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Nonlinear frequency shift of electrostatic waves in general collisionless plasma: unifying theory of fluid and kinetic nonlinearities

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The nonlinear frequency shift is derived in a transparent asymptotic form for intense Langmuir waves in general collisionless plasma. The formula describes both fluid and kinetic effects simultaneously. The fluid nonlinearity is expressed through the plasma dielectric function, and the kinetic nonlinearity accounts for both smooth distributions and trapped-particle beams. The various known limiting scalings are reproduced as special cases. The calculation does not involve solving any differential equations and can be extended straightforwardly to other nonlinear plasma waves.

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I. INTRODUCTION

It has long been known that the nonlinear frequency shifts $\delta\omega$ of collisionless plasma waves can depend on the wave amplitude a in various ways, even at small a. The contributions to $\delta\omega$ that stem from fluid effects typically scale as $\propto a^2$ [1–5], whereas kinetic effects can produce contributions scaling as $\propto \sqrt{a}$ [6–21] or even decreasing with a [22–28]. The interest to such nonlinear dispersion relations (NDRs) has been revived recently in connection with laser-plasma interactions [29] and the evolution of energetic-particle modes in tokamaks [30]. However, each of the existing theories that offers an explicit formula for $\delta\omega$ describes just one type of nonlinearity. The possible effect of multiple coexisting nonlinearities that yield different scalings in a was explored in Ref. [31], but only heuristically and for an artificial plasma model. The more general existing theories, such as that of Bernstein-Greene-Kruskal (BGK) waves [32], rely on formal solutions of the Vlasov-Maxwell equations and are not always easy to apply in practice due to their complexity. But is it possible to derive a comprehensive and yet tractable asymptotic NDR for general collisionless plasma?

The answer is yes, and here we show how to do it explicitly. We build on the theory that was developed recently in Refs. [33–37] and represents a fully nonlinear kinetic version of Whitham's average-Lagrangian approach [38, 39]. (We assume that, when present, resonant particles are phase-mixed, so the corresponding waves are of the BGK type [40].) This allows deriving the NDR directly from the wave Lagrangian, which is known, without solving any differential equations. In fact, all wave properties can be traced to the properties of a single function characterizing individual particles, namely, the normalized action j of a particle as a function of its normalized energy r. We focus on Langmuir waves in one-dimensional electron plasma, but extending the theory to general waves is straightforward to do.

Unlike in Refs. [34–37], where a related theory was constructed under the sinusoidal-wave approximation, we now allow for a nonzero amplitude of the second harmonic (higher-order harmonics are assumed negligible) and find this amplitude self-consistently. Kinetic and fluid nonlinearities are hence treated on the same footing. In particular, we show that the fluid nonlinearity can be expressed in terms of the plasma dielectric function and, contrary to the common presumption, can have either sign. The underlying assumptions are that a is small enough and that the particle distribution function, f_0, is smooth near the resonance. (For cold plasmas, this model is exact because f_0 is identically zero in the resonance vicinity and thus is perfectly flat there.) Abrupt distributions of trapped beams superimposed on smooth f_0 are described separately and produce additive contributions to the NDR. For example, we revisit the case of flat-top trapped beams [26, 27] and show that our asymptotic theory yields predictions for both the wave frequency and the second-harmonic amplitude that are virtually indistinguishable from those given by the exact solution of the Vlasov-Poisson system. Our theory then can be considered as advantageous over such solutions. This is because, while offering a reasonable precision, it is more transparent and flexible, i.e., can be used also when exact analytical formulas do not exist.

The paper is organized as follows. In Sec. II, we briefly review the underlying variational approach and introduce basic notation. In Sec. III, we describe the wave model and asymptotics of some auxiliary functions derived from $j(r)$. In Sec. IV, we derive the general expression for $\delta\omega$ for smooth f_0 and present examples, including the cases of cold, waterbag, kappa, and Maxwellian f_0. In Sec. V, we discuss the effect of trapped-particle beams superimposed on a smooth f_0. In Sec. VI, we summarize the main results of our work. Some auxiliary calculations are also presented in appendixes.

II. BASIC CONCEPTS AND NOTATION

A. Wave Lagrangian density

As reviewed in Refs. [33], the dynamics of an adiabatic plasma wave can be derived from the least action principle, $\delta\Lambda = 0$, where $\Lambda = \int \mathfrak{L}\, dt\, d^3x$ is the action integral, and $\mathfrak{L}$ is the wave Lagrangian density given by

$$\mathfrak{L} = \langle \mathfrak{L}_{\text{em}} \rangle - \sum_{\mathsf{s}} \bar{n}_{\mathsf{s}} \langle\!\langle \mathcal{H}_{\mathsf{s}} \rangle\!\rangle. \tag{1}$$

Here $\mathfrak{L}_{\text{em}}$ is the Lagrangian density of electromagnetic field in vacuum, $\mathcal{H}_{\mathsf{s}}$ are the oscillation-center (OC) Hamiltonians of single particles of type s, the summation is taken over all particle types, $\bar{n}_{\mathsf{s}}$ are the corresponding average densities, $\langle \ldots \rangle$ denotes averaging over rapid oscillations in time and space, and $\langle\!\langle \ldots \rangle\!\rangle$ denotes averaging over the local distributions of the particle OC canonical momenta. Keep in mind that these distributions must be treated as prescribed when the least action principle is applied to derive the NDR.

Below, we limit our consideration to electrostatic oscillations in one-dimensional nonrelativistic electron plasma. In this case, $\mathfrak{L}_{\text{em}} = E^2/8\pi$, where $E = -\partial_x \phi$ is the electric field, and ϕ is the electrostatic potential. Also, the OC Hamiltonian of passing particles (to be denoted with index p) and trapped particles (to be denoted with index t) then can be expressed as follows:

$$\mathcal{H}_{\mathsf{p}} = Pu + \mathcal{E} - mu^2/2, \quad \mathcal{H}_{\mathsf{t}} = \mathcal{E} - mu^2/2. \tag{2}$$

Here $u \doteq \omega/k$ is the wave phase velocity (we use the symbol $\doteq$ for definitions), $\omega \doteq -\partial_t \xi$ is the local frequency, $k \doteq \partial_x \xi$ is the local wave number, and ξ is the wave phase. Also, P is the OC canonical momentum of a passing particle in the laboratory frame $\mathcal{K}$,

$$\mathcal{E} \doteq mw^2/2 + e\phi \tag{3}$$

is the particle total energy in the frame $\mathcal{K}'$ that moves with respect $\mathcal{K}$ at velocity u, w is the particle velocity in $\mathcal{K}'$, and m and e are the particle mass and charge. Notice also, for future references, that, in at small enough ϕ, $\mathcal{H}_{\mathsf{p}}$ becomes the well known ponderomotive Hamiltonian,

$$\mathcal{H}_{\mathsf{p}} \approx \frac{P^2}{2m} + \Phi, \tag{4}$$

where the "ponderomotive potential" Φ is given by

$$\Phi = \frac{e^2 k^2 \phi_1^2}{4m(\omega - kP/m)^2}, \tag{5}$$

and ϕ_1 is the amplitude of ϕ. The OC canonical momentum P can then be expressed as

$$P \approx mV - \partial_V \Phi, \tag{6}$$

where V is the average velocity, or the OC velocity.

Using Eqs. (2), one can cast Eq. (1) as

$$\mathfrak{L} = \frac{\langle E^2 \rangle}{8\pi} - \bar{n} \langle\!\langle \mathcal{E} \rangle\!\rangle - \bar{n}_{\mathsf{p}} \langle\!\langle P \rangle\!\rangle u + \frac{\bar{n} m u^2}{2}, \tag{7}$$

where $\bar{n} \doteq \bar{n}_{\mathsf{p}} + \bar{n}_{\mathsf{t}}$ is the total average density. Let us also introduce J as the particle action variable in $\mathcal{K}'$, i.e., the (appropriately normalized) phase space area that is swept by the particle trajectory on a single period, $J \propto m \oint w \, dx$. The separatrix action, J_*, can be estimated as $J_* \sim \hat{J} \sqrt{a_p}$, where we introduced

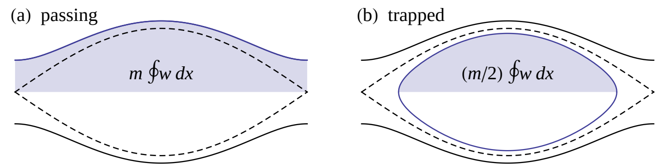

FIG. 1: Schematic of particle trajectories in phase space, illustrating the definition of $2\pi J$ (shaded area): (a) for a passing particle, (b) for a trapped particle. For J to be continuous at the separatrix (dashed), with the passing-particle action defined as $2\pi J = m \oint w \, dx$, for trapped particles one must use the definition $2\pi J = (m/2) \oint w \, dx$. The figure is an adjustment of Fig. 1 from Ref. [36].

$$\hat{J} \doteq \frac{m\omega}{k^2}, \quad a_p \doteq \frac{ek^2 \phi_1}{m\omega_p^2}. \tag{8}$$

Trapped particles, for which J is the OC canonical momentum, have $0 < J < J_*$. Passing particles, for which

$$P = mu + kJ \operatorname{sgn}(w), \tag{9}$$

have $J > J_*$. One can then rewrite Eq. (7) as follows:

$$\mathfrak{L} = \frac{\langle E^2 \rangle}{8\pi} - \bar{n} \langle\!\langle \mathcal{E} \rangle\!\rangle + \Delta \mathfrak{L}. \tag{10}$$

Here $\Delta \mathfrak{L}$ is independent of the wave amplitude, so it has no effect on the NDR, as will become clear shortly.

