivasan & Young (2012), and references therein]. In this approximation, the small-scale turbulence is represented by merely an external white-noise forcing; otherwise, "eddy–eddy interac-

[95]Such an analogy was already recognized by Kadomtsev [Sec. I.2(a)], who discussed an example of plasma convection.
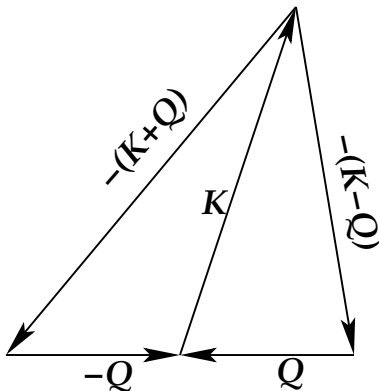
FIGURE 6. Wave vectors for the modulational instability of a fixed pump wave $\boldsymbol{K}$ to fluctuations at $\boldsymbol{Q}$; the sidebands are $\boldsymbol{P}_\pm \doteq -(\boldsymbol{K} \pm \boldsymbol{Q})$. The mode coupling obeys $\boldsymbol{K} + \boldsymbol{P}_\pm \pm \boldsymbol{Q} = \boldsymbol{0}$. This process can be extracted from the general dispersion relation for zonostrophic instability by inserting the background spectrum $C_{\boldsymbol{k}} \propto \delta(\boldsymbol{k} - \boldsymbol{K}) + \delta(\boldsymbol{k} + \boldsymbol{K})$.

tions" are neglected and the retained interactions are between only the mean fields and the turbulence. It is left to the reader to learn more about this closure [see the discussion below as well as various articles in Galperin & Read (2015)] and to ponder why the dispersion relations should be equivalent.

In the general notation of Eqs. (2.6)–(2.8), the CE2 closure omits the terms of $O(\delta\psi^2)$ in Eq. (2.7) or the third-order cumulant T in Eq. (2.8)—those terms describe the eddy–eddy interactions—while keeping Eq. (2.6) for the mean field intact. With white-noise forcing, Eq. (2.8) closes in terms of $\langle\psi\rangle(t)$, $C(t,t)$, and the known strength of the forcing; the truncated form of Eq. (2.7) ensures that this approximation is realizable. Tobias *et al.* (2011), Tobias & Marston (2013), and Marston *et al.* (2015) have referred to the numerical solutions of such truncated cumulant systems as "direct statistical simulations."

Realizability is an important property of CE2. While one can define higher-order approximations CEn for $n > 2$, none of those are realizable.[96] Marston (2012) and Marston *et al.* (2015) have explored several phenomenologically modified versions of CE3 that appear to pragmatically restore realizability. The presence of eddy–eddy interactions in those closures leads to results that are superior to those from CE2 in some situations. However, a more systematic approach is desirable.

Studies of CEn closures (by that name) were preceded by the research of Farrell and coworkers on the so-called *stochastic structural stability theory* (SSST or S3T), the fundamental paper being that of Farrell & Ioannou (2003); see also Farrell & Ioannou (2015) for an overview. One version of the theory is mathematically identical to CE2, a philosophical difference being the interpretation of the random forcing. In strict CE2, the forcing is viewed as being entirely extrinsic—for example, due to modes that are otherwise unaccounted for. In S3T, the forcing is frequently said to represent the effects of the eddy–eddy interactions as well as any extrinsic noise. That would be in accord with a Langevin representation of the turbulence provided that an energy-conserving coherent damping due to the eddy–eddy terms were also included. Sometimes that has been done, as in the work of DelSole (2001) and Farrell & Ioannou (2009*b*); other times it has been

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[96]This follows from a theorem due to Marcinkiewicz (1939), which states that a realizable PDF is either Gaussian (i.e., it is composed of just the first two cumulants) or possesses an infinite number of cumulants.

omitted, as in the application by Farrell & Ioannou (2009a) of S3T to the Hasegawa–Wakatani system. Omission of that damping fosters uncertainty about some of the details reported in the latter work (without, however, vitiating the principal message).

Farrell & Ioannou (2015) have reviewed some calculations using S3T and have also embedded that approximation into a more general framework that they call *statistical state dynamics*. The basic philosophy is that working with an ensemble (even one that is characterized by only the first few cumulants) provides information about cooperative and self-consistent behaviour that is superior to that gained from studies of individual realizations. A technical justification for a simple stochastic parameterization of the eddy–eddy interactions involves the non-normality[97] of the linear operator in the presence of mean fields; consult Farrell & Ioannou (2015, and references therein) for further discussion of the implications of non-normality.

In the original applications of these types of closures, coupled equations for mean fields and eddy covariance were obtained by applying a zonal average to the original PDE. However, Bakas *et al.* (2015) have generalized the procedure to show that one can consider the onset of both zonal and nonzonal perturbations; many detailed results about barotropic beta-plane turbulence are contained in that beautiful paper, which has implications for drift-wave physics as well.

There are many opportunities for further research. For example, some current work focuses on applying CE2-type ideas to the study of dynamo action in accretion disks (Squire & Bhattacharjee 2015). Also, the kinds of inhomogeneous states described in this section, which involve steady or quasi-steady zonal jets, do not appear to be relevant for microturbulence in tokamaks; see footnote 90 on page 52 as well as challenge 6 in Sec. 8.

7. Highlights of some research threads

Now that the reader has gained an overview of the subject from a modern perspective, I wish to provide additional representative references that chronicle the progression of some lines of plasma turbulence research that emanate from the mid-1960s, as well as to mention some ideas and techniques that arose only later. It is fascinating to watch the emergence and development of ideas in the course of a half century of research. Unfortunately, I cannot give a complete review of the subject, which is vast. Any one of the subsections below merits a full-length review in its own right, and I have omitted some important areas altogether; here I merely want to whet the reader's interest and provide some entry points to the literature. I restrict my attention to basic conceptual topics. The consequences of turbulence for plasma confinement are beyond the scope of this article, although they are obviously of paramount importance.[98]

7.1. *Some ideas discussed by Kadomtsev*

7.1.1. Bifurcations

Early in his book, Kadomtsev introduced (p. 5) the topic of bifurcations, either "soft" (supercritical) or "hard" (subcritical). Hinton & Horton (1971) did a nice bifurcation calculation of a collisional drift-wave instability. Much more recently, Kolesnikov &

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[97]A non-normal matrix $\boldsymbol{L}$ is one that does not commute with its Hermitian adjoint. The linear system $\partial_t \boldsymbol{\psi} = \boldsymbol{L} \cdot \boldsymbol{\psi}$ can exhibit transient growth even when all of the eigenvalues of $\boldsymbol{L}$ are negative.

[98]Early results were reviewed in Kadomtsev's Chapter V, and Kadomtsev & Pogutse (1970) reviewed turbulence in toroidal systems a few years later. A wealth of more up-to-date information can be found in the book by Horton (2012).

Krommes (2005a,b) attempted to consider the transition to ion-temperature-gradient-driven turbulence in the present of zonal flows, motivated by the numerically observed phenomenon of the *Dimits shift*[99] (Dimits *et al.* 2000). Although they used modern techniques such as the centre-manifold theorem (Guckenheimer & Holmes 1983; Kuznetsov 1998), the results were ultimately unsatisfactory; considerable further work is indicated.

Also in the context of bifurcations, Kadomtsev (p. 6) referred to Landau's scenario for the transition to turbulence. A revolution occurred when Ruelle & Takens (1971) argued for a qualitatively different scenario involving strange attractors, which had been discovered without fanfare years earlier by Lorenz (1963). Some good discussion of issues and phenomenology related to modern nonlinear dynamics is given by Holmes *et al.* (1996).

Subcritical bifurcation provides a route to *subcritical* or *submarginal turbulence*, a difficult topic about which I have said little. The phenomenon was subsequently observed in a variety of computer simulations, representative examples of which include the works of Scott (1992) and Drake *et al.* (1995).[100] Itoh *et al.* (1996) and Itoh *et al.* (1999) discussed the topic in terms of nonlinear dispersion relations that follow from crude statistical closures. There is much more work to do; see challenge 4 in Sec. 8.

7.1.2. Quasilinear theory

Kadomtsev (Sec. I.3) reviewed the basic ingredients of the plasma quasilinear theory, which had just appeared a few years earlier (Vedenov *et al.* 1962; Drummond & Pines 1962). About five years later, the monograph of Sagdeev & Galeev (1969, based on lectures a few years earlier) treated the subject in somewhat more detail. They displayed an appreciation of the importance of stochastic phase-space trajectories, and they cited Dupree (1966) for the most rigorous justification of the diffusion operator.[101] However, none of those authors discussed what is now called the *Chirikov criterion* (island overlap) for the onset of stochasticity;[102] Chirikov's thesis only appeared a few years later (Chirikov 1969). That was the tip of the iceberg associated with the emerging awareness by the plasma community of Hamiltonian chaos and nonlinear dynamics, which gained force during the 1970s. Modern understanding is described in the books by Lichtenberg & Lieberman (1992) and (more briefly but very elegantly) by Diacu & Holmes (1996).

7.1.3. Wave kinetics

Kadomtsev discussed wave–wave coupling in his Chapter I.3. That theme was also taken up by Sagdeev & Galeev (1969); a modern book is by Nazarenko (2011). Recent advances in the theory of plasma waves are described by Dodin (2014a,b).

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[99]Given a drive parameter λ (e.g., temperature gradient), one might naively expect that turbulent heat flux should ensue when λ exceeds the threshold λ_c for linear instability. In fact, the flux was numerically observed to almost vanish until λ achieved a value λ_* that significantly exceeded λ_c. The distance $\lambda_* - \lambda_c$ is the Dimits shift. The suppression of flux in the Dimits-shift regime is due to the excitation of zonal flows (Rogers *et al.* 2000; Kolesnikov & Krommes 2005a,b).

[100]For many more references to work on subcritical turbulence, see Yoshizawa *et al.* (2001).

[101]An important later paper on quasilinear theory by Kaufman (1972) should also be noted here.

[102]The relationships of the Chirikov criterion and Dupree's resonance broadening to plasma quasilinear theory were described by Krommes (1984, Sec. 5.5.3, p. 195).

7.1.4. Noise due to particle discreteness, and the transition to unstable, nonlinearly
saturated turbulence in plasmas

By the early 1960s, one had a clear picture of the consequences of particle discreteness for a weakly coupled stable plasma in the form of the Balescu–Lenard and Landau collision operators. The theory was briefly described by Kadomtsev [Sec. II.2(b)] and treated in more detail by Montgomery & Tidman (1964).

To discuss the qualitative transition from stable to unstable plasma, Kadomtsev considered (p. 46) the equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 2\gamma I + q - \alpha I^2, \tag{7.1}$$

where I denotes spectral intensity, γ is a typical linear growth rate, q is a source of fluctuations due to moving discrete particles, and α is a mode-coupling coefficient. For stable plasma, he described the discreteness-induced noise level as a balance between emission and absorption:[103] $I = q/(2|\gamma|)$. As the modal growth rate transitions through zero to positive values, nonlinear mode-coupling effects enter and the steady state is given by $I = 2\gamma/\alpha$; see Kadomtsev's Fig. 11. Detailed treatment of this transition is difficult even in weak-turbulence theory (Rogister & Oberman 1968, 1969). In principle, it can be described by the general equations of Rose (1979).