B. Action distribution

The ensemble averaging can be expressed through the action distribution $F(J)$ that includes both trapped and passing particles, $F(J) = F_{\mathsf{t}}(J) + F_{\mathsf{p}}(J)$, where $F_{\mathsf{t,p}}(J \gtrless J_*) \equiv 0$. We will assume the normalization such that $\int_0^\infty F_{\mathsf{t,p}}(J) \, dJ = \bar{n}_{\mathsf{t,p}}/\bar{n}$; then, $\int_0^\infty F(J) \, dJ = 1$, and

$$\langle\!\langle \ldots \rangle\!\rangle = \int_0^\infty (\ldots) \, F(J) \, dJ. \tag{11}$$

We will also assume that a wave is excited by a driver with homogeneous amplitude, so the OC canonical momenta P are conserved and equal the initial momenta,

$$P = mV_0. \tag{12}$$

From Eq. (9), gets

$$J = |1 - V_0/u| \hat{J}, \tag{13}$$

so the action distribution is given by

$$F(J) = \frac{k}{m} \left[f_0 \left(u + \frac{kJ}{m} \right) + f_0 \left(u - \frac{kJ}{m} \right) \right], \tag{14}$$

where f_0 is the initial velocity distribution. This is known as the adiabatic-excitation model [9, 21, 29].

C. Nonlinear Doppler shift

Note that the conservation of P implies that, as a result of the wave excitation, the plasma generally changes its average velocity, $\langle\!\langle V\rangle\!\rangle$, by some $\langle\!\langle \Delta V\rangle\!\rangle$. This leads to a nonlinear Doppler shift, $k\langle\!\langle \Delta V\rangle\!\rangle$, by which the frequency ω in the laboratory frame $\mathcal{K}$ differs from that in the plasma rest frame [5, 41]. The shift can be calculated using that $V_{\mathsf{t}} = u$ and $V_{\mathsf{p}} = \partial_P \mathcal{H}_{\mathsf{p}} = u + (\partial_J \mathcal{E})\,\mathrm{sgn}(w)$, where $\mathcal{E}$ is considered as a function of J and of the wave parameters [33]. However, the very concept of the plasma rest frame is most meaningful when the resonant population is negligible, in which case the following simple derivation is possible. (Also see Appendix B for an alternative derivation.)

By combining Eq. (6) with Eq. (12), one gets for an initially-resting plasma that

$$\langle\!\langle \Delta V\rangle\!\rangle = \langle\!\langle V - V_0\rangle\!\rangle \approx \frac{\langle\!\langle \partial_V \Phi\rangle\!\rangle}{m} \approx \frac{ua^2}{2}\,\langle\!\langle (1 - V_0/u)^{-3}\rangle\!\rangle. \tag{15}$$

One can also reexpress the right hand side as follows:

$$\begin{aligned}\frac{\langle\!\langle \partial_V \Phi\rangle\!\rangle}{m} &\approx \frac{e^2k^2\phi_1^2}{4m^2}\int_{-\infty}^{\infty}[\partial_{V_0}(\omega - kV_0)^{-2}]f_0(V_0)\,dV_0\\ &\approx -\frac{e^2k\phi_1^2}{4m^2}\,\frac{\partial}{\partial\omega}\int_{-\infty}^{\infty}\frac{f_0(V_0)}{(V_0-\omega/k)^2}\,dV_0\\ &= \frac{a^2}{4}\,\frac{\omega^4}{k\omega_p^2}\,\frac{\partial\epsilon(\omega,k)}{\partial\omega},\end{aligned} \tag{16}$$

where

$$\begin{aligned}\epsilon(\omega,k) &\doteq 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{f_0(V_0)}{(V_0-\omega/k)^2}\,dV_0\\ &= 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{f_0'(V_0)}{V_0-\omega/k}\,dV_0.\end{aligned} \tag{17}$$

This leads to

$$\langle\!\langle \Delta V\rangle\!\rangle = \frac{a^2}{4}\,\frac{\omega^4}{k\omega_p^2}\,\frac{\partial\epsilon(\omega,k)}{\partial\omega}, \tag{18}$$

where, within the adopted accuracy, the right hand side must be evaluated at the linear frequency, ω_0. Hence, one eventually arrives at the following expression for the nonlinear Doppler shift:

$$k\langle\!\langle \Delta V\rangle\!\rangle \approx \frac{a_0^2}{4}\,\frac{\omega_0^4}{\omega_p^2}\,\frac{\partial\epsilon(\omega_0,k)}{\partial\omega_0}, \tag{19}$$

where $a_0 \doteq ek^2\phi_1/(m\omega_0^2)$.

D. Dielectric function

It is to be noted that integration by parts in Eq. (17) is justified only by the assumption of having no resonant particles, in which case the integrand is analytic. More generally, we will define ϵ as the following real function,

$$\epsilon(\omega,k) \doteq 1 - \frac{\omega_p^2}{k^2}\,\mathsf{P}\int_{-\infty}^{\infty}\frac{f_0'(v)}{v-\omega/k}\,dw, \tag{20}$$

where P denotes the Cauchy principal value. For the adiabatic waves of interest, which are by definition phase-mixed and exhibit no Landau damping, such ϵ can be recognized as the linear dielectric function [42, Chap. 8].

III. WAVE MODEL

A. General dispersion relation

Since large amplitudes typically result in rapid deterioration of waves through various nonlinear instabilities, the very concept of the NDR is of interest mainly for waves with small enough amplitudes, i.e., when the wave shape is close to sinusoidal. Harmonics of order $\ell > 1$ (which are phase-locked to the fundamental harmonic, $\ell = 1$) can then be treated as perturbations, and the spectrum decreases rapidly with ℓ [43]. Typical calculations of the trapped-particle nonlinearities, which are reviewed in Ref. [36], are restricted to the sinusoidal wave model, which neglects all harmonics with $\ell > 1$. Below, we extend those calculations by introducing a "bisinusoidal" wave model, which retains the first *and* second harmonic while neglecting those with $\ell > 2$.

Within the bisinusoidal wave model, one can search for the wave electrostatic potential in the form

$$\phi = \phi_1\cos(\xi) + \phi_2\cos(2\xi + \chi), \tag{21}$$

where ϕ_1, ϕ_2, and χ are slow functions of (t, x). (We will assume that ϕ_2 is small enough, so there is only one minimum of the potential energy per wavelength; see below.) It is then convenient to adopt ξ, χ, and

$$a_1 \doteq e\phi_1k^2/(m\omega^2), \quad a_2 \doteq e\phi_2k^2/(m\omega^2) \tag{22}$$

as four independent variables. Minimizing Λ with respect to ξ leads to a dynamic equation for the wave amplitude [33], which is not of interest in the context of this paper. The remaining Euler-Lagrange equations are as follows:

$$\partial_{a_1}\mathfrak{L} = 0, \quad \partial_{a_2}\mathfrak{L} = 0, \quad \partial_{\chi}\mathfrak{L} = 0, \tag{23}$$

where $\mathfrak{L} = \mathfrak{L}(a_1, a_2, \chi, \omega, k)$. By substituting

$$\langle E^2\rangle = \frac{a_1^2 + 4a_2^2}{2}\left(\frac{m\omega u}{e}\right)^2, \tag{24}$$

one hence obtains the following three equations,

$$0 = \frac{\partial\langle\!\langle \mathcal{E}\rangle\!\rangle}{\partial a_1} - \frac{a_1}{8\pi\bar{n}}\left(\frac{m\omega u}{e}\right)^2, \tag{25}$$

$$0 = \frac{\partial\langle\!\langle \mathcal{E}\rangle\!\rangle}{\partial a_2} - \frac{a_2}{2\pi\bar{n}}\left(\frac{m\omega u}{e}\right)^2, \tag{26}$$

$$0 = \frac{\partial\langle\!\langle \mathcal{E}\rangle\!\rangle}{\partial \chi}, \tag{27}$$

where $\mathcal{E} = \mathcal{E}(J, a_1, a_2, \chi, \omega, k)$. After eliminating a_2 and χ and introducing $a \doteq a_1$, one arrives at a single equation for $\omega(k, a)$, which constitutes the NDR.

B. Eliminating χ

Our first step is to eliminate χ. To do this, consider Eq. (27) in the following form:

$$\langle\!\langle \partial_\chi \mathcal{E}(J, a_1, a_2, \chi, \omega, k) \rangle\!\rangle = 0. \tag{28}$$

In order to express $\mathcal{E}$ as a function of J, we will need an explicit formula for J. A universal formula that applies to both trapped and passing particles is [33]

$$J = \mathrm{Re} \int_0^{2\pi/k} \sqrt{2m[\mathcal{E} - e\phi(x)]}\, \frac{dx}{2\pi}. \tag{29}$$

Using the notation

$$z \doteq \frac{a_2}{2a_1}, \quad r \doteq \frac{1}{2}\left(\frac{\mathcal{E}}{mu^2 a} + 1\right), \tag{30}$$

it is convenient to cast Eq. (29) as

$$J = \frac{\hat{J}\sqrt{a}}{\pi}\, \mathrm{Re} \int_{-\pi}^{\pi} \left[r - \sin^2(\theta/2) - z\cos(2\theta + \chi)\right]^{1/2} d\theta.$$

Hence we can rewrite Eq. (28) as follows [44]:

$$\langle\!\langle \partial_\chi J / \partial_r J \rangle\!\rangle = 0, \tag{31}$$

where $J = J(r, a, z, \chi, \omega, k)$. Notice now that

$$(\partial_\chi J)|_{\chi=0} = \frac{\hat{J}\sqrt{a}}{\pi}\, \mathrm{Re} \int_{-\pi}^{\pi} d\theta \times z\sin(2\theta)\left[r - \sin^2(\theta/2) - z\cos(2\theta)\right]^{-1/2},$$

which is zero, because the integrand is an odd function of θ. Therefore, Eq. (31) has an obvious solution, $\chi = 0$ (or $\chi = \pi$, but the difference between these solutions can be absorbed in the sign of z). We will assume this solution without searching for others, because it corresponds to what seems to be the only (relatively) stable equilibrium seen in simulations, even for large-amplitude waves [45].

The remaining equations that constitute the NDR [Eqs. (25) and (26)] can then be represented as follows,

$$\mathcal{G}_1 - \frac{a_1\omega^2}{2\omega_p^2} = 0, \quad \mathcal{G}_2 - \frac{2a_2\omega^2}{\omega_p^2} = 0, \tag{32}$$

or, equivalently,

$$1 - 2\mathcal{G}_1/a_p = 0, \quad 8z\mathcal{G}_1 - \mathcal{G}_2 = 0. \tag{33}$$

Here $\omega_p \doteq (4\pi\bar{n}e^2/m)^{1/2}$, $\mathcal{G}_{1,2} \doteq \langle\!\langle G_{1,2} \rangle\!\rangle$, and

$$G_{1,2} \doteq \frac{\partial}{\partial a_{1,2}}\left[\frac{\mathcal{E}(J, a_1, a_2, \omega, k)}{mu^2}\right]. \tag{34}$$

Notably, $\partial_{a_2} G_1(J, a_1, a_2, \omega, k) = \partial_{a_1} G_2(J, a_1, a_2, \omega, k)$ by definition, and the same applies to $\mathcal{G}_{1,2}$. Also notably, $G_{1,2}$ have a clear physical meaning, which is as follows. The function $2G_1/a$ can be understood as the relative contribution of a particle with given action (or energy) to the wave squared frequency, ω^2, in units ω_p^2. The function $G_2/2$ can be understood as the particle relative contribution to the second harmonic amplitude, ϕ_2, in units $m\omega_p^2/(ek^2)$ [because $a_2\omega^2/\omega_p^2 = e\phi_2 k^2/(m\omega_p^2)$].

C. Normalized action

It is convenient to represent the particle action as $J = \hat{J}\sqrt{a}\, j(r, z)$, where the "normalized action", $j(r, z)$, is a dimensionless function given by

$$j(r, z) \doteq \frac{2}{\pi}\, \mathrm{Re} \int_0^{\pi} \sqrt{\mathcal{R}(r, z, \theta)}\, d\theta, \tag{35}$$

$$\mathcal{R}(r, z, \theta) \doteq r - \sin^2(\theta/2) - z\cos(2\theta). \tag{36}$$

Then, $G_{1,2}$ can be expressed through j alone,

$$\begin{aligned} G_1 &= 2r - 1 + 2a_1\, \partial_{a_1} r(J, a_1, a_2, \omega, k) \\ &= 2r - 1 - 2a_1\, \frac{\partial_{a_1} J(r, a_1, a_2, \omega, k)}{\partial_r J(r, a_1, a_2, \omega, k)} \\ &= 2r - 1 - \frac{j - 2zj_z}{j_r}, \end{aligned} \tag{37}$$

$$\begin{aligned} G_2 &= 2a_1\, \partial_{a_2} r(J, a_1, a_2, \omega, k) \\ &= -2a_1\, \frac{\partial_{a_2} J(r, a_1, a_2, \omega, k)}{\partial_r J(r, a_1, a_2, \omega, k)} \\ &= -\frac{j_z}{j_r}, \end{aligned} \tag{38}$$

where $j \equiv j(r, z)$, and the indexes r and z denote partial derivatives. Explicit formulas for $j(r, z)$ and $G_{1,2}(r, z)$ are given in Appendix A. Characteristic plots of $G_{1,2}$ as functions of r are presented in Fig. 2.

D. Restrictions on r and z: reflection points and the trapping condition

Let us also introduce an alternative, all-real representation of j,

$$j(r, z) = \frac{2}{\pi} \int_0^{\theta_0} \sqrt{\mathcal{R}(r, z, \theta)}\, d\theta. \tag{39}$$

Here, for passing particles θ_0 equals π and for trapped particles θ_0 equals the first positive solution of

$$\mathcal{R}(r, z, \theta_0) = 0. \tag{40}$$

Equation (40) can be expressed as the following quadratic equation for $y \doteq \cos\theta_0$,

$$4zy^2 - y - 2r + 1 - 2z = 0. \tag{41}$$

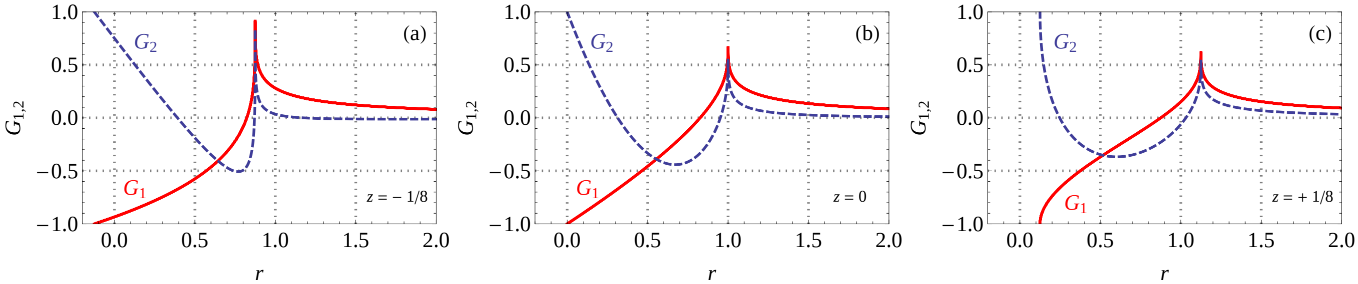

FIG. 2: (Color online) $G_1(r,z)$ (solid) and $G_2(r,z)$ (dashed) vs r at fixed z: (a) $z=-1/8$, (b) $z=0$, (c) $z=+1/8$. Notice the singularity at $r=1+z$, which corresponds to the boundary between passing and trapped trajectories (Sec. III D).

That gives $y=(1\pm D)/(8z)$, where

$$D \doteq \sqrt{1-16z+32rz+32z^2}. \tag{42}$$

The solution for y must be combined with the conditions

$$-1 \leqslant y \leqslant 1, \quad -1/8 \leqslant z \leqslant 1/8, \tag{43}$$

which flow, respectively, from the definition of y and from the assumption of having a single minimum of the potential energy per wavelength (Fig. 3), which we adopted earlier. That leaves only one of the roots,

$$\theta_0 = \arccos\left(\frac{1-D}{8z}\right), \tag{44}$$

and leads to the following trapping condition, which also guarantees that D is real:

$$z \leqslant r < 1+z. \tag{45}$$

Passing particles satisfy $r > 1+z$. Note that, in this case, D is real for all r when $z \geqslant 0$ and for $r \leqslant 1/2 + |z| + 1/(32|z|)$ when $z<0$.