Discreteness-related noise is also important in particle-in-cell (PIC) simulations of plasmas. For unmagnetized plasmas, the relevant kinetic theory is described by Birdsall & Langdon (1985, Chapter 12). The theory of noise in gyrokinetic plasma is especially interesting because the gyrokinetic noise level is substantially lower than that of the true many-body plasma. Theoretical analysis of gyrokinetic noise in both electrostatic (Krommes *et al.* 1986) and electromagnetic (Krommes 1993) regimes were followed by detailed calculations of the noise level in gyrokinetic PIC simulations (Nevins *et al.* 2005); the latter work inspired a vigorous and useful debate about signal-to-noise ratios and resolution requirements in the simulations. The theory was reviewed by Krommes (2007).

7.2. *Some ideas discussed by Dupree and Weinstock*

7.2.1. Energy conservation

Dupree's original resonance-broadening theory did not conserve energy; as I noted in footnote 37 on page 17, it was appropriate for passive, not self-consistent problems. This and other difficulties were discussed by Orszag & Kraichnan (1967), whose paper on variants of the DIA for kinetic plasma turbulence was full of important insights. Unfortunately, that paper appears to have been mostly ignored for about a decade until the rise of interest in the mid-1970s in systematically renormalized closures by DuBois and coworkers, Krommes, and others. Dupree & Tetreault (1978) also discussed the problem with energy conservation in a restricted context.

7.2.2. Phase-space granulation

Subsequently, Dupree focused his research on considerations of small-scale granulations and the related topics of "clumps," "holes," and coherent structures (areas not discussed by Kadomtsev). Boutros-Ghali & Dupree (1981) gave a thorough discussion of the consequences of the one- and two-time equations for the two-point correlation function, with

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[103]Kadomtsev's discussion may foster some confusion. The interpretation of γ as a *modal* growth rate makes sense only when Landau damping is weak ($k\lambda_D \ll 1$). That is not the case for most of the scales that fit inside of a Debye sphere.

reference to coherent and incoherent response. They challenged some of the conclusions reached by DuBois & Espedal (1978) relating to phase-space granulation and clumps. However, in the opinion of the present author, DuBois and Espedal were correct.[104] For fluctuations in phase space, the role of coherent structures remains of interest today (Diamond *et al.* 2010; Lesur & Diamond 2013). The application of clump-related ideas to fluid problems is rather subtle, as explained by Krommes (1997). His discussion was in part motivated by the celebrated *Kraichnan model* (Kraichnan 1994) of passive advection involving a random coefficient delta correlated in time but with specified wave-number spectrum. That model proved to be enormously useful and ultimately led to the discoveries about anomalous scaling reviewed by Falkovich *et al.* (2001)—truly one of the major triumphs of the modern statistical theory of turbulent fluids. To be clear, these latter results were not motivated by Dupree's early work; they are just beautiful physics, and they set a very high standard for the theoretical analysis of turbulent systems.

7.2.3. Averaging and projection operators

The papers of Weinstock (1969, 1970) introduced the concept of an *averaging operator*, with which he showed how to interpret and generalize the analyses of Dupree (1966, 1967). [The systematic use of cumulants (Weinstock 1968) was also important in the discussion.] An averaging operator is a special case of a linear projection operator P, which obeys $P^2 = P$. Define also the orthogonal projector $Q \doteq 1 - P$. Then a dynamical equation for a field ψ can be projected onto a "relevant" subspace with P, and onto everything else (the orthogonal subspace) with Q. Elimination of the irrelevant dynamics $Q\psi$ then leads to a formally closed equation for the P projection. Such methodology was used in a famous paper by Mori (1965), who showed how to use projection operators to derive generalized Langevin equations, and it figures importantly in modern derivations of linear and nonlinear theories of transport in many-body systems (Zwanzig 2001).

The elimination of $Q\psi$ formally solves the closure problem! Of course, the devil is in the details. Exactly how to best define *relevant* is unclear. And even given such a definition, for nonlinear systems the elimination of the irrelevant dynamics is nontrivial; ultimately, the various approximate techniques for carrying that out amount to a restatement of the kinds of approaches employed in traditional closure theory (with all of their difficulties). Nevertheless, Weinstock's averaging-operator approach elucidated the mechanics of Dupree's formalism and clarified the structure of plasma turbulence theory. Projection-operator methods should be in the toolbox of a well-rounded turbulence theorist, although they are definitely not a panacea and should be used only with the greatest of care.

7.3. *Some miscellaneous topics related to analytical methods*

The following subsections mention representative topics that have emerged subsequent to the early works by Kadomtsev, Dupree, and Weinstock.

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[104]Boutros-Ghali and Dupree wrote, "If by the direct interaction approximation (as applied to plasma turbulence), we understand a scheme which iterates the coherent response only then this procedure will break down." This statement is confusing; there is no ambiguity about the definition of the complete DIA for a plasma in the absence of particle discreteness effects. As I have explained, the DIA contains both coherent and incoherent response. However, I confess to possibly having contributed to the confusion by coauthoring the paper by Krommes & Kleva (1979), in which a "coherent approximation to the DIA" was considered for the purpose of elucidating some specific aspects of weak-turbulence theory. (In that paper it was clearly pointed out that incoherent response was deliberately neglected in most of the discussion.)

7.3.1. Entropy, dissipation, and phase-space cascades

It is implicit in the spectral balance equations that a forced steady state cannot be achieved in the absence of dissipation; see, for example, the presence of ν^{-1} in Eq. (2.36b). A basic paper on the general relationship between dissipation, entropy, and spectral cascades is by Krommes & Hu (1994).[105] That work focused on fluid models. However, modern astrophysical and space-physics applications, enlightened by high-quality telescope and satellite data, have led to considerable interest in *cascades of free energy in phase space*. Some beautiful and detailed calculations were done by Schekochihin *et al.* (2008) and Schekochihin *et al.* (2009).

7.3.2. Optimal variables

The statistical closures described in this article are generally applied directly to a given PDE and are couched in terms of the original independent variables of that equation. It is not clear that this procedure is optimal from the point of view of capturing interesting physics. For example, the most elegant formulation of quasilinear theory is in terms of *oscillation-centre variables* (Dewar 1973) that remove the nonresonant particle oscillations in the wave fields. Dewar (1976, 1985) invested considerable effort in trying to generalize such techniques to turbulent situations. Although that work was not entirely successful, it was well motivated and highly original; some perspectives and discussion are given by Krommes (2012a). Further results in this area would be very welcome.

7.3.3. Fractional diffusion

While many aspects of turbulence phenomenology can be satisfactorily described in terms of classical Brownian motion, Langevin equations, and Fokker–Planck equations, others cannot. For example, if the conditions of the standard central limit theorem (which assumes finite variance) are violated, generalized random walks (Lévy processes) become possible. Those can be described by generalized Fokker–Planck equations that involve fractional derivatives. Good introductions are given by Klafter *et al.* (1996), Balescu (1997), Metzler & Klafter (2000), and Zaslavsky (2002). An example of work along these lines in the plasma context is by del Castillo-Negrete *et al.* (2004).

7.3.4. Variational methods

At the heart of the MSR formalism is a cumulant generating functional. Normally that is used to obtain formally closed Dyson equations that are then approximated. An alternate approach is to approximate the generating functional directly. One example of a serious attempt in that direction is by Spineanu & Vlad (2005).

7.3.5. Saddle-point methods

Although I have said little in this introductory article about methods aimed at entire PDFs, I do not mean to imply that they are not useful or unimportant. They are well suited to situations involving large intermittency, which manifests as non-Gaussian tails on PDFs. One technique for addressing that problem is the (saddle-point) method of instantons (Zinn-Justin 1996); for some plasma-related work, see, for example, Kim *et al.* (2003) or Anderson & Xanthopoulos (2010).

7.3.6. MHD turbulence

A major area of current interest is the description and role of electromagnetic effects in turbulence. Those are relevant to microturbulence in modern, high-pressure toroidal

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[105] The key balances discussed in that paper were verified by the numerical simulations of Watanabe & Sugama (2004) and Candy & Waltz (2006).

devices, and basic electromagnetic corrections to drift-wave theory were already discussed by Kadomtsev (p. 82). However, he specifically omitted discussion of "problems of astrophysical application" (p. 4). Recently the theory of macroscopic MHD turbulence, especially in astrophysics, has developed rapidly; some reviews are by Sridhar (2010) and Tobias *et al.* (2013).

7.3.7. Critical balance

MHD turbulence is anisotropic in the presence of a background magnetic field; gyrokinetic turbulence is also intrinsically anisotropic. While the statistical closures I have discussed in this article, such as the DIA, are generally valid for anisotropic turbulence, the lack of complete symmetry makes the extraction of practical information and understanding from them entirely nontrivial. A crucial part of the modern insights about anisotropic turbulence, both for MHD (Goldreich & Sridhar 1995) and more generally (Nazarenko & Schekochihin 2011), is the conjecture of *critical balance*. The concept is defined slightly differently depending on the author; in essence, the turbulence is supposed to adjust so that the correlation times associated with parallel and perpendicular effects are comparable. The idea has had a huge impact on one's understanding of both MHD and microturbulence in realistic, magnetized situations. For further references to the extensive MHD literature, see the above-mentioned reviews. In research on magnetic fusion, the conjecture is supported by both simulations (Barnes *et al.* 2011) and experiments (Grim *et al.* 2013). From the point of view of analytical methods, on which the present section 7.3 focuses, the justification of critical balance has been argued by detailed analyses of the nonlinear mode coupling (see especially the MHD references) as well as with the aid of stochastic-oscillator models (Lithwick & Goldreich 2003).

8. Summary; outstanding issues

The new material presented in this article includes (i) arguments based on Novikov's theorem that provide a heuristic, $\boldsymbol{x}$-space derivation of the direct-interaction approximation (Sec. 4.1); and (ii) an algorithmic approach to the determination of the dielectric response function $\mathcal{D}$ in the DIA to a general quadratically nonlinear PDE (Sec. 4.2).

The general goal of the tutorial was to describe some of the highlights of the statistical theory of turbulence, motivations being the seminal insights of Kadomtsev on strong turbulence and the pioneering attempts by Dupree and Weinstock to obtain and deal with the consequences of a nonlinear $\mathcal{D}$. I mentioned nonlinear incoherent noise and coherent damping (both present in passive problems) as well as self-consistent dielectric polarization. Dielectric response is subsumed by the infinitesimal response function R, which participates along with the covariance C in the steady-state spectral balance for a self-consistent turbulent system. An approximation to R also determines the dispersion relation for the bifurcation of homogeneous turbulence into inhomogeneous turbulence with nonvanishing mean fields; that is of great interest for the problem of zonal-flow generation. The CE2 and S3T closures (more generally, studies of statistical state dynamics) have had important successes in elucidating some physics of the interactions of zonal flows with turbulence.

Specific facts about $\mathcal{D}$ are as follows: (i) $\mathcal{D}^{-1}$ is related to the first-order mean response to perturbations of self-consistent turbulence. (ii) $\mathcal{D}$ is a nonlinear functional of the fluctuation spectrum through all orders. (iii) Nonlinear corrections to $\mathcal{D}$ are essential because they guarantee that the total growth rate (linear plus nonlinear) is negative in a steady state, as it must be to balance the positive-definite nonlinear noise that arises from mode coupling. (iv) In a kinetic description, $\mathcal{D}$ includes the wave–wave–particle

interactions of weak-turbulence theory as well as part of the n-wave coupling processes. (v) A $\mathcal{D}$ can be defined not only for plasmas but for turbulent fluids and additional nonlinear systems as well. (vi) $\mathcal{D}$ is subsumed by the infinitesimal response function R.

Specific facts about nonlinear noise are as follows: (i) Internal nonlinear noise is present in the statistical description of all nonlinear systems. (ii) Nonlinear noise is necessary in order to guarantee, along with coherent damping, the conservation of robust nonlinear invariants. (iii) Nonlinear noise includes part of the n-wave coupling processes of weak-turbulence theory. (iv) Furthermore, it is required in order to ensure the proper small-distance behaviour of relative diffusion coefficients.

Some outstanding challenges related to fundamental statistical descriptions include the following:

1. There is a difficult literature on the kinetic plasma DIA (DuBois & Pesme 1985) that has more or less ground to a halt in the absence of numerical solutions. It would be interesting to revisit that, given advances in modern computing; further insights about the interpretation of quasilinear theory (Laval & Pesme 1984) and the importance of mode coupling would be forthcoming.

2. Although it is clear how to define the operator $\Sigma^f \doteq \partial \delta \overline{f} \cdot \mathbb{E}$ that defines the nonlinear corrections to the background distribution function in the renormalized dielectric, the physical interpretation of those terms is still largely absent for kinetic problems (mostly for lack of trying, although see DuBois & Pesme 1985). A hint comes from the work of Krommes (2009), where it was shown how the notions of oscillation-centre variables and ponderomotive force are buried in the renormalizations. An outstanding question is whether such analysis can be extended to $\delta \overline{f}$. Results here would bear on the question of optimal variables and might suggest more physically useful approximations.

3. Second-order closures such as the DIA or its Markovian relatives are limited in their ability to deal with the consequences of coherent structures and intermittency (Chen *et al.* 1989; Krommes 1996), for which it is better to focus on entire PDFs. Kraichnan has proposed an ingenious *mapping closure* for PDFs (Kraichnan 1991; Das & Kaw 1995), but many questions remain open. For example, it would be of great interest to develop a workable mapping closure for the Hasegawa–Wakatani system. (There are difficulties associated with the treatment of the perpendicular Laplacian.)

4. The statistical theory of subcritical turbulence should be much further developed. One needs to meld insights and techniques from nonlinear dynamics (Waleffe 1995, 1997) with systematic statistical closure. Because non-normality is required for this phenomenon, some of the knowledge that has been gained from research on CE2/S3T closures might be useful.

5. The theory of the instability of homogeneous turbulence (Sec. 6.2) is in its infancy. Definite calculations have been done in the CE2 approximation; however, that neglects eddy–eddy interactions. The forced, dissipative CE2 solution for the homogeneous turbulence regime is physically simplistic; one should study perturbations of more realistic homogeneous steady states. One should also study the consequences of dispersion relations that go beyond Eq. (6.13). Such analysis should ultimately be conducted with models that include some toroidal effects.

6. Some solutions of the CE2/S3T closure reach a time-independent steady state (a stable fixed point), which seems to provide a reasonable explanation of the

basically time-independent zonal jets observed on the large planets. But in other regimes, solutions can be quasiperiodic or even stochastic (Farrell & Ioannou 2009a, 2015). Stochastic solutions would appear to be relevant to the kinds of microturbulence observed in magnetic confinement devices, but additional work needs to be done in order to understand the meaning of such solutions, their relation to homogeneous ensembles, and their implications for the physics of zonal flows in tokamaks.

7. Further research should be done to incorporate the salient effects of eddy–eddy interactions into generalizations of the CE2 closure. First steps have been taken by Marston et $al.$ (2015), but they raise a variety of questions that remain unanswered. It is tantalizing to contemplate workable hybrid numerical-plus-closure schemes, long a holy grail, that are built on CEn. Those too should be formulated in toroidal geometry.

In this tutorial I have covered considerable ground, from the simplest scaling estimates of transport coefficients to advanced, renormalized statistical closures. All of this is part of the subject, and a theorist wishing to work in the field should be aware of all of its facets. But it is worth inquiring again whether advanced formalism is really necessary. A stark comparison is between the physics of magnetically confined fusion plasmas and of the Lamb shift in quantum electrodynamics. Precise calculations of that shift are in agreement with measurements to a remarkable precision of perhaps ten kilohertz out of a gigahertz. That is a triumph that will never be matched in fusion research, where complicated geometries, rich mixtures of physical processes, and hostile environments for diagnostics lead to substantial uncertainties in both theoretical descriptions and experimental measurements. But appreciating the *structure* of the formalism is still important. For example, an understanding of the form and content of the DIA gives some perspective to the CE2/S3T closures, which in turn have been useful in interpreting direct numerical simulations and experimental observations of zonal flows in the natural world. Nevertheless, if the current state of statistical plasma turbulence theory does not satisfy your needs, I encourage you to work at improving it. You will enjoy the ride.

In conclusion, I have given a taste of the complications that arise when a statistical description of a nonlinear PDE is attempted. Kadomtsev closed his review article by saying (p. 139),

"One must hope that it will be possible in the course of the next few years of hectic development of the theory, in conjunction with detailed and accurate experiments, to set up a complete picture of the turbulence of plasmas."

Unfortunately, fifty years and billions of hours of supercomputer processor time later, we are still not there—but we are definitely closer. Perhaps this tutorial will provide a useful launch point for further study of the rich and challenging theory of turbulence in both plasmas and fluids.

Appendix A. Common quantities in drift-wave theory

A basic reference quantity in drift-wave theory is the sound speed $c_{\mathrm{s}} \doteq (ZT_e/m_i)^{1/2}$, where Z is the atomic number. From that and the ion gyrofrequency $\omega_{ci} \doteq (qB/mc)_i$, one can form the *sound radius* $\rho_{\mathrm{s}} \doteq c_{\mathrm{s}}/\omega_{ci}$. This is the characteristic fluctuation scale that appears in the Hasegawa–Mima equation (5.1).

Because the electron response to nonzonal perturbations is adiabatic (in the sense of slow variations), $\delta n_e/n_e \approx e\delta\phi/T_e$, it is useful to introduce the normalized potential $\varphi \doteq e\phi/T_e$. Then the $\boldsymbol{E}\times\boldsymbol{B}$ velocity is $\boldsymbol{V}_E \doteq (c/B)\widehat{\boldsymbol{b}}\times\boldsymbol{\nabla}\phi = D_{\mathrm{B}}\widehat{\boldsymbol{b}}\times\boldsymbol{\nabla}\varphi$, where the Bohm diffusion coefficient (*sans* factor of 1/16) is $D_{\mathrm{B}} \doteq cT_e/eB = \rho_{\mathrm{s}}c_{\mathrm{s}}$. The $\boldsymbol{E}\times\boldsymbol{B}$ advection of a background ion density profile, described by $\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\langle n_i\rangle$, thus becomes $\langle n_i\rangle V_*\partial_y\varphi$, where $V_* \doteq (\rho_{\mathrm{s}}/L_n)c_{\mathrm{s}} = cT_e/(eBL_n)$ and it is assumed that $\boldsymbol{\nabla}\ln\langle n_i\rangle = -L_n^{-1}\widehat{\boldsymbol{x}}$. [If one defines $V_{*s} \doteq -cT_s/(q_sBL_n)$, then $V_* \equiv V_{*e}$.] Although V_* is numerically equal to the diamagnetic flow speed of the electron fluid for constant T_e, the physics of the basic drift wave has nothing to do with electron diamagnetic flow; as discussed in Sec. 5.2.2, the appearance of V_* describes the generation of ion density fluctuations by $\boldsymbol{E}\times\boldsymbol{B}$ advection of the background ion density profile. The Fourier transform of $V_*\partial_y\varphi$ is $i\omega_*\varphi_{\boldsymbol{k}}$, where $\omega_* \doteq k_y V_*$ is known as the *diamagnetic frequency*. One has $\omega_* = k_y\rho_{\mathrm{s}}c_{\mathrm{s}}/L_n = O(c_{\mathrm{s}}/L_n)$ for fluctuations with $k_\perp\rho_{\mathrm{s}} = O(1)$. This identifies L_n/c_{s} (basically the time for a sound wave to cross the system) as a characteristic time for drift-wave dynamics.

Dimensional analysis of the HME shows that the scaling of the basic turbulent diffusion coefficient $D_\perp$ is not with D_{B} but rather with the *gyro-Bohm diffusion coefficient* $D_{\mathrm{gB}} \doteq (\rho_{\mathrm{s}}/L_n)D_{\mathrm{B}}$. This is much smaller than D_{B} and has the more favourable scaling B^{-2} instead of Bohm's B^{-1}. (In fact, $D_\perp$ vanishes altogether in the strict, time-reversible HME. However, the basic gyro-Bohm scaling is reinstated when growth and damping are introduced, as discussed in footnote 86 on page 50.)

Appendix B. The Furutsu–Novikov theorem

For completeness, I give here a brief derivation of Novikov's theorem. This is equivalent to the original procedure of Novikov (1964), but is algebraically more concise. Let $\widetilde{\phi}$ be a centred Gaussian random variable with variance σ^2, and let $\mathcal{F}(\widetilde{\phi})$ be an arbitrary differentiable function of $\widetilde{\phi}$. Then consider the average

$$\langle\mathcal{F}(\widetilde{\phi})\widetilde{\phi}\rangle \equiv \int_{-\infty}^{\infty} \mathrm{d}\phi\,\mathcal{F}(\phi)\phi\frac{\mathrm{e}^{-\phi^2/2\sigma^2}}{(2\pi\sigma^2)^{1/2}}. \tag{B1}$$

Upon using a generating-function approach, one has

$$\langle\mathcal{F}(\widetilde{\phi})\widetilde{\phi}\rangle = \frac{\partial}{\partial\eta}\langle\mathcal{F}(\widetilde{\phi})\mathrm{e}^{\widetilde{\phi}\eta}\rangle\Big|_{\eta=0}. \tag{B2}$$

Upon completing the square, one is led to

$$\langle\mathcal{F}(\widetilde{\phi})\mathrm{e}^{\widetilde{\phi}\eta}\rangle = \int_{-\infty}^{\infty}\mathrm{d}\phi\,\mathcal{F}(\phi)\frac{\mathrm{e}^{-(\phi-\sigma^2\eta)^2/2\sigma^2}}{(2\pi\sigma^2)^{1/2}}\mathrm{e}^{\sigma^2\eta^2/2} \tag{B3a}$$

$$= \langle\mathcal{F}(\widetilde{\phi}+\sigma^2\eta)\rangle\mathrm{e}^{\sigma^2\eta^2/2} \tag{B3b}$$

$$= \langle\mathcal{F}(\widetilde{\phi})\rangle + \left\langle\frac{\mathrm{d}\mathcal{F}(\widetilde{\phi})}{\mathrm{d}\widetilde{\phi}}\right\rangle\sigma^2\eta + O(\eta^2). \tag{B3c}$$

Upon evaluating the derivative required by Eq. (B 2), one finds the desired result

$$\langle \mathcal{F}(\widetilde{\phi})\widetilde{\phi}\rangle = \left\langle \frac{\mathrm{d}\mathcal{F}(\widetilde{\phi})}{\mathrm{d}\widetilde{\phi}}\right\rangle \sigma^2. \tag{B 4}$$

The essence of this procedure is the introduction of η to create an ensemble with nonzero mean, followed by inquiry about the differential behaviour of statistics in the modified ensemble. Nontrivial generalizations of this procedure are used in modern field-theoretic methods such as the MSR formalism (Martin *et al.* 1973).