E. Asymptotics

Below, we will also need asymptotics of j and $G_{1,2}$. For deeply trapped particles ($r \to z$), one can show that

$$j(r\to z)=0, \quad G_{1,2}(r\to z) = \mp 1, \tag{46}$$

whereas, for passing particles far from the resonance ($r \gg 1$), the asymptotics are derived as follows [46]. Away from the separatrix, $j(r,z)$ is an analytic function of z. Assuming $z \ll 1$ (which is verified *a posteriori*), one can hence replace $j(r,z)$ with its Taylor expansion,

$$j(r,z)=\sum_{n=0}^{\infty} z^n I_n(r). \tag{47}$$

The coefficients I_n are calculated (Appendix A) using

$$I_n(r)=\frac{2}{\pi}\,\frac{(-1)^n}{n!}\,\frac{d^n}{dr^n}\int_0^\pi \cos^n(2\theta)\,\sqrt{r-\sin^2(\theta/2)}\,d\theta.$$

By also expanding these functions in $1/r$, one gets

$$j(r,z)=\sqrt{r}\bigg[2-\frac{1}{2r}-\frac{3+4z^2}{32r^2}-\frac{5+3z+12z^2}{128r^3} - \frac{5\left(35+48z+144z^2\right)}{8192r^4}\dots\bigg], \tag{48}$$

where we omitted terms of higher orders (in both $1/r$ and z) as insignificant for our purposes. The inverse series is then found to be

$$r(j,z)=\frac{j^2}{4}+\frac{1}{2}+\frac{1+4z^2}{8j^2}+\frac{3z}{8j^4}+\frac{5(1+32z^2)}{128j^6}+\dots \tag{49}$$

This leads to the following equation for $\mathcal{E}$,

$$\frac{\mathcal{E}}{mu^2}=\frac{(J/\hat{J})^2}{2}+\frac{a_1^2+a_2^2}{4(J/\hat{J})^2}+\frac{3a_1^2a_2}{8(J/\hat{J})^4} + \frac{5(a_1^4+8a_1^2a_2^2)}{64(J/\hat{J})^6}+\dots, \tag{50}$$

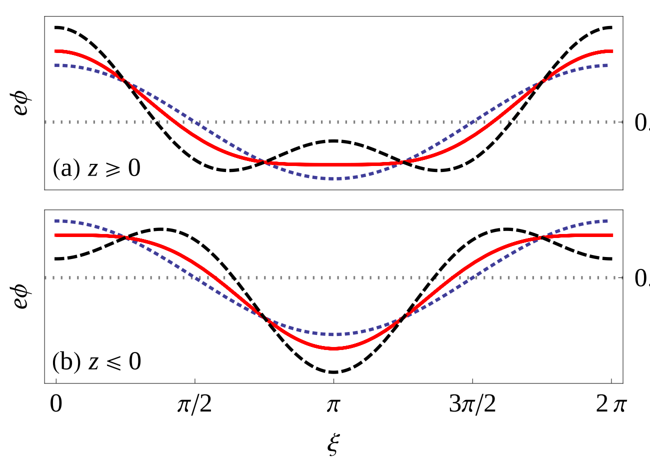

FIG. 3: (Color online) Potential energy, $e\phi$, vs the wave phase, ξ, for $z=0$ (dotted blue), $|z|=1/8$ (solid red), and $|z|=1/3$ (dashed black): (a) $z \geqslant 0$, (b) $z \leqslant 0$. A second (per period) minimum of $e\phi$ appears at $|z|>1/8$.

so Eqs. (34) lead to

$$G_1 = \frac{1}{2j^2} + \frac{3z}{2j^4} + \frac{5(1+16z^2)}{16j^6} + \ldots, \tag{51}$$

$$G_2 = \frac{z}{j^2} + \frac{3}{8j^4} + \frac{5z}{2j^6} + \ldots \tag{52}$$

Note also that, to the extent that z can be neglected, Eq. (49) can be cast in the following transparent form,

$$\mathcal{E} \approx \frac{(P')^2}{2m} + \frac{e^2k^2\phi_1^2}{4m(kP'/m)^2}. \tag{53}$$

Here $P' \doteq -kJ$, which serves as the canonical OC momentum in $\mathcal{K}'$. [This is seen if one writes Eq. (9) in $\mathcal{K}'$, where the frequency is zero.] Hence, the second term on the right-hand side is recognized as the ponderomotive potential (5) produced by a zero-frequency wave in $\mathcal{K}'$, and the above formula for $\mathcal{E}$ is recognized as an approximate formula for the canonical OC energy, Eq. (4).

IV. SMOOTH DISTRIBUTIONS

A. Basic equations

First, let us suppose that F is smooth compared to G, i.e., has a characteristic scale much larger than J_* (Fig. 4a), at least near the resonance. This allows approximating $\mathcal{G}_{1,2}$ as shown in Appendix C. Specifically, using Eq. (C23) with coefficients c_q inferred by comparing Eq. (C2) with Eqs. (51) and (52), one obtains

$$\mathcal{G}_1 \approx \frac{a_p}{2}\bigg\{[1-\epsilon(\omega,k)] - \omega^2\,\frac{\partial^2\epsilon(\omega,k)}{\partial\omega^2}\,\frac{za}{2} - \omega^4\,\frac{\partial^4\epsilon(\omega,k)}{\partial\omega^4}\,\frac{a^2}{192} + 4\eta_1\sqrt{a}\,u^3 f_0^{(2)}(u)\,\frac{\omega_p^2}{\omega^2}\bigg\}, \tag{54}$$

$$\mathcal{G}_2 \approx a_p\bigg\{[1-\epsilon(\omega,k)]z - \omega^2\,\frac{\partial^2\epsilon(\omega,k)}{\partial\omega^2}\,\frac{a}{16} + 2\eta_2\sqrt{a}\,u^3 f_0^{(2)}(u)\,\frac{\omega_p^2}{\omega^2}\bigg\}, \tag{55}$$

where we kept only the terms that will be relevant for our discussion (the applicability conditions will be discussed below; also see Appendix C) and introduced

$$\eta_1 \doteq \int_0^\infty [\Psi_1(j,z)j - 1/2]\,dj, \tag{56}$$

$$\eta_2 \doteq \int_0^\infty [\Psi_2(j,z)j - z]\,dj, \tag{57}$$

where $\Psi_{1,2}(j,z) \doteq -\int_0^j G_{1,2}(r(\tilde{j}),z)\,d\tilde{j}$. Hence, the NDR [Eqs. (33)] can be cast as follows:

$$0 = \epsilon(\omega,k) + \omega^2\,\frac{\partial^2\epsilon(\omega,k)}{\partial\omega^2}\,\frac{za}{2} + \omega^4\,\frac{\partial^4\epsilon(\omega,k)}{\partial\omega^4}\,\frac{a^2}{192} - 4\eta_1\sqrt{a}\,u^3 f_0^{(2)}(u)\,\frac{\omega_p^2}{\omega^2}, \tag{58}$$

$$0 = 3z + \omega^2\,\frac{\partial^2\epsilon(\omega,k)}{\partial\omega^2}\,\frac{a}{16} - 2\eta_2\sqrt{a}\,u^3 f_0^{(2)}(u)\,\frac{\omega_p^2}{\omega^2}. \tag{59}$$

The linear frequency is then found as a solution of

$$\epsilon(\omega_0,k) = 0. \tag{60}$$

Also one finds that $z = o(1)$, namely,

$$z \approx -\omega_0^2\,\frac{\partial^2\epsilon(\omega_0,k)}{\partial\omega_0^2}\,\frac{a_0}{48} + \frac{2}{3}\,\eta_2\sqrt{a_0}\,f_0^{(2)}(u_0)\,\frac{\omega_0\omega_p^2}{k^3} \tag{61}$$

(where $u_0 \doteq \omega_0/k$), and the nonlinear nonlinear frequency shift, $\delta\omega \doteq \omega - \omega_0$, is given by

$$\delta\omega \approx \left[\frac{\partial\epsilon(\omega_0,k)}{\partial\omega_0}\right]^{-1}\bigg[-\omega_0^2\,\frac{\partial^2\epsilon(\omega_0,k)}{\partial\omega_0^2}\,\frac{za_0}{2} - \omega_0^4\,\frac{\partial^4\epsilon(\omega_0,k)}{\partial\omega_0^4}\,\frac{a_0^2}{192} + 4\eta_1\sqrt{a_0}\,f_0^{(2)}(u_0)\,\frac{\omega_0\omega_p^2}{k^3}\bigg]. \tag{62}$$

The coefficients $\eta_{1,2}$ can be evaluated at $z = 0$:

$$\eta_1 \approx \int_0^\infty [\Psi_1(j,0)j - 1/2]\,dj \approx -0.27, \tag{63}$$

which is taken from Ref. [36], and

$$\eta_2 \approx \int_0^\infty \Psi_2(j,0)j\,dj = \frac{1}{2}\int_0^\infty G_2(r(j,0),0)j^2\,dj = \frac{1}{2}\int_0^\infty G_2(r,0)j^2(r,0)\,j_r(r,0)\,dr \approx 0.11, \tag{64}$$

which is calculated numerically. Below, we demonstrate how these equations are applied to specific distributions and show how our results correspond to those that were reported in the literature previously.

B. Fluid limit

Let us start with the fluid limit, when a is relatively large (but, still, $a \ll 1$), and $k\lambda_D \ll 1$; here $\lambda_D \doteq v_T/\omega_p$ is the Debye length, and v_T is the thermal speed. Then, $f_0(u_0)$ is negligibly small, so Eq. (61) yields

$$z = -\omega_0^2\,\frac{\partial^2\epsilon(\omega_0,k)}{\partial\omega_0^2}\,\frac{a_0}{48}. \tag{65}$$

By substituting this into Eq. (62) and ignoring the kinetic correction there too, we also obtain

$$\delta\omega \approx \frac{a_0^2}{96}\left[\frac{\partial\epsilon(\omega_0,k)}{\partial\omega_0}\right]^{-1} \times\left\{\left[\omega_0^2\,\frac{\partial^2\epsilon(\omega_0,k)}{\partial\omega_0^2}\right]^2 - \frac{\omega_0^4}{2}\,\frac{\partial^4\epsilon(\omega_0,k)}{\partial\omega_0^4}\right\}. \tag{66}$$

The effect $\delta\omega \propto a^2$ is called a fluid nonlinearity. To our knowledge, Eq. (66) is the first generalization of fluid nonlinearities to plasmas with arbitrary ϵ. Below we consider several special cases for illustration.

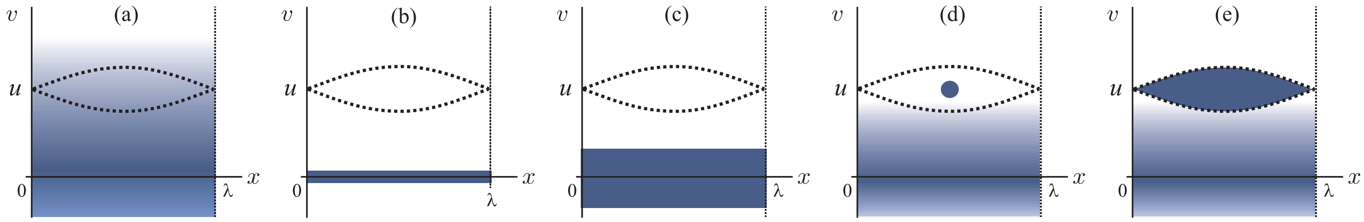

FIG. 4: Schematics of the specific phase space distributions that are discussed in the present paper: (a) arbitrary smooth distribution; (b) cold limit; (c) finite-temperature waterbag (flat) distribution; (d) deeply trapped particles; (e) flat distribution of trapped particles. In the cases of (d) and (e), details of the passing particle distribution are inessential as long as the beam nonlinearity (Sec. V) dominates. The dashed line is the boundary (separatrix) separating passing trajectories (outside) from trapped trajectories (inside). Also, x is the coordinate, $\lambda \doteq 2\pi/k$, and $v \doteq w + u$ is the velocity in the laboratory frame, $\mathcal{K}$.

1. *Cold limit*

In the limit of cold plasma that is initially at rest (Fig. 4b), one has

$$\epsilon(\omega, k) = 1 - \omega_p^2/\omega^2. \tag{67}$$

Then Eq. (60) gives $\omega_0 = \omega_p$, and Eqs. (65) and (66) give

$$z = a_0/8, \quad \delta\omega = \omega_p a_0^2/2, \tag{68}$$

in agreement with Ref. [5]. In this case, $\delta\omega$ consists entirely of the nonlinear Doppler shift (Sec. II), as can be seen by substituting Eq. (67) into Eq. (19).

2. *Waterbag distribution*

Now suppose the waterbag distribution discussed in Ref. [31]. Specifically, assume that the initial velocities of particles are distributed homogeneously within the interval $(-\bar{v}, \bar{v})$, where $\bar{v}$ is some constant (Fig. 4c). The corresponding dielectric function is calculated by directly taking the integral in Eq. (17) [47] and is given by

$$\epsilon(\omega, k) = 1 - \frac{\omega_p^2}{\omega^2 - k^2\bar{v}^2}. \tag{69}$$

Then Eq. (60) yields $\omega_0^2 = \omega_p^2 + k^2\bar{v}^2$, and Eqs. (65) and (66) lead to

$$z \approx \frac{a_0(3+\bar{\alpha})}{24(1-\bar{\alpha})^2}, \tag{70}$$

$$\delta\omega \approx \omega_0 a_0^2\, \frac{(6 + 9\bar{\alpha} + \bar{\alpha}^2)}{12(1-\bar{\alpha})^3}, \tag{71}$$

where $\bar{\alpha} \doteq (k\bar{v}/\omega_0)^2$. At $\bar{\alpha} \to 0$, Eq. (71) reproduces the result that we presented in Sec. IV B 1 for cold plasma.

One may notice that Eq. (71) is at variance with the result obtained in Ref. [31]. This is due to the fact that, in Ref. [31], the NDR is calculated not in the laboratory frame but rather in the frame where the average velocity is zero; hence the nonlinear Doppler shift is not included. For the distribution in question, Eq. (19) gives

$$k\langle\!\langle \Delta V \rangle\!\rangle \approx \frac{\omega_0 a^2}{2(1-\bar{\alpha})^2}.$$

Subtracting this from Eq. (71) leads to $\delta\omega = \omega_0 a^2 \bar{\alpha}(15 + \bar{\alpha})/[12(1-\bar{\alpha})^3]$, which reproduces the result of Ref. [31].

3. *Kappa distribution*

We also calculated the effect of the fluid nonlinearity on the dispersion of a nonlinear Langmuir wave [48] in electron plasma with the kappa distribution [49],

$$f_0(V_0) = \frac{\Gamma(\kappa+1)/\Gamma(\kappa-1/2)}{v_T \kappa\sqrt{\pi(2\kappa-3)}} \left[1 + \frac{V_0^2}{(2\kappa-3)v_T^2}\right]^{-\kappa}, \tag{72}$$

where $\kappa > 3/2$ is a constant dimensionless parameter, $v_T^2 = \int_{-\infty}^{\infty} v^2 f_0(v)\, dv$ is the thermal speed squared, and Γ is the gamma function. [The distribution (72) approaches the Maxwellian distribution at $\kappa \gg 1$, but has more pronounced tails at smaller κ.] The results are shown in Fig. 5. Notably, z is not monotonic in k, and $\delta\omega$ can have either sign. This is in variation with Ref. [31], where the fluid frequency shift was reported as strictly positive.

C. Kinetic limit

Now let us consider the opposite limit, when the amplitude is relatively small, whereas $k\lambda_D$ is substantial. In this case,

$$z = \frac{2}{3}\,\eta_2 \sqrt{a_0}\, f_0^{(2)}(u_0)\, \frac{\omega_0\omega_p^2}{k^3}, \tag{73}$$

$$\delta\omega \approx 4\eta_1 \sqrt{a_0}\, f_0^{(2)}(u_0)\, \frac{\omega_0\omega_p^2}{k^3} \left[\frac{\partial\epsilon(\omega_0,k)}{\partial\omega_0}\right]^{-1}. \tag{74}$$

The effect $\delta\omega \propto \sqrt{a_0}$ is called a kinetic nonlinearity. In particular, Eq. (73) shows that $\phi_2 \propto \phi_1^{3/2}$. This is in

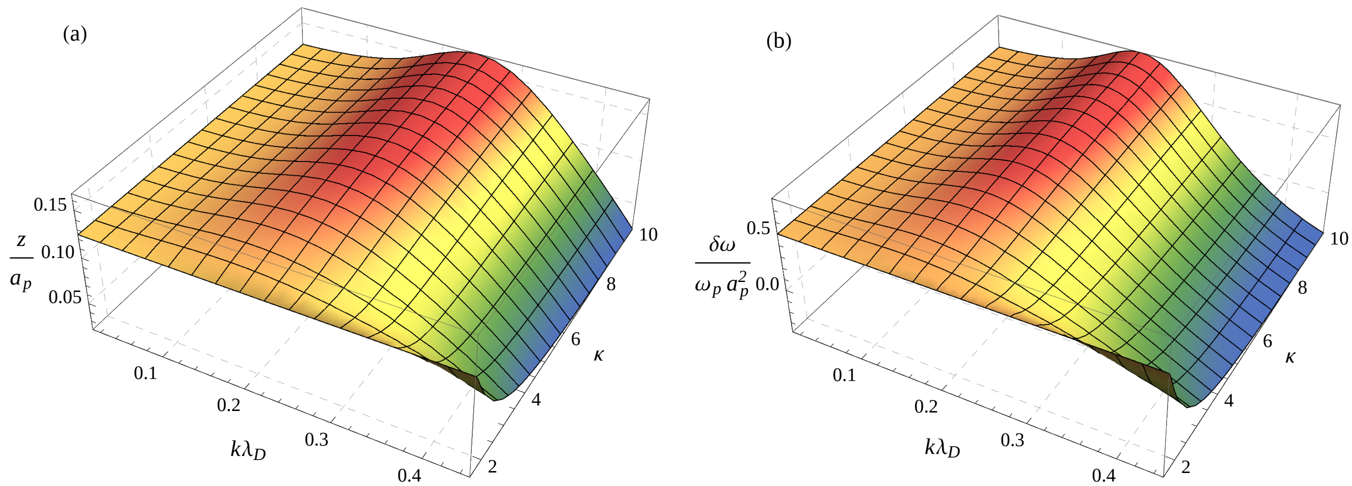

FIG. 5: (Color online) The effect of the fluid nonlinearity on the dispersion of a nonlinear Langmuir wave in electron plasma with the kappa distribution, Eq. (72): (a) normalized amplitude of the second harmonic, $z \doteq \phi_2/(2\phi_1)$, in units a_p; (b) nonlinear frequency shift, $\delta\omega$, in units $\omega_p a_p^2$. The horizontal axes are $k\lambda_D$ and the distribution parameter κ. At $\kappa \gg 1$, the distribution is approximately Maxwellian.

qualitative agreement with the (not self-consistent) estimate in Ref. [11], but notice that our numerical coefficient is different. As regarding Eq. (74), it is in precise agreement (modulo typos) with Refs. [34, 36] and also with the results reported in Ref. [9] for the "adiabatic excitation".