The theorem can be generalized to a finite collection of multivariate centred Gaussian variables $\widetilde{\phi}_i$:

$$\langle \mathcal{F}(\widetilde{\boldsymbol{\phi}})\widetilde{\phi}_i\rangle = \sum_j \left\langle \frac{\partial \mathcal{F}}{\partial \widetilde{\phi}_j}\right\rangle C_{ji}, \tag{B 5}$$

where $C_{ij} \doteq \langle \delta\phi_i \delta\phi_j\rangle$ is the covariance matrix. Finally, a further generalization to a Gaussian random field $\widetilde{\phi}(\boldsymbol{x}, t) \equiv \widetilde{\phi}(x)$, where $\{\boldsymbol{x}, t\} \equiv x$ are nonrandom observer coordinates in spacetime, leads to

$$\langle \mathcal{F}[\widetilde{\phi}]\widetilde{\phi}(x)\rangle = \int_{-\infty}^{\infty} \mathrm{d}\overline{x} \left\langle \frac{\delta\mathcal{F}}{\delta\widetilde{\phi}(\overline{x})}\right\rangle C(\overline{x}, x), \tag{B 6}$$

where square brackets denote functional dependence, $\delta\mathcal{F}/\delta\widetilde{\phi}$ denotes the functional derivative [which satisfies $\delta\phi(x)/\delta\phi(x') = \delta(x - x')$], and $C(x, x') \doteq \langle \delta\phi(x)\delta\phi(x')\rangle$.

If $\widetilde{\phi}$ has nonzero mean, it is easy to show that the previous results generalize straightforwardly to include the term $\langle \mathcal{F}[\widetilde{\phi}]\rangle\langle\widetilde{\phi}\rangle$.

Appendix C. Notation

— Symbols —

a — acceleration

α — switch that vanishes for zonal modes and equals one otherwise

$\boldsymbol{B}$ — magnetic field

$\widehat{\boldsymbol{b}}$ — unit vector in direction of $\boldsymbol{B}$

β — standard deviation of random coefficient at infinite $\mathcal{K}$

$C(1, 2)$ — two-point correlation function

c — speed of light

c_{s} — sound speed: $c_{\mathrm{s}} \doteq (ZT_e/m_i)^{1/2}$

χ — propagator

$\boldsymbol{D}$ — dielectric tensor

$\boldsymbol{D}$ — electric displacement: $\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}$

D — diffusion coefficient

D_v — velocity-space diffusion coefficient

D_{B} — Bohm diffusion coefficient: $D_{\mathrm{B}} \doteq cT_e/eB$

D_{gB} — gyro-Bohm diffusion coefficient: $D_{\mathrm{gB}} \doteq (\rho_{\mathrm{s}}/L_n)D_{\mathrm{B}}$

d — dimensionality of space

Δ — infinitesimal perturbation

Δt — characteristic time interval or mode period

δ — nonadiabatic response $\times$ i

$\delta\overline{f}$ — nonlinear correction to distribution function

δk — mode spacing: $\delta k \doteq 2\pi/L$

$\mathcal{D}$ — dielectric function

$\mathcal{D}_\perp$ — permittivity of gyrokinetic vacuum: $\mathcal{D}_\perp \doteq \rho_{\mathrm{s}}^2/\lambda_{\mathrm{D}e}^2$

$\boldsymbol{\partial}$ — $(q/m)\partial/\partial\boldsymbol{v}$

$\boldsymbol{E}$ — electric field

$\mathbb{E}$ — electric-field operator: $\boldsymbol{E} = \mathbb{E}f$

$\mathcal{E}$ — fluctuation spectrum of electric field

e — magnitude of electron charge

ϵ — positive infinitesimal

ϵ_{p} — plasma parameter: $\epsilon_{\mathrm{p}} \doteq 1/(n\lambda_{\mathrm{D}}^3)$

ε — rate of energy input

η — nonlinear coherent damping

$\widehat{\eta}$ — additive source in mean equation

F — covariance of random forcing or incoherent noise; gyrocenter distribution function

f — one-particle distribution function

$\overline{f}$ — effective distribution in renormalized theory (not a PDF)

$\widetilde{f}$ — random forcing (covariance F)

g — particle Green's function

Γ — flux (e.g., of particles)

γ — growth rate

H — Heaviside unit step function

$\mathcal{K}$ — Kubo number

$\boldsymbol{k}$ — Fourier wave vector conjugate to position $\boldsymbol{x}$

$\overline{k}$ — characteristic wave number

k_{D} — Debye wave number: $k_{\mathrm{D}}^2 \doteq \sum_s k_{\mathrm{D}s}^2$, where $k_{\mathrm{D}s} \doteq [4\pi(nq^2/T)_s]^{1/2}$

κ — L_n^{-1}

L — box size

$L_{\boldsymbol{k}}$ — linear operator

L_n — density gradient scale length

L_{ac} — autocorrelation length

ℓ — mixing length

λ_{D} — Debye length: $\lambda_{\mathrm{D}} \doteq k_{\mathrm{D}}^{-1}$

M — mode-coupling coefficient

m — mass

μ — kinematic viscosity

$\widetilde{N}$ — Klimontovich microdensity

$\mathcal{N}$ — numerator of spectral balance relation; nonlinear noise

n — density

$\overline{n}$ — mean density

ν — drag or damping coefficient

$\Omega_{\boldsymbol{k}}$ — real part of mode frequency

ω — Fourier variable conjugate to time t

ω_{c} — gyrofrequency: $\omega_{cs} \doteq q_s B/m_s c$

ω_* — diamagnetic frequency: $\omega_* \doteq k_y V_*$

$\omega_{\boldsymbol{k}}$ — complex mode frequency: $\omega_{\boldsymbol{k}} \doteq \Omega_{\boldsymbol{k}} + i\gamma_{\boldsymbol{k}}$

$\boldsymbol{P}$ — polarization vector

P — pressure

P — projection operator

$\boldsymbol{p}$ — wave vector

ϕ — electrostatic potential

φ — normalized potential: $\varphi \doteq e\phi/T_e$

Q — orthogonal projection operator

$\boldsymbol{q}$ — wave vector (often used for zonal modes)

q — particle charge ($q_e = -e$)

R — mean infinitesimal response function

$\mathcal{R}$ — Reynolds number

ρ — charge density; gyroradius: $\rho \doteq v_\perp/\omega_{\mathrm{c}}$

ρ_{m} — mass density: $\rho_{\mathrm{m}} \doteq nm$

ρ_{s} — sound radius: $\rho_{\mathrm{s}} \doteq c_{\mathrm{s}}/\omega_{ci}$

$\boldsymbol{S}$ — rate-of-strain tensor

s — species index

Σ — mass operator (coherent turbulent collision operator)

σ — weight factor relating covariance to energy

T — temperature

$T(1,2,3)$ — three-point correlation function

t — time

$\boldsymbol{\tau}$ — Reynolds stress

τ_{ac} — autocorrelation time

θ — triad interaction time

U_n — nth-order coupling coefficient

$\boldsymbol{u}$ — fluid velocity

$\boldsymbol{V}_E$ — $\boldsymbol{E} \times \boldsymbol{B}$ velocity: $\boldsymbol{V}_E \doteq (c/B)\boldsymbol{E} \times \widehat{\boldsymbol{b}}$

V — volume of entire system

$\overline{V}$ — characteristic velocity

V_* — diamagnetic speed

$\boldsymbol{v}$ — particle velocity

v_{t} — thermal velocity: $v_{\mathrm{t}} \doteq (T/m)^{1/2}$

W — covariance of random coefficient

$w(z)$ — $Z(z)/(\mathrm{i}\sqrt{\pi})$

$\boldsymbol{X}$ — susceptibility tensor

ξ — auxiliary random variable in Langevin representation

Z — atomic number

$Z(z)$ — plasma dispersion function

$Z(1)$ — mean field

— **Miscellaneous notation** —

$\widetilde{A}$ — random variable

$\widetilde{A}'$ — fluctuation: $\widetilde{A}' \doteq \widetilde{A} - \langle\widetilde{A}\rangle$. See also δA.

$\widehat{A}(\boldsymbol{k},\omega)$ — Fourier transform of $A(\boldsymbol{x},t)$

68

A^*	— complex conjugate of A

A^* — complex conjugate of A
$A_+(t)$ — $H(t)A(t)$
$A_\parallel$ — $\boldsymbol{A} \cdot \widehat{\boldsymbol{b}}$
$\boldsymbol{A}_\perp$ — $\boldsymbol{A} - A_\parallel \widehat{\boldsymbol{b}} = \widehat{\boldsymbol{b}} \times (\boldsymbol{A} \times \widehat{\boldsymbol{b}})$
$A[\psi]$ — A is a functional of ψ
$\boldsymbol{A}^{\mathrm{T}}$ — transpose of $\boldsymbol{A}$
δA — fluctuation: $\delta A \doteq \widetilde{A} - \langle \widetilde{A} \rangle$. See also A'.
$\delta(\tau)$ — Dirac delta function
$\delta_{i,j}$ — Kronecker delta function
$\delta/\delta\eta$ — functional derivative
$\psi(1)$ — $\psi_{s_1}(\boldsymbol{x}_1, \boldsymbol{v}_1, t_1)$
$\doteq$ — definition
$\langle \ldots \rangle$ — ensemble average

— **Abbreviations and acronyms** —

CEn — nth-order cumulant expansion
DIA — Direct-interaction approximation
ext — external
int — internal
lin — linear
pol — polarization
PDF — probability density function
RBT — resonance-broadening theory
S3T — stochastic structural stability theory (also SSST)
tot — total
ZF — zonal flow

REFERENCES

ABEL, I. G., PLUNK, G. G., WANG, E., BARNES, M., COWLEY, S. C., DORLAND, W. & SCHEKOCHIHIN, A. A. 2013 Multiscale gyrokinetics for rotating tokamak plasmas: Fluctuations, transport and energy flows. *Rep. Prog. Phys.* **76**, 116201 (69 pages).

ANDERSON, J. & XANTHOPOULOS, P. 2010 Signature of a universal statistical description for drift-wave plasma turbulence. *Phys. Plasmas* **17**, 110702 (4 pages).

BAKAS, N. A., CONSTANTINOU, N. C. & IOANNOU, P. J. 2015 S3T stability of the homogeneous state of barotropic beta-plane turbulence. *J. Atmos. Sci.* **72**, 1689–1712.

BALESCU, R. 1997 *Statistical Dynamics: Matter out of Equilibrium*. London: Imperial College Press.

BALL, J., PARRA, F. I., BARNES, M., DORLAND, W., HAMMETT, G. W., RODRIGUES, P. & LOUREIRO, N. F. 2014 Intrinsic momentum transport in up–down asymmetric tokamaks. *Plasma Phys. Control. Fusion* **56**, 095014 (20 pages).

BARENBLATT, G. I. 1996 *Scaling, Self-similarity, and Intermediate Asymptotics*. Cambridge: Cambridge University Press.

BARNES, M., PARRA, F. I. & SCHEKOCHIHIN, A. A. 2011 Critically balanced ion temperature gradient turbulence in fusion plasmas. *Phys. Rev. Lett.* **107**, 115003 (5 pages).

BERAN, M. M. 1968 *Statistical Continuum Theories*. New York: Interscience.

BIRDSALL, C. K. & LANGDON, A. B. 1985 *Plasma Physics via Computer Simulation*. New York: McGraw–Hill.

BOFFETTA, G. & ECKE, R. E. 2012 Two-dimensional turbulence. *Annu. Rev. Fluid Mech.* **44**, 427–451.

BOHM, D. 1949 Qualitative description of the arc plasma in a magnetic field. In *The Characteristics of Electrical Discharges in Magnetic Fields* (ed. A. Guthrie & R. K. Wakerling), chap. 1, pp. 1–12. New York: McGraw–Hill.