D. Fluid and kinetic nonlinearities combined

We also calculated the combined effect of the fluid and kinetic nonlinearities for a nonlinear Langmuir wave [48] in electron plasma with the Maxwellian distribution,

$$f_0(V_0) = \frac{1}{v_T\sqrt{2\pi}} \exp\left(-\frac{V_0^2}{2v_T^2}\right). \tag{75}$$

In this case,

$$\epsilon(\omega, k) = 1 - \frac{1}{2k^2\lambda^2}\, Z'\left(\frac{\omega}{kv_T\sqrt{2}}\right), \tag{76}$$

where $Z(\zeta) = -2S(\zeta)$, and S is the Dawson function, $S(\zeta) \doteq \exp(-\zeta^2)\int_0^\zeta \exp(y^2)\, dy$ [42, Chap. 8].

Two sets of results are presented. Figure 6 shows the results obtained by application of the approximate formulas (61) and (62). Figure 7 shows the result of the direct numerical solution of the more exact NDR, Eq. (33). It is seen that the approximate formulas provide a reasonable approximation to the exact solution, even though the applicability conditions of the asymptotic theory, (C24), are satisfied only marginally. As anticipated, one can see the following trends. At small enough $k\lambda_D$, the fluid nonlinearity dominates, leading to Eqs. (68). (The fact that, in the figure, $\delta\omega$ remains finite at vanishing $k\lambda_D \propto v_T$ is due to the choice of the units in which the amplitude is measured.) At larger $k\lambda_D$, the kinetic nonlinearity becomes dominating. That said, comparison with Fig. 5 shows that the presence of the kinetic nonlinearity does not affect the picture qualitatively, contrary to Ref. [31].

V. BEAM DISTRIBUTIONS

In addition to fluid and kinetic nonlinearities discussed above, waves with trapped particles can also exhibit nonlinearities of a third, beam type. Those emerge when, *on top* of a distribution F_0 that is smooth or negligibly small near the resonance, an additional distribution F_b of a trapped beam is superimposed that has abrupt boundaries within or at the edge of the trapping island. Such distributions, with both positive F_b (clumps) and negative F_b (holes), are known in various contexts [22–28, 37, 50, 51]. Below, we consider some of them in detail.

A. Basic equations

The presence of F_b results in the appearance of additional terms,

$$\tilde{\mathcal{G}}_{1,2} \doteq \int G_{1,2} F_\mathsf{b}(J)\, dJ, \tag{77}$$

in Eqs. (54) and (55). For simplicity, we will suppose that F_b is large enough, such that its nonlinear effect on the NDR dominates over the nonlinear effect produced by F_0. Then,

$$\mathcal{G}_1 \approx (a_p/2)[1 - \epsilon(\omega, k)] + \tilde{\mathcal{G}}_1, \tag{78}$$

$$\mathcal{G}_2 \approx a_p z[1 - \epsilon(\omega, k)] + \tilde{\mathcal{G}}_2, \tag{79}$$

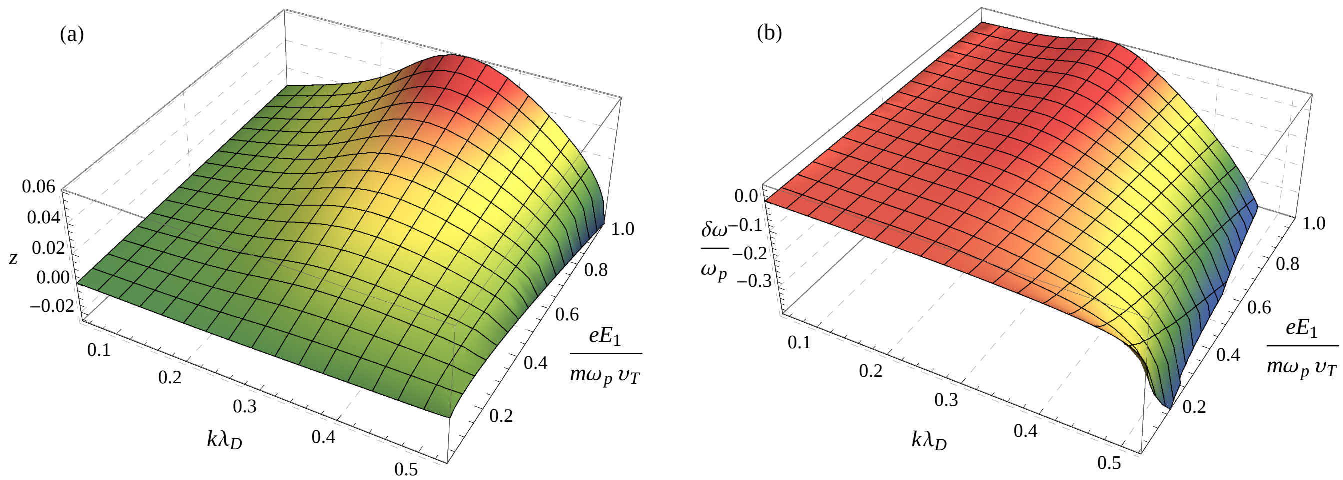

FIG. 6: (Color online) The combined effect of the fluid and kinetic nonlinearities on the dispersion of a nonlinear Langmuir wave in electron plasma with the Maxwellian distribution, Eq. (75): (a) normalized amplitude of the second harmonic, $z \doteq \phi_2/(2\phi_1)$; (b) nonlinear frequency shift, $\delta\omega$, in units ω_p. The horizontal axes are $k\lambda_D$ and $eE_1/(m\omega_p v_T) = v_{\rm osc}/v_T$, where $E_1 \doteq k\phi_1$ is the amplitude of the first harmonic of the wave electric field, and $v_{\rm osc} \doteq eE_1/(m\omega_p)$ is the characteristic amplitude of the electron velocity oscillations. The results are obtained by application of the approximate formulas (61) and (62).

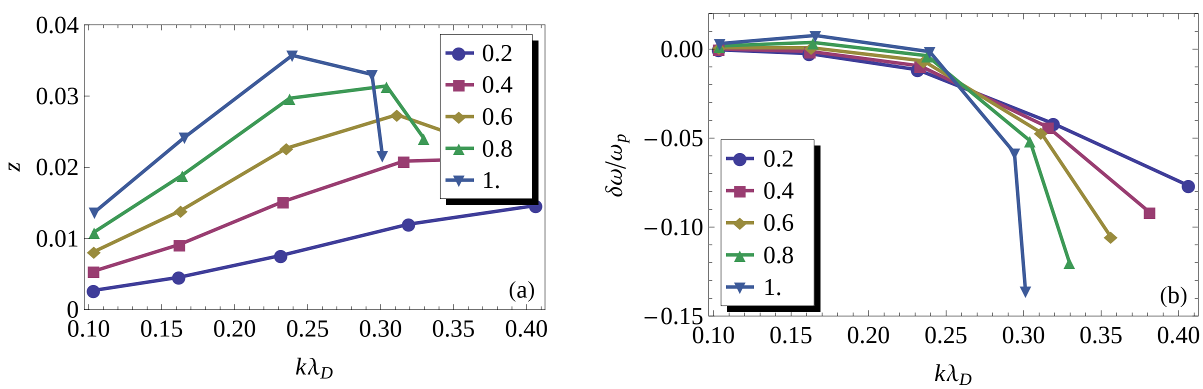

FIG. 7: (Color online) Same as in Fig. 6 but using the more exact NDR, Eq. (33), and only for selected values of $eE/(m\omega_p v_T)$ that are specified in the legend.

so Eqs. (33) yield the NDR in the form

$$z \approx \frac{\tilde{\mathcal{G}}_2}{3a_p}, \quad \delta\omega \approx \frac{2\tilde{\mathcal{G}}_1}{a_p}\left[\frac{\partial\epsilon(\omega_0,k)}{\partial\omega_0}\right]^{-1}. \tag{80}$$

Notice that the wave is close to linear in this case not when a_p is small, like before, but rather when $\beta \doteq \bar{n}_{\mathsf{b}}/(\bar{n}a_p)$ is small, where $\bar{n}_{\mathsf{b}}$ is the beam density. (In other words, a_p must be large enough.) The NDRs for specific distributions are derived as follows.