BOUTROS-GHALI, T. & DUPREE, T. H. 1981 Theory of two-point correlation function in a Vlasov plasma. *Phys. Fluids* **24**, 1839–1858.

BOWMAN, J. C. 1992 Realizable Markovian Statistical Closures: General Theory and Application to Drift-wave Turbulence. PhD thesis, Princeton University.

BOWMAN, J. C. & KROMMES, J. A. 1997 The realizable Markovian closure and realizable test-field model. II: Application to anisotropic drift-wave turbulence. *Phys. Plasmas* **4**, 3895–3909.

BOWMAN, J. C., KROMMES, J. A. & OTTAVIANI, M. 1993 The realizable Markovian closure. I. General theory, with application to three-wave dynamics. *Phys. Fluids B* **5**, 3558–3589.

BRIZARD, A. J. & HAHM, T. S. 2007 Foundations of nonlinear gyrokinetic theory. *Rev. Mod. Phys.* **79**, 421–468.

Brown, L. M., ed. 1993 *Renormalization: From Lorentz to Landau (and Beyond)*. New York: Springer.

Brown, R. 1828 A brief account of microscopical observations made in the months of June, July, and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *Philos. Mag.* **4**, 161–173.

Candy, J. & Waltz, R. E. 2006 Velocity-space resolution, entropy production, and upwind dissipation in Eulerian gyrokinetic simulations. *Phys. Plasmas* **13**, 032310 (11 pages).

Carnevale, G. F. & Martin, P. C. 1982 Field theoretical techniques in statistical fluid dynamics: With application to nonlinear wave dynamics. *Geophys. Astrophys. Fluid Dyn.* **20**, 131–163.

del Castillo-Negrete, D., Carreras, B. A. & Lynch, V. E. 2004 Fractional diffusion in plasma turbulence. *Phys. Plasmas* **11**, 3854–3864.

Catto, P. J. 1978*a* Adiabatic modifications to plasma turbulence theories. *Phys. Fluids* **21**, 147–148.

Catto, P. J. 1978*b* Linearized gyro-kinetics. *Plasma Phys.* **20**, 719–722.

Cercignani, C. 1998 *Ludwig Boltzmann: The man who trusted atoms*. Oxford: Clarendon.

Charney, J. G. & Stern, M. E. 1962 On the stability of internal baroclinic jets in a rotating atmosphere. *J. Atmos. Sci.* **19**, 159–172.

Chen, H., Herring, J. R., Kerr, R. M. & Kraichnan, R. H. 1989 Non-Gaussian statistics in isotropic turbulence. *Phys. Fluids A* **1**, 1844–1854.

Chirikov, B. V. 1969 Research concerning the theory of non-linear resonance and stochasticity. Nuclear Physics Institute of the Siberian Section of the USSR Academy of Sciences Report 267, translated by A. T. Sanders (CERN Trans. 71–40, 1971).

Connaughton, C. P., Nadiga, B. T., Nazarenko, S. V. & Quinn, B. E. 2010 Modulational instability of Rossby and drift waves and generation of zonal jets. *J. Fluid Mech.* **654**, 207–231.

Connor, J. W. 1988 Invariance principles and plasma confinement. *Plasma Phys. Control. Fusion* **30**, 619–650.

Connor, J. W. & Pogutse, O. P. 2001 On the relationship between mixing length and strong turbulence estimates for transport due to drift turbulence. *Plasma Phys. Control. Fusion* **43**, 155–175.

Connor, J. W. & Taylor, J. B. 1977 Scaling laws for plasma confinement. *Nucl. Fusion* **17**, 1047–1055.

Cook, I. 1978 Application of the Novikov–Furutsu theorem to the random acceleration problem. *Plasma Phys.* **20**, 349–359.

Das, A. & Kaw, P. 1995 Mapping closure for probability distribution function in low frequency magnetized plasma turbulence. *Phys. Plasmas* **2**, 1497–1505.

Davidson, R. C. 1972 *Methods in Nonlinear Plasma Theory*. New York: Academic.

Dawson, J. 1962 Investigation of the double-stream instability. In *Proceedings of the Conference on Plasma Physics and Controlled Nuclear Fusion Research (Salzburg, 1961) [Nucl. Fusion Suppl. Pt. 3]*, pp. 1033–1043. Vienna: International Atomic Energy Agency.

DelSole, T. 2001 A theory for the forcing and dissipation in stochastic turbulence models. *J. Atmos. Sci.* **58**, 3762–3775.

Dewar, R. L. 1973 Oscillation center quasilinear theory. *Phys. Fluids* **16**, 1102–1107.

Dewar, R. L. 1976 Renormalized canonical perturbation theory for stochastic propagators. *J. Phys. A* **9**, 2043–2057.

Dewar, R. L. 1985 Optimal oscillation-center transformations. *Physica D* **17D**, 37–53.

Dewar, R. L. & Abdullatif, R. F. 2006 Zonal flow generation by modulational instability. In *Turbulence and Coherent Structures in Fluids, Plasmas and Nonlinear Mediums: Proceedings of the COSNET Workshop* (ed. J. Denier & J. Frederiksen), pp. 415–430. Singapore: World Scientific.

Diacu, F. & Holmes, P. 1996 *Celestial Encounters: The Origins of Chaos and Stability*. Princeton, NJ: Princeton University Press.

Diamond, P. H. & Carreras, B. A. 1987 On mixing length theory and saturated turbulence. *Comm. Plasma Phys. Control. Fusion* **10**, 271–278.

Diamond, P. H., Itoh, S.-I. & Itoh, K. 2010 *Modern Plasma Physics*, vol. 1: Physical Kinetics of Turbulent Plasmas. Cambridge: Cambridge University Press.

DIAMOND, P. H., ITOH, S.-I., ITOH, K. & HAHM, T. S. 2005 Zonal flows in plasma—a review. *Plasma Phys. Control. Fusion* **47** (5), R35–R161.

DIAMOND, P. H., ROSENBLUTH, M. N., HINTON, F. L., MALKOV, M., FLEISCHER, J. & SMOLYAKOV, A. 1998 Dynamics of zonal flows and self-regulating drift-wave turbulence. In *17th IAEA Fusion Energy Conference*, pp. 1421–1428. Vienna: International Atomic Energy Agency, IAEA–CN–69/TH3/1.

DIMITS, A. M., BATEMAN, G., BEER, M. A., COHEN, B. I., DORLAND, W., HAMMETT, G. W., KIM, C., KINSEY, J. E., KOTSCHENREUTHER, M., KRITZ, A. H., LAO, L. L., MANDREKAS, J., NEVINS, W. M., PARKER, S. E., REDD, A. J., SHUMAKER, D. E., SYDORA, R. & WEILAND, J. 2000 Comparisons and physics basis of tokamak transport models and turbulence simulations. *Phys. Plasmas* **7**, 969–983.

DODIN, I. Y. 2014a Geometric view on noneikonal waves. *Phys. Lett. A* **378**, 1598–1621.

DODIN, I. Y. 2014b On variational methods in the physics of plasma waves. *Fusion Sci. Technol.* **65**, 54–78.

DORLAND, W. & HAMMETT, G. W. 1993 Gyrofluid turbulence models with kinetic effects. *Phys. Fluids B* **5**, 812–835.

DRAKE, J. F., ZEILER, A. & BISKAMP, D. 1995 Nonlinear self-sustained drift-wave turbulence. *Phys. Rev. Lett.* **75**, 4222–4225.

DRUMMOND, W. E. 1966 Review of *Plasma Turbulence* by B. B. Kadomtsev. *Am. Sci.* **54**, 350A–352A.

DRUMMOND, W. E. & PINES, D. 1962 Non-linear stability of plasma oscillations. In *Proceedings of the Conference on Plasma Physics and Controlled Nuclear Fusion Research (Salzburg, 1961) [Nucl. Fusion Suppl. Pt. 3]*, pp. 1049–1057. Vienna: International Atomic Energy Agency.

DUBIN, D. H. E., KROMMES, J. A., OBERMAN, C. R. & LEE, W. W. 1983 Nonlinear gyrokinetic equations. *Phys. Fluids* **26**, 3524–3535.

DUBOIS, D. F. & ESPEDAL, M. 1978 Direct interaction approximation and plasma turbulence theory. *Plasma Phys.* **20**, 1209–1239.

DUBOIS, D. F. & PESME, D. 1985 Direct interaction approximation for Vlasov turbulence from the Kadomtsev weak coupling approximation. *Phys. Fluids* **28**, 1305–1317.

DUPREE, T. H. 1966 A perturbation theory for strong plasma turbulence. *Phys. Fluids* **9**, 1773–1782.

DUPREE, T. H. 1967 Nonlinear theory of drift-wave turbulence and enhanced diffusion. *Phys. Fluids* **10**, 1049–1055.

DUPREE, T. H. 1972 Theory of phase space density granulation in plasma. *Phys. Fluids* **15**, 334–344.

DUPREE, T. H. & TETREAULT, D. J. 1978 Renormalized dielectric function for collisionless drift wave turbulence. *Phys. Fluids* **21**, 425–433.

DYSON, F. J. 1949 The radiation theories of Tomonaga, Schwinger, and Feynman. *Phys. Rev.* **75**, 486–502.

ECKER, G. 1972 *Theory of Fully Ionized Plasmas*. New York: Academic.

EINSTEIN, A. 1905 Über die von der molekularkinetischen Theori der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Ann. Phys. (Leipzig)* **17**, 549–60, A. Einstein, *Investigations on the theory of the Brownian Movement*, edited by R. Fürth, translated by A. D. Cowper (Dover, New York, 1956), pp. 1–18.

EYINK, G. & FRISCH, U. 2011 Robert H. Kraichnan. In *A Voyage Through Turbulence* (ed. P. A. Davidson, Y. Kaneda, K. Moffatt & K. R. Sreenivasan), chap. 10, pp. 329–372. Cambridge: Cambridge University Press.

EYINK, G. L. & SREENIVASAN, K. R. 2006 Onsager and the theory of hydrodynamic turbulence. *Rev. Mod. Phys.* **78**, 87–135.

FALKOVICH, G. 2006 Introduction to developed turbulence. In *Turbulence and Coherent Structures in Fluids, Plasmas and Nonlinear Media* (ed. M. Shats & H. Punzmann), pp. 1–20. Singapore: World Scientific.

FALKOVICH, G., GAWĘDZKI, K. & VERGASSOLA, M. 2001 Particles and fields in fluid turbulence. *Rev. Mod. Phys.* **73**, 913–975.

FARRELL, B. F. & IOANNOU, P. J. 2003 Structural stability of turbulent jets. *J. Atmos. Sci.* **60**, 2101–2118.

FARRELL, B. F. & IOANNOU, P. J. 2009a A stochastic structural stability theory model of the drift wave–zonal flow system. *Phys. Plasmas* **16**, 112903 (19 pages).

FARRELL, B. F. & IOANNOU, P. J. 2009b A theory of baroclinic turbulence. *J. Atmos. Sci.* **66**, 2444–2454.

FARRELL, B. F. & IOANNOU, P. J. 2015 Statistical State Dynamics: a new perspective on turbulence in shear flow. In *Zonal Jets* (ed. B. Galperin & P. Read), chap. V.2.2. Cambridge: Cambridge University Press, in press.

FOX, R. F. 1978 Gaussian stochastic processes in physics. *Phys. Rev.* **48**, 179–283.

FOX, R. F. & UHLENBECK, G. E. 1970 Contributions to non-equilibrium thermodynamics. I. Theory of hydrodynamical fluctuations. *Phys. Fluids* **13**, 1893–1902.

FREDERIKSEN, J. S., DAVIES, A. G. & BELL, R. C. 1994 Closure theories with non-Gaussian restarts for truncated two-dimensional turbulence. *Phys. Fluids* **6**, 3153–3163.

FRIEDLANDER, S. K. & TOPPER, L., ed. 1961 *Turbulence: Classic Papers on Statistical Theory*, New York. Interscience.

FRIEMAN, E. A. & CHEN, L. 1982 Nonlinear gyrokinetic equations for low-frequency electromagnetic waves in general plasma equilibria. *Phys. Fluids* **25**, 502–508.

FRISCH, U. 1995 *Turbulence*. Cambridge: Cambridge University Press.

FRISCH, U. & BOURRET, R. 1970 Parastochastics. *J. Math. Phys.* **11**, 364–390.

FUJISAWA, A. 2009 A review of zonal flow experiments. *Nucl. Fusion* **49**, 013001 (42 pages).

GALPERIN, B. & READ, P., ed. 2015 *Zonal Jets*. Cambridge: Cambridge University Press, in press.

GARBET, X., IDOMURA, Y., VILLARD, L. & WATANABE, T. H. 2010 Gyrokinetic simulations of turbulent transport. *Nucl. Fusion* **50**, 043002 (30 pages).

GARDINER, C. W. 2004 *Handbook of Stochastic Methods*, 3rd edn. Berlin: Springer.

GERMANO, M., PIOMELLI, U., MOIN, P. & CABOT, W. H. 1991 A dynamic subgrid-scale eddy viscosity model. *Phys. Fluids A* **3**, 1760–1765.

GOLDENFELD, N. 1992 *Lectures on Phase Transitions and the Renormalization Group*. Reading, MA: Addison–Wesley.

GOLDREICH, P. & SRIDHAR, S. 1995 Toward a theory of interstellar turbulence. II. Strong Alfvénic turbulence. *Ap. J.* **438**, 763–775.

GRANT, H. L., STEWART, R. W. & MOILLIET, A. 1962 Turbulence spectra from a tidal channel. *J. Fluid Mech.* **12**, 241–268.

GRIM, Y.-C., SCHEKOCHIHIN, A. A., FIELD, A. R., ABEL, I. G., BARNES, M., COLYER, G., COWLEY, S. C., PARRA, F. I., DUNAI, D. & ZOLETNIK, S. 2013 Experimental signatures of critically balanced turbulence in MAST. *Phys. Rev. Lett.* **110**, 145002 (6 pages).

GUCKENHEIMER, J. & HOLMES, P. 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. New York: Springer.

HAMMETT, G. W., BEER, M. A., DORLAND, W., COWLEY, S. C. & SMITH, S. A. 1993 Developments in the gyrofluid approach to tokamak turbulence simulations. *Plasma Phys. Control. Fusion* **35**, 973–985.

HAMMETT, G. W., DORLAND, W. & PERKINS, F. W. 1992 Fluid models of phase mixing, Landau damping, and nonlinear gyrokinetic dynamics. *Phys. Fluids B* **4**, 2052–2061.

HAMMETT, G. W. & PERKINS, F. W. 1990 Fluid moment models for Landau damping with application to the ion-temperature-gradient instability. *Phys. Rev. Lett.* **64**, 3019–3022.

HASEGAWA, A. & MIMA, K. 1978 Pseudo-three-dimensional turbulence in magnetized nonuniform plasma. *Phys. Fluids* **21**, 87–92.

HASEGAWA, A. & WAKATANI, M. 1983 Plasma edge turbulence. *Phys. Rev. Lett.* **50**, 682–686.

HATCH, D. R., TERRY, P. W., JENKO, F., MERZ, F. & NEVINS, W. M. 2011a Saturation of gyrokinetic turbulence through damped eigenmodes. *Phys. Rev. Lett.* **106**, 115003 (4 pages).

HATCH, D. R., TERRY, P. W., JENKO, F., MERZ, F., PUESCHEL, M. J., NEVINS, W. M. & WANG, E. 2011b Role of subdominant stable modes in plasma microturbulence. *Phys. Plasmas* **18**, 055706 (11 pages).

HERSHCOVITCH, A. & POLITZER, P. A. 1979 Suppression of an instability by the introduction of external turbulence. *Phys. Fluids* **22**, 249–256.

HINTON, F. L. & HORTON, JR., C. W. 1971 Amplitude limitation of a collisional drift wave instability. *Phys. Fluids* **14**, 116–123.

HOLMES, P., LUMLEY, J. L. & BERKOOZ, G. 1996 *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*. Cambridge: Cambridge University Press.

HORTON, W. 2012 *Turbulent Transport in Magnetized Plasmas*. Singapore: World Scientific.

HORTON, W. & CHOI, D. I. 1979 Renormalized turbulence theory for the ion acoustic problem. *Phys. Rep.* **49**, 273–410.

HU, G., KROMMES, J. A. & BOWMAN, J. C. 1995 Resistive drift-wave plasma turbulence and the realizable Markovian closure. *Phys. Lett. A* **202**, 117–125.

HU, G., KROMMES, J. A. & BOWMAN, J. C. 1997 Statistical theory of resistive drift-wave turbulence and transport. *Phys. Plasmas* **4**, 2116–2133.

ICHIMARU, S. 1973 *Basic Principles of Plasma Physics—A Statistical Approach*. New York: Benjamin.

ITOH, K., ITOH, S.-I. & FUKUYAMA, A. 1999 *Transport and Structural Formation in Plasmas*. Bristol: Institute of Physics Publishing.

ITOH, K., ITOH, S.-I., YAGI, M. & FUKUYAMA, A. 1996 Subcritical excitation of plasma turbulence. *J. Phys. Soc. Jpn.* **65**, 2749–2752.

KADOMTSEV, B. B. 1962 Anomalous diffusion of a low-density current-carrying plasma in a magnetic field. *Zh. Eksp. Teor. Fiz.* **43**, 1688–1696 [Sov. Phys. JETP **16**, 1191–1197 (1963)].

KADOMTSEV, B. B. 1965 *Plasma Turbulence*. New York: Academic, translated by L. C. Ronson from *Problems in Plasma Theory*, Vol. 4, edited by M. A. Leontoich (Atomizdat, Moscow, 1964) pp. 188–339. Translation edited by M. C. Rusbridge.

KADOMTSEV, B. B. & PETVIASHVILI, V. J. 1963 Weakly turbulent plasma in a magnetic field. *Zh. Eksp. Teor. Fiz.* **43**, 2234 [Sov. Phys. JETP **16**, 1578–1585 (1963)].

KADOMTSEV, B. B. & POGUTSE, O. P. 1970 Turbulence in toroidal systems. In *Reviews of Plasma Physics, Vol. 5*, p. 249. New York: Consultants Bureau.

KAUFMAN, A. N. 1972 Reformulation of quasi-linear theory. *J. Plasma Phys.* **8**, 1–5.

KIM, E. J. & DIAMOND, P. H. 2004 Random shearing by zonal flows and transport reduction. *Phys. Plasmas* **11**, L77–L80.

KIM, EUN-JIN, DIAMOND, P. H., MALKOV, M., HAHM, T. S., ITOH, K., ITOH, S.-I., CHAMPEAUX, S., GRUZINOV, I., GURCAN, O., HOLLAND, C., ROSENBLUTH, M. N. & SMOLYAKOV, A. 2003 Non-perturbative models of intermittency in drift-wave turbulence: towards a probabilistic theory of anomalous transport. *Nucl. Fusion* **43**, 961–968.

KLAFTER, J., SHLESINGER, M. F. & ZUMOFEN, G. 1996 Beyond Brownian motion. *Phys. Today* **49** (2), 33–39.

KLIMONTOVICH, Y. L. 1967 *The Statistical Theory of Non-equilibrium Processes in a Plasma*. Cambridge, MA: MIT Press, edited by D. ter Harr, translated by H. S. H. Massey and O. M. Blunn.

KOLESNIKOV, R. A. & KROMMES, J. A. 2005a Bifurcation theory of the transition to collisionless ion-temperature-gradient-driven plasma turbulence. *Phys. Plasmas* **12**, 122302 (25 pages).

KOLESNIKOV, R. A. & KROMMES, J. A. 2005b The transition to collisionless ion-temperature-gradient-driven plasma turbulence: A dynamical systems approach. *Phys. Rev. Lett.* **94**, 235002 (4 pages).

KOLMOGOROV, A. N. 1941 The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *C. R. Acad. Sci. USSR* **30**, 301–305, reproduced with emendations in Proc. R. Soc. London, Ser. A **434**, 9–13 (1991).

KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5**, 497–543.

KRAICHNAN, R. H. 1961 Dynamics of nonlinear stochastic systems. *J. Math. Phys.* **2**, 124–148; **3**, 205(E).

KRAICHNAN, R. H. 1962 The closure problem of turbulence theory. *Proc. Symp. Appl. Math.* **13**, 199–225.

KRAICHNAN, R. H. 1964a Approximations for steady-state isotropic turbulence. *Phys. Fluids* **7**, 1163–1168.

KRAICHNAN, R. H. 1964b Direct-interaction approximation for shear and thermally driven turbulence. *Phys. Fluids* **7**, 1048–1062.

KRAICHNAN, R. H. 1964c Kolmogorov's hypotheses and Eulerian turbulence theory. *Phys. Fluids* **7**, 1723–1734.

KRAICHNAN, R. H. 1965 Lagrangian-history closure approximation for turbulence. *Phys. Fluids* **8**, 575–598; **9**, 1884(E).

KRAICHNAN, R. H. 1970 Convergents to turbulence functions. *J. Fluid Mech.* **41**, 189–217.

KRAICHNAN, R. H. 1971 An almost-Markovian Galilean-invariant turbulence model. *J. Fluid Mech.* **47**, 513–524.

KRAICHNAN, R. H. 1977 Eulerian and Lagrangian renormalization in turbulence theory. *J. Fluid Mech.* **83**, 349–374.

KRAICHNAN, R. H. 1980 Realizability inequalities and closed moment equations. In *Nonlinear Dynamics* (ed. R. H. G. Helleman), pp. 37–46. New York: New York Academy of Sciences.

KRAICHNAN, R. H. 1991 Stochastic modeling of isotropic turbulence. In *New Perspectives in Turbulence* (ed. L. Sirovich), chap. 1, pp. 1–54. New York: Springer.

KRAICHNAN, R. H. 1994 Anomalous scaling of a randomly advected passive scalar. *Phys. Rev. Lett.* **72**, 1016–1019.

KROMMES, J. A. 1978 Turbulence, clumps, and the Bethe–Salpeter equation. In *Theoretical and Computational Plasma Physics*, pp. 405–417. Vienna: International Atomic Energy Agency.

KROMMES, J. A. 1981 Comments on 'Adiabatic modifications to plasma turbulence theory'. *Phys. Fluids* **24**, 1762.

KROMMES, J. A. 1982 The direct-interaction approximation and statistically steady states of three nonlinearly coupled modes. *Phys. Fluids* **25**, 1393–1395.

KROMMES, J. A. 1984 Statistical descriptions and plasma physics. In *Handbook of Plasma Physics* (ed. A. A. Galeev & R. N. Sudan), vol. 2, chap. 5.5, pp. 183–268. Amsterdam: North–Holland.

KROMMES, J. A. 1993 Dielectric response and thermal fluctuations in gyrokinetic plasma. *Phys. Fluids B* **5**, 1066–1100.

KROMMES, J. A. 1996 Non-Gaussian statistics, classical field theory, and realizable Langevin models. *Phys. Rev. E* **53**, 4865–4882.