B. Deeply trapped particles

Suppose that trapped particles reside at the very bottom of the wave troughs (Fig. 4d); i.e.,

$$F_{\mathsf{b}}(J) = (\bar{n}_{\mathsf{b}}/\bar{n})\,\delta(J). \tag{81}$$

[Strictly speaking, the delta function here must be understood as the limit of $\delta(J - J_c)$ at $J_c \to 0$.] Hence, all trapped particles have $r = z$, so Eqs. (46), (77), and (81) yield $\tilde{\mathcal{G}}_{1,2} = \mp\bar{n}_{\mathsf{b}}/\bar{n}$. Then, Eqs. (80) give

$$z \approx \frac{\beta}{3}, \quad \frac{\delta\omega}{\omega_p} \approx -\frac{2\beta}{\omega_p}\left[\frac{\partial\epsilon(\omega_0,k)}{\partial\omega_0}\right]^{-1}. \tag{82}$$

The obtained expression for $\delta\omega$ agrees with the well known result [22, 23]. It is also easy to see that the value of z is in agreement with what one can infer from the Vlasov-Poisson system directly [52].

C. Homogeneous clumps and holes

1. Basic equations

Let us also consider the case when F_{b} is flat across the whole trapping area (Fig. 4e). In this case, $F_{\mathsf{b}}(J < J_*) = \hat{F}$, where $\hat{F}$ is some constant. Then,

$$\tilde{\mathcal{G}}_{1,2} = \hat{J}\sqrt{a}\hat{F}\int_z^{1+z} G_{1,2}(r,z)\, j_r(r,z)\, dr = a_p\mathcal{N}\gamma_{1,2}(z) \approx a_p\mathcal{N}\gamma_{1,2}(0), \quad (83)$$

where we introduced the dimensionless quantities

$$\mathcal{N} \doteq \frac{m\omega_p}{k}\frac{\hat{F}}{\sqrt{a_p}} \sim \beta, \quad (84)$$

$$\gamma_{1,2}(z) \doteq \int_z^{1+z} G_{1,2}(r,z)\, j_r(r,z)\, dr. \quad (85)$$

Hence, one obtains

$$z \approx \frac{1}{3}\gamma_2(0)\mathcal{N}, \quad (86)$$

$$\delta\omega \approx 2\gamma_1(0)\mathcal{N}\left[\frac{\partial\epsilon(\omega_0,k)}{\partial\omega_0}\right]^{-1}. \quad (87)$$

We find numerically that $\gamma_2(0) \approx -0.085$, and $\gamma_1(0) = -4/(3\pi)$ was reported already in Ref. [36].

2. Discussion

Since $\tilde{\mathcal{G}}_1$ given by Eq. (83) is independent of ω, one can also derive a more precise analytic expression for ω from Eq. (78) when ϵ is a simple enough function. For instance, for cold plasma [Eq. (67)], one gets

$$\frac{\omega}{\omega_p} = \left(1 + \frac{8\mathcal{N}}{3\pi}\right)^{-1/2} = 1 - \frac{4\mathcal{N}}{3\pi} + \frac{8\mathcal{N}^2}{3\pi^2} + O(\mathcal{N}^3). \quad (88)$$

Notably, phase space clumps in this case correspond to $\delta\omega < 0$, whereas phase space holes correspond to $\delta\omega > 0$.

Let us compare results predicted by Eq. (88) for ω, as well as those by Eq. (86) for z, with the exact kinetic solution, which happens to exist in this special case. [The word "exact" here refers to the description of trapped particles, whereas the bulk plasma is still modeled with a linear dielectric function (67).] Assuming the notation $\alpha \doteq (\pi/2)(\omega_p/\omega)$, the exact solution is given by [53]

$$z = \frac{\pi^2 - 4\alpha^2}{32\pi^2 - 8\alpha^2}, \quad (89)$$

$$\mathcal{N}^{-2} = \frac{2\tan\alpha}{3\alpha}\frac{\pi^4}{\pi^4 - 5\pi^2\alpha^2 + 4\alpha^4}, \quad (90)$$

or, more explicitly for small $\mathcal{N}$,

$$z = -\frac{4\mathcal{N}}{45\pi} + O(\mathcal{N}^2), \quad (91)$$

$$\frac{\omega}{\omega_p} = 1 - \frac{4\mathcal{N}}{3\pi} + \frac{68\mathcal{N}^2}{27\pi^2} + O(\mathcal{N}^3). \quad (92)$$

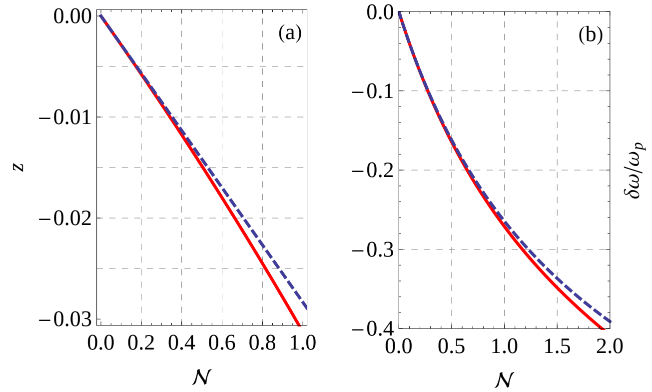

FIG. 8: (Color online) The effect of the beam nonlinearity on the dispersion of a nonlinear Langmuir wave in cold electron plasma for the case when the trapping islands contain homogeneous clumps; i.e., $F_{\mathsf{b}}(J < J_*) = \hat{F} > 0$ (Fig. 4e): (a) normalized amplitude of the second harmonic, $z \doteq \phi_2/(2\phi_1)$; (b) nonlinear frequency shift, $\delta\omega$, in units ω_p. The coordinate $\mathcal{N}$, which is given by Eq. (84), represents a dimensionless measure of the trapped-particle phase space density. The red solid curves correspond to the exact solutions, Eqs. (89) and (90). The blue dashed curves correspond to our asymptotic solutions, Eqs. (86) and (88).

As seen in Fig. 8, the results of our asymptotic theory are virtually indistinguishable from the predictions of Eqs. (89) and (90) at $\mathcal{N} \lesssim 1$ and even beyond. One conclusion from this is that the long-range sweeping of energetic-particle modes, which is described in Ref. [27] by means of the aforementioned exact solution, can be described with fidelity just using a simple asymptotic NDR, Eq. (88). [In fact, Eq. (88) was first proposed in Ref. [26].] Using the theory presented here, asymptotic NDRs can be derived also when exact analytic solutions do not exist. For example, in addition to beam nonlinearities, one may need to account for fluid nonlinearities when $a_p^2 \sim \mathcal{N}$, and for kinetic nonlinearities when $a_p \sim \mathcal{N}^2$ and $k\lambda_D \sim 1$. That is done simply by retaining *all* the terms in the expressions for $\mathcal{G}_{1,2}$, including not only $\tilde{\mathcal{G}}_{1,2}$ but also the other terms that appear in Eqs. (54) and (55). To the leading order, the nonlinear effects stemming from fluid, kinetic, and beam effects then enter the NDR additively.

VI. CONCLUSIONS

In summary, we derived a transparent asymptotic expression for the nonlinear frequency shift of intense Langmuir waves in general collisionless plasma. The expression [Eq. (62)] describes kinetic and fluid effects simultaneously. The kinetic nonlinearity accounts for both smooth distributions [Eq. (74)] and trapped-particle beams (Sec. V). The fluid nonlinearity is cast in terms of the plasma dielectric function [Eq. (66)] and, contrary to the common presumption, can have either sign. This formulation is benchmarked against the many previously known NDRs and is shown to reproduce them as special cases of a single unifying theory. For example, we calculate the NDRs for Langmuir waves in plasmas with cold, waterbag, kappa, and Maxwellian distributions. These specific results and our general method are applicable, e.g., to waves produced at intense laser-plasma interactions and to energetic-particle modes in tokamaks. Our asymptotic formulation, in fact, may be advantageous over exact kinetic solutions for such waves. While offering a reasonable precision, it leads to simpler (and thus more elucidating) results, does not involve solving any differential equations, and can be extended straightforwardly to other nonlinear plasma waves.

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APPENDIX A: FORMULAS FOR j AND $G_{1,2}$

The explicit formulas for the auxiliary functions introduced in the main text can be expressed in terms of the following special functions,

$$\mathsf{K}(\zeta) \doteq \int_0^{\pi/2} (1-\zeta\sin^2\theta)^{-1/2} d\theta,$$
$$\mathsf{E}(\zeta) \doteq \int_0^{\pi/2} (1-\zeta\sin^2\theta)^{1/2} d\theta,$$
$$\mathsf{\Pi}(\vartheta,\zeta) \doteq \int_0^{\pi/2} (1-\vartheta\sin^2\theta)^{-1}(1-\zeta\sin^2\theta)^{-1/2} d\theta,$$

which are the complete elliptic integrals of the first, second, and third kind, respectively [46].