KROMMES, J. A. 1997 The clump lifetime revisited: Exact calculation of the second-order structure function for a model of forced, dissipative turbulence. *Phys. Plasmas* **4**, 655–673.

KROMMES, J. A. 2002 Fundamental statistical theories of plasma turbulence in magnetic fields. *Phys. Rep.* **360**, 1–352.

KROMMES, J. A. 2006 Analytical descriptions of plasma turbulence. In *Turbulence and Coherent Structures in Fluids, Plasmas and Nonlinear Media* (ed. M. Shats & H. Punzmann), pp. 115–232. Singapore: World Scientific.

KROMMES, J. A. 2007 Nonequilibrium gyrokinetic fluctuation theory and sampling noise in gyrokinetic particle-in-cell simulations. *Phys. Plasmas* **14**, 090501 (26 pages).

KROMMES, J. A. 2009 Nonlinear theoretical tools for fusion-related microturbulence: Historical evolution, and recent applications to stochastic magnetic fields, zonal-flow dynamics, and intermittency. *J. Phys. Conf. Ser.* **169**, 012005 (35 pages).

KROMMES, J. A. 2012a Bob Dewar and turbulence theory: Lessons in creativity and courage. *Plasma Phys. Control. Fusion* **54**, 014001 (8 pages).

KROMMES, J. A. 2012b The gyrokinetic description of microturbulence in magnetized plasmas. *Annu. Rev. Fluid Mech.* **44**, 175–201.

KROMMES, J. A. & HAMMETT, G. W. 2013 Report of the Study Group GK2 on Momentum Transport in Gyrokinetics. *Tech. Rep.* PPPL–4945. Plasma Physics Laboratory, Princeton University, `http://bp.pppl.gov/pub_report/2014/PPPL-4945-abs.html`, 147 pages.

KROMMES, J. A. & HU, G. 1994 The role of dissipation in simulations of homogeneous plasma turbulence, and resolution of the entropy paradox. *Phys. Plasmas* **1**, 3211–3238.

KROMMES, J. A. & KIM, C.-B. 1988 Magnetic fluctuations can contribute to plasma transport, 'self-consistency constraints' notwithstanding. *Phys. Fluids* **31**, 869–886.

KROMMES, J. A. & KIM, C.-B. 2000 Interactions of disparate scales in drift-wave turbulence. *Phys. Rev. E* **62**, 8508–8539.

KROMMES, J. A. & KLEVA, R. 1979 Aspects of a renormalized plasma turbulence theory. *Phys. Fluids* **22**, 2168–2177.

KROMMES, J. A., LEE, W. W. & OBERMAN, C. 1986 Equilibrium fluctuation energy of gyrokinetic plasma. *Phys. Fluids* **29**, 2421–2425.

KROMMES, J. A. & PARKER, J. B. 2015 Statistical closures and zonal flows. In *Zonal Jets* (ed. B. Galperin & P. Read), chap. V.1.1. Cambridge: Cambridge University Press, in press.

KROMMES, J. A. & SIMILON, P. 1980 Dielectric response in guiding center plasma. *Phys. Fluids* **23**, 1553–1558.

KROMMES, J. A. & SMITH, R. A. 1987 Rigorous upper bounds for transport due to passive advection by inhomogeneous turbulence. *Ann. Phys. (N.Y.)* **177**, 246–329.

KUBO, R. 1959 Some aspects of the statistical-mechanical theory of irreversible processes. In *Lectures in Theoretical Physics* (ed. W. E. Brittin & L. G. Dunham), vol. I, pp. 120–203. New York: Interscience.

KUBO, R. 1962 Generalized cumulant expansion method. *J. Phys. Soc. Jpn.* **17**, 1100–1120.

KUBO, R. 1966 The fluctuation–dissipation theorem. *Rep. Prog. Phys.* **29**, 255–284.

KUZNETSOV, Y. A. 1998 *Elements of Applied Bifurcation Theory*, 2nd edn. New York: Springer.

LAVAL, G. & PESME, D. 1984 Self-consistency effects in quasilinear theory: A model for turbulent trapping. *Phys. Rev. Lett.* **53**, 270–273.

LEBOWITZ, J. L. 1993 Boltzmann's entropy and time's arrow. *Phys. Today* **46(9)**, 32–38.

LEBOWITZ, J. L. 2007 From time-symmetric microscopic dynamics to time-asymmetric macroscopic behavior: An overview. In *Proceedings of the Vienna Conference on Boltzmann*, pp. 63–89. Los Alamos National Laboratory, http://arXiv.org/abs/0709.0724.

LEE, W. W. 1983 Gyrokinetic approach in particle simulation. *Phys. Fluids* **26**, 556–562.

LEITH, C. E. 1971 Atmospheric predictability and two-dimensional turbulence. *J. Atmos. Sci.* **28**, 145–161.

LEMONS, D. S. & GYTHIEL, A. 1997 Paul Langevin's 1908 paper 'On the theory of Brownian motion' ['Sur la théori du mouvement brownien,' C. R. Acad. Sci. (Paris) **146**, 530–533 (1908)]. *Am. J. Phys.* **65**, 1079–1081.

LESIEUR, M. 1997 *Turbulence in Fluids*, 3rd edn. Dordrecht: Kluwer Academic.

LESLIE, D. C. 1973 *Developments in the Theory of Turbulence*. Oxford: Clarendon.

LESUR, M. & DIAMOND, P. H. 2013 Nonlinear instabilities driven by coherent phase-space structures. *Phys. Rev. E* **87**, 031101 (4 pages).

LICHTENBERG, A. J. & LIEBERMAN, M. A. 1992 *Regular and Chaotic Dynamics*, 2nd edn. New York: Springer.

LILLY, D. K. 1992 A proposed modification of the Germano subgrid-scale closure method. *Phys. Fluids A* **4**, 633–635.

LIN, Z., ETHIER, S., HAHM, T. S. & TANG, W. M. 2002 Size scaling of turbulent transport in magnetically confined plasmas. *Phys. Rev. Lett.* **88**, 195004 (4 pages).

LIN, Z., ETHIER, S., HAHM, T. S. & TANG, W. M. 2012 Verification of gyrokinetic particle simulation of device size scaling of turbulent transport. *Plasma Sci. Technol.* **14**, 1125–1126.

LITHWICK, Y. & GOLDREICH, P. 2003 Imbalanced weak magnetohydrodynamic turbulence. *Ap. J.* **582**, 1220–1240.

LODESTRO, L. L., COHEN, B. I., COHEN, R. H., DIMITS, A. M., MATSUDA, Y., NEVINS, W. M., NEWCOMB, W. A., WILLIAMS, T. J., KONIGES, A. E., DANNEVIK, W. P., CROTINGER, J. A., SYDORA, R. D., DAWSON, J. M., MA, S., DECYK, V. K., LEE, W. W., HAHM, T. S., NAITOU, H. & KAMIMURA, T. 1991 Comparison of simulations and theory of low-frequency plasma turbulence. In *Plasma Physics and Controlled Nuclear Fusion Research, 1990*, vol. II, pp. 31–43. Vienna: International Atomic Energy Agency.

LORENZ, E. N. 1963 Deterministic nonperiodic flow. *J. Atmos. Sci.* **20**, 130–141.

MARCINKIEWICZ, J. 1939 Sur une propriété de la loi de gauß. *Mathematische Zeitschrift* **44**, 612–618.

MARSTON, J. B. 2012 Planetary atmospheres as nonequilibrium condensed matter. *Annu. Rev. Condens. Matter Phys.* **3**, 285–310.

MARSTON, J. B., QI, W. & TOBIAS, S. M. 2015 Direct statistical simulation of a jet. In *Zonal Jets* (ed. B. Galperin & P. Read), chap. V.1.2. Cambridge: Cambridge University Press, in press.

MARTIN, P. C. 1968 *Measurements and correlation functions*. New York: Gordon and Breach,

The body of this book also appeared as an article in *Many Body Physics*, edited by C. deWitt and R. Balian (Gordon and Breach, New York, 1968) pp. 37–136.

MARTIN, P. C., SIGGIA, E. D. & ROSE, H. A. 1973 Statistical dynamics of classical systems. *Phys. Rev. A* **8**, 423–437.

McCOMB, W. D. 1990 *The Physics of Fluid Turbulence*. Oxford: Clarendon.

McCOMB, W. D. 2004 *Renormalization Methods: A Guide for Beginners*. Oxford: Oxford University Press.

McCOMB, W. D. 2014 *Homogeneous, Isotropic Turbulence*. Oxford: Oxford University Press.

McCULLAGH, P. & KOLASSA, J. 2009 Cumulants. *Scholarpedia* **4** (3), 4699 [http://www.scholarpedia.org/article/Cumulants].

METZLER, R. & KLAFTER, J. 2000 The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339**, 1–77.

MONIN, A. S. & YAGLOM, A. M. 1971 *Statistical Fluid Mechanics*. Cambridge, MA: MIT Press.

MONTGOMERY, D. C. & TIDMAN, D. A. 1964 *Plasma Kinetic Theory*. New York: McGraw–Hill.

MOREL, P., NAVARRO, A. BAÑÓN, ALBRECHT-MARC, M., CARATI, D., MERZ, F., GÖRLER, T. & JENKO, F. 2011 Gyrokinetic large eddy simulations. *Phys. Plasmas* **18**, 072301 (9 pages).

MOREL, P., NAVARRO, A. BAÑÓN, ALBRECHT-MARC, M., CARATI, D., MERZ, F., GÖRLER, T. & JENKO, F. 2012 Dynamic procedure for filtered gyrokinetic simulations. *Phys. Plasmas* **19**, 012311 (10 pages).

MORI, H. 1965 Transport, collective motion, and Brownian motion. *Prog. Theor. Phys.* **33**, 423–455.

MOU, C.-Y. & WEICHMAN, P. B. 1993 Spherical model for turbulence. *Phys. Rev. Lett.* **70**, 1101–1104.

MYNICK, H. E. 1988 The generalized Balescu–Lenard collision operator. *J. Plasma Phys.* **38**, 303–317.

NAZARENKO, S. 2011 *Wave Turbulence*, vol. 825 of *Lecture Notes in Physics*. Heidelberg: Springer.

NAZARENKO, S. V. & SCHEKOCHIHIN, A. A. 2011 Critical balance in magnetohydrodynamic, rotating and stratified turbulence: towards a universal scaling conjecture. *J. Fluid Mech.* **677**, 134–153.

NEVINS, W. M., HAMMETT, G. W., DIMITS, A. M., DORLAND, W. & SHUMAKER, D. E. 2005 Discrete particle noise in particle-in-cell simulations of plasma microturbulence. *Phys. Plasmas* **12**, 122305 (16 pages).

NOVIKOV, E. A. 1964 Functionals and the random-force method in turbulence theory. *Zh. Eksp. Teor. Fiz.* **47**, 1919–1926 [Sov. Phys. JETP **20**, 1290–1294 (1965)].

OGURA, Y. 1962*a* Energy transfer in a normally distributed and isotropic turbulent velocity field in two dimensions. *Phys. Fluids* **5**, 395–401.

OGURA, Y. 1962*b* Energy transfer in an isotropic turbulent flow. *J. Geophys. Res.* **67**, 3143–3149.

OGURA, Y. 1963 A consequence of the zero-fourth-cumulant approximation in the decay of isotropic turbulence. *J. Fluid Mech.* **16**, 33–40.

ORSZAG, S. A. 1969 Stochastic acceleration by strong electric fields. In *Turbulence of Fluids and Plasmas* (ed. J. Fox), pp. 17–28. Brooklyn, NY: Polytechnic Press.

ORSZAG, S. A. 1970 Anaytical theories of turbulence. *J. Fluid Mech.* **41**, 363–386.

ORSZAG, S. A. 1977 Lectures on the statistical theory of turbulence. In *Fluid Dynamics* (ed. R. Balian & J.-L. Peube), pp. 235–374. New York: Gordon and Breach.

ORSZAG, S. A. & KRAICHNAN, R. H. 1967 Model equations for strong turbulence in a Vlasov plasma. *Phys. Fluids* **10**, 1720–1736.

PARKER, J. B. & KROMMES, J. A. 2013 Zonal flow as pattern formation: Merging jets and the ultimate jet length scale. *Phys. Plasmas* **20**, 100703 (4 pages).

PARKER, J. B. & KROMMES, J. A. 2014 Generation of zonal flows through symmetry breaking of statistical homogeneity. *New J. Phys.* **16**, 035006 (28 pages).

PARKER, J. B. & KROMMES, J. A. 2015 Zonal flow as pattern formation. In *Zonal Jets* (ed. B. Galperin & P. Read), chap. V.2.4. Cambridge: Cambridge University Press, in press.

PARRA, F. I., BARNES, M. & PEETERS, A. G. 2011 Up-down symmetry of the turbulent transport of toroidal angular momentum in tokamaks. *Phys. Plasmas* **18**, 062501 (14 pages).

Perkins, F. W., Barnes, Cris W., Johnson, D. W., Scott, S. D., Zarnstorff, M. C., Bell, M. G., Bell, R. E., Bush, C. E., Grek, B., Hill, K. W., Mansfield, D. K., Park, H., Ramsey, A. T., Schivell, J., Stratton, B. C. & Synakowski, E. 1993 Nondimensional transport scaling in the Tokamak Fusion Test Reactor: Is tokamak transport Bohm or gyro-Bohm? *Phys. Fluids B* **5**, 477–498.

Prandtl, L. 1925 Bericht über Untersuchungen zur ausgebildeten Turbulenz. *Z. Angew. Math. Mech.* **5**, 136–139.

Rogers, B. N., Dorland, W. & Kotschenreuther, M. 2000 Generation and stability of zonal flows in ion-temperature-gradient mode turbulence. *Phys. Rev. Lett.* **85**, 5336–5339.

Rogister, A. & Oberman, C. 1968 On the kinetic theory of stable and weakly unstable plasma. Part 1. *J. Plasma Phys.* **2**, 33–49.

Rogister, A. & Oberman, C. 1969 On the kinetic theory of stable and weakly unstable plasma. Part 2. *J. Plasma Phys.* **3**, 119–147.

Rose, H. A. 1979 Renormalized kinetic theory of nonequilibrium many-particle classical systems. *J. Stat. Phys.* **20**, 415–447.

Rose, H. A. 1985 An efficient non-Markovian theory of non-equilibrium dynamics. *Physica D* **14**, 216–226.

Rudnick, J. & Gaspari, G. 2004 *Elements of the Random Walk: An introduction for Advanced Students and Researchers*. Cambridge: Cambridge University Press.

Ruelle, D. & Takens, F. 1971 On the nature of turbulence. *Commun. Math. Phys.* **20**, 167–192.

Sagdeev, R. Z. 1979 The 1976 Oppenheimer lectures: Critical problems in plasma astrophysics. I. Turbulence and nonlinear waves. *Rev. Mod. Phys.* **51**, 1–9.

Sagdeev, R. Z. & Galeev, A. A. 1969 *Nonlinear Plasma Theory*. New York: Benjamin.

Schekochihin, A. A., Cowley, S. C., Dorland, W., Hammett, G. W., Howes, G. G., Plunk, G. G., Quataert, E. & Tatsuno, T. 2008 Gyrokinetic turbulence: a nonlinear route to dissipation through phase space. *Plasma Phys. Control. Fusion* **50**, 124024 (13 pages).

Schekochihin, A. A., Cowley, S. C., Dorland, W., Hammett, G. W., Howes, G. G., Quataert, E. & Tatsuno, T. 2009 Astrophysical gyrokinetics: Kinetic and fluid turbulent cascades in magnetized weakly collisional plasmas. *Astrophys. J. Suppl. Ser.* **182**, 310–377.

Scott, B. D. 1992 The mechanism of self-sustainment in collisional drift wave turbulence. *Phys. Fluids B* **4**, 2468–2494.

Smagorinsky, J. 1963 General circulation experiments with the primitive equations. *Mon. Wea. Rev.* **91**, 99–164.

Spineanu, F. & Vlad, M. 2005 Statistical properties of an ensemble of vortices interacting with a turbulent field. *Phys. Plasmas* **12**, 112303 (19 pages).

Squire, J. & Bhattacharjee, A. 2015 Statistical simulation of the magnetorotational dynamo. *Phys. Rev. Lett.* **114**, 085002 (5 pages).

Sridhar, S. 2010 Magnetohydrodynamic turbulence in a strongly magnetised plasma. *Astron. Nachr.* **331**, 93–100.

Srinivasan, K. & Young, W. R. 2012 Zonostrophic instability. *J. Atmos. Sci.* **69**, 1633–1656.

Stoltzfus-Dueck, T., Scott, B. D. & Krommes, J. A. 2013 Nonadiabatic electron response in the Hasegawa-Wakatani equations. *Phys. Plasmas* **20**, 082314 (13 pages).

Sudan, R. N. & Pfirsch, D. 1985 On the relation between 'mixing length' and 'direct interaction approximation' theories of turbulence. *Phys. Fluids* **28**, 1702–1718.

Taylor, G. I. 1921 Diffusion by continuous movements. *Proc. London Math. Soc., Ser. 2* **20**, 196–212, reprinted in *Turbulence: Classic Papers on Statistical Theory*, edited by S. K. Friedlander and L. Topper (Interscience, New York, 1961), pp. 1–17.

Taylor, J. B. 1961 Diffusion of plasma across a magnetic field. *Phys. Rev. Lett.* **6**, 262–263.

Taylor, J. B. 1974 Dielectric function and diffusion of a guiding-center plasma. *Phys. Rev. Lett.* **32**, 199–202.

Tennekes, H. & Lumley, J. L. 1972 *A First Course in Turbulence*. Cambridge, MA: MIT Press.

Terry, P. & Horton, W. 1982 Stochasticity and the random phase approximation for three electron drift waves. *Phys. Fluids* **25**, 491–501.

TERRY, P. W. 2000 Suppression of turbulence and transport by sheared flow. *Rev. Mod. Phys.* **72**, 109–165.

TOBIAS, S. M., CATTANEO, F. & BOLDYREV, S. 2013 MHD dynamos and turbulence. In *Ten Chapters in Turbulence* (ed. P. A. Davidson, Y. Kaneda & K. R. Sreenivasan), chap. 9, pp. 351–404. Cambridge: Cambridge University Press.

TOBIAS, S. M., DAGON, K. & MARSTON, J. B. 2011 Astrophysical fluid dynamics via direct statistical simulation. *Astrophys. J.* **727**, 127–138.

TOBIAS, S. M. & MARSTON, J. B. 2013 Direct statistical simulation of out-of-equilibrium jets. *Phys. Rev. Lett.* **110**, 104502 (5 pages).

VAN KAMPEN, N. G. 1976 Stochastic differential equations. *Phys. Rep.* **24**, 171–228.

VAN KAMPEN, N. G. 1981 *Stochastic Processes in Physics and Chemistry*. Amsterdam: North–Holland.

VAN KAMPEN, N. G. & FELDERHOF, B. U. 1967 *Theoretical Methods in Plasma Physics*. Amsterdam: North–Holland.

VEDENOV, A., VELIKHOV, E. & SAGDEEV, R. 1962 The quasi-linear theory of plasma oscillations. In *Proceedings of the Conference on Plasma Physics and Controlled Nuclear Fusion Research (Salzburg, 1961) [Nucl. Fusion Suppl. Pt. 2]*, pp. 465–475. Vienna: International Atomic Energy Agency, translated in U.S.A.E.C. Division of Technical Information document AEC–tr–5589 (1963), pp. 204–237.

VELIKHOV, E. P., GINZBURG, V. L., GAPONOV-GREKHOV, A. V., DYKHNE, A. M., KELDYSH, L. V., KLIMONTOVICH, YU.L., KOGAN, V. I., MENSKIĬ, M. B., PITAEVSKIĬ, L. P., FORTOV, V. E., CHERNOPLEKOV, N. A. & SHAFRANOV, V. D. 1998 In memory of Boris Borisovich Kadomtsev. *Phys. Usp.* **41**, 1155–1156.

WALEFFE, F. 1995 Transition in shear flows. Non-linear normality versus non-normal linearity. *Phys. Fluids* **7**, 3060–3066.

WALEFFE, F. 1997 On a self-sustaining process in shear flows. *Phys. Fluids* **9**, 883–900.

WALTZ, R. E. 1983 Numerical study of drift wave turbulence with simple models for wave–wave coupling. *Phys. Fluids* **26**, 169–179.

WANG, E., NEVINS, W. M., CANDY, J., HATCH, D., TERRY, P. & GUTTENFELDER, W. 2011 Electron heat transport from stochastic fields in gyrokinetic simulations. *Phys. Plasmas* **18**, 056111 (7 pages).

WANG, M. C. & UHLENBECK, G. E. 1945 On the theory of the Brownian motion II. *Rev. Mod. Phys.* **17**, 323–342, reprinted in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954), pp. 113–132.

WATANABE, T.-H. & SUGAMA, H. 2004 Kinetic simulation of steady states of ion temperature gradient driven turbulence with weak collisionality. *Phys. Plasmas* **11**, 1476–1483.

WEINSTOCK, J. 1968 Turbulent diffusion, particle orbits, and field fluctuations in a plasma in a magnetic field. *Phys. Fluids* **11**, 1977–1981.

WEINSTOCK, J. 1969 Formulation of a statistical theory of strong plasma turbulence. *Phys. Fluids* **12**, 1045–1058.

WEINSTOCK, J. 1970 Turbulent plasmas in a magnetic field—A statistical theory. *Phys. Fluids* **13**, 2308–2316.

WEINSTOCK, J. & CATTO, P. J. 1981 Comment on 'Adiabatic modifications to plasma turbulence theories'. *Phys. Fluids* **24**, 1763–1764.

WEINSTOCK, J. & WILLIAMS, R. H. 1971 Nonlinear theory of macroinstabilities and enhanced transport in plasmas. *Phys. Fluids* **14**, 1472–1480.

YOSHIZAWA, A., ITOH, S.-I., ITOH, K. & YOKOI, N. 2001 Turbulence theories and modelling of fluids and plasmas. *Plasma Phys. Control. Fusion* **43**, R1–R144.

ZASLAVSKY, G. M. 2002 Chaos, fractional kinetics, and anomalous transport. *Phys. Rev.* **371**, 461–580.

ZINN-JUSTIN, J. 1996 *Quantum Field Theory and Critical Phenomena*, 3rd edn. Oxford: Oxford University Press.

ZWANZIG, R. 1961a Memory effects in irreversible thermodynamics. *Phys. Rev.* **124**, 983–992.

ZWANZIG, R. 1961b Statistical mechanics of irreversibility. In *Lectures in Theoretical Physics* (ed. W. E. Brittin, B. W. Downs & J. Downs), vol. 3, pp. 106–141. New York: Interscience.

ZWANZIG, R. 2001 *Nonequilibrium Statistical Mechanics*. Oxford: Oxford University Press.

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