1. Normalized action

The explicit formulas for the function j are as follows. For trapped particles ($z \leqslant r < 1+z$), one has

$$j = \frac{2}{\pi}\left[A_\mathsf{t}^\mathsf{K}\mathsf{K}(R_\mathsf{t}) + A_\mathsf{t}^\mathsf{E}\mathsf{E}(R_\mathsf{t}) + A_\mathsf{t}^\mathsf{\Pi}\mathsf{\Pi}(N_\mathsf{t},R_\mathsf{t})\right], \quad \text{(A1)}$$
$$A_\mathsf{t}^\mathsf{K} = -D^{1/2}\,\frac{2r+6z-1-D}{8z+1-D}, \quad \text{(A2)}$$
$$A_\mathsf{t}^\mathsf{E} = D^{1/2}, \quad \text{(A3)}$$
$$A_\mathsf{t}^\mathsf{\Pi} = D^{-1/2}\,\frac{2r+6z-1-D}{8z+1-D}, \quad \text{(A4)}$$
$$R_\mathsf{t} = \frac{2r+6z-1+D}{2D}, \quad \text{(A5)}$$
$$N_\mathsf{t} = \frac{8z-1+D}{2D}. \quad \text{(A6)}$$

For passing particles ($r > 1+z$), one has

$$j = \frac{2}{\pi}\left[A_\mathsf{p}^\mathsf{K}\mathsf{K}(R_\mathsf{p}) + A_\mathsf{p}^\mathsf{E}\mathsf{E}(R_\mathsf{p}) + A_\mathsf{p}^\mathsf{\Pi}\mathsf{\Pi}(N_\mathsf{p},R_\mathsf{p})\right], \quad \text{(A7)}$$
$$A_\mathsf{p}^\mathsf{K} = \frac{\sqrt{2}(r-z-1)}{\sqrt{2r+6z-1+D}}\,\frac{8z-1+D}{8z+1+D}, \quad \text{(A8)}$$
$$A_\mathsf{p}^\mathsf{E} = \sqrt{\frac{2r+6z-1+D}{2}}, \quad \text{(A9)}$$
$$A_\mathsf{p}^\mathsf{\Pi} = \frac{2\sqrt{2}(r-z-1)}{\sqrt{2r+6z-1+D}}\,\frac{1}{8z+1+D}, \quad \text{(A10)}$$
$$R_\mathsf{p} = \frac{2D}{2r+6z-1+D}, \quad \text{(A11)}$$
$$N_\mathsf{p} = \frac{1-8z+D}{2(r-z)}. \quad \text{(A12)}$$

In particular, the first three coefficients in Eq. (47) are

$$I_1 = \frac{4}{\pi}\sqrt{r}\,\mathsf{E}(r^{-1}), \quad \text{(A13)}$$

$$I_2 = \frac{1}{3\pi\sqrt{r}}\,[16r(-1+2r)\mathsf{E}(r^{-1}) - 2\left(3-16r+16r^2\right)\mathsf{K}(r^{-1})], \quad \text{(A14)}$$

$$I_3 = \frac{1}{10\pi(r-1)\sqrt{r}} \times [(-5+224r-1248r^2+2048r^3-1024r^4)\mathsf{E}(r^{-1}) + 16(5-47r+138r^2-160r^3+64r^4)\mathsf{K}(r^{-1})].$$

It is also to be noted that, in the limit $z \to 0$, these formulas are commonly known; for example, see Ref. [36].

2. Functions $G_{1,2}$

The explicit formulas for the functions $G_{1,2}$ are as follows. For trapped particles ($z \leqslant r < 1+z$), one has

$$G_1 = \bar{A}_\mathsf{t}^\mathsf{K} + \frac{\bar{A}_\mathsf{t}^\mathsf{\Pi}\mathsf{\Pi}(N_\mathsf{t},R_\mathsf{t})}{\mathsf{K}(R_\mathsf{t})}, \quad \text{(A15)}$$
$$G_2 = \bar{B}_\mathsf{t}^\mathsf{K} + \frac{\bar{B}_\mathsf{t}^\mathsf{E}\mathsf{E}(R_\mathsf{t}) + \bar{B}_\mathsf{t}^\mathsf{\Pi}\mathsf{\Pi}(N_\mathsf{t},R_\mathsf{t})}{\mathsf{K}(R_\mathsf{t})}, \quad \text{(A16)}$$
$$\bar{A}_\mathsf{t}^\mathsf{K} = -\frac{1+D}{8z}, \quad \text{(A17)}$$
$$\bar{A}_\mathsf{t}^\mathsf{\Pi} = \frac{1-8z+D}{8z}, \quad \text{(A18)}$$
$$\bar{B}_\mathsf{t}^\mathsf{K} = \frac{1-32z^2+(1+8z)D}{32z^2}, \quad \text{(A19)}$$
$$\bar{B}_\mathsf{t}^\mathsf{E} = -\frac{D}{2z}, \quad \text{(A20)}$$
$$\bar{B}_\mathsf{t}^\mathsf{\Pi} = -\frac{1-8z+D}{32z^2}. \quad \text{(A21)}$$

For passing particles ($r > 1+z$), one has

$$G_1 = \bar{A}_{\mathsf{p}}^{\mathsf{K}} + \frac{\bar{A}_{\mathsf{p}}^{\Pi}\Pi(N_{\mathsf{p}}, R_{\mathsf{p}})}{\mathsf{K}(R_{\mathsf{p}})}, \tag{A22}$$

$$G_2 = \bar{B}_{\mathsf{p}}^{\mathsf{K}} + \frac{\bar{B}_{\mathsf{p}}^{\mathsf{E}}\mathsf{E}(R_{\mathsf{p}}) + \bar{B}_{\mathsf{p}}^{\Pi}\Pi(N_{\mathsf{p}}, R_{\mathsf{p}})}{\mathsf{K}(R_{\mathsf{p}})}, \tag{A23}$$

$$\bar{A}_{\mathsf{p}}^{\mathsf{K}} = -\frac{1-D}{8z}, \tag{A24}$$

$$\bar{A}_{\mathsf{p}}^{\Pi} = \frac{1+8z-D}{8z}, \tag{A25}$$

$$\bar{B}_{\mathsf{p}}^{\mathsf{K}} = \frac{1-8z+16rz+16z^2-D}{32z^2}, \tag{A26}$$

$$\bar{B}_{\mathsf{p}}^{\mathsf{E}} = \frac{1-2r-6z-D}{4z}, \tag{A27}$$

$$\bar{B}_{\mathsf{p}}^{\Pi} = -\frac{1+8z-D}{32z^2}. \tag{A28}$$

It is to be noted that, in the limit $z \to 0$, the function G_1 was reported also in Ref. [36].

APPENDIX B: BULK PLASMA ACCELERATION BY HOMOGENEOUS WAVE EXCITATION

When a homogeneous electrostatic wave is excited in plasma, the plasma generally changes its average velocity by some $\langle\!\langle \Delta V \rangle\!\rangle$. One derivation of this effect was presented in Sec. II C using the OC formalism. Here, we show how a hydrodynamic model yields the same result.

Consider the equation for the plasma momentum density, $\mathcal{P} = mnv$, namely,

$$\partial_t \mathcal{P} + \partial_x (mnv^2) = neE - \partial_x \Pi, \tag{B1}$$

where n is the particle density, v is the flow velocity, and Π is the pressure. By averaging over x, which we denote with $\langle \ldots \rangle_x$, one gets

$$\partial_t \langle \mathcal{P} \rangle_x = \langle neE \rangle_x \approx \langle \tilde{n} e \tilde{E} \rangle_x \tag{B2}$$

(the tilde here denotes quantities of the first order in the field), where we used that the gradient of a periodic function averages to zero. Note now that

$$e\tilde{E} = m\partial_t \tilde{v} + \partial_x \tilde{\Pi}/\bar{n}. \tag{B3}$$

Assuming $\Pi = \Pi(n)$, we have $\tilde{n}\,\partial_x \tilde{\Pi}/\bar{n} = \text{const} \times \partial_x \tilde{n}^2$, so

$$\langle \tilde{n} e \tilde{E} \rangle_x = m\langle \tilde{n} \partial_t \tilde{v} \rangle_x = m\partial_t \langle \tilde{n}\tilde{v} \rangle_x - m\langle \tilde{v} \partial_t \tilde{n} \rangle_x. \tag{B4}$$

From the continuity equation, we also have $\partial_t \tilde{n} = -\bar{n}\partial_x \tilde{v}$. This leads to

$$\begin{aligned} \langle \tilde{n} e \tilde{E} \rangle_x &= m\partial_t \langle \tilde{n}\tilde{v} \rangle_x + m\bar{n}\langle \tilde{v}\partial_x \tilde{v} \rangle_x \\ &= \partial_t \langle m\tilde{n}\tilde{v} \rangle_x + \langle \partial_x (m\bar{n}\tilde{v}^2/2) \rangle_x \\ &= \partial_t \langle m\tilde{n}\tilde{v} \rangle_x. \end{aligned} \tag{B5}$$

Thus, $\partial_t \langle \mathcal{P} \rangle_x = \partial_t \langle m\tilde{n}\tilde{v} \rangle_x$, or, after integrating over t,

$$\langle \Delta \mathcal{P} \rangle_x = \langle m\tilde{n}\tilde{v} \rangle_x = mu\langle \tilde{n}^2 \rangle_x/\bar{n} = m\bar{n}\langle \tilde{v}^2 \rangle_x/u, \tag{B6}$$

where we also substituted the continuity equation, $\tilde{n} \approx \bar{n}\tilde{v}/u$. Then, $\langle\!\langle \Delta V \rangle\!\rangle \doteq \langle \Delta \mathcal{P} \rangle_x/(m\bar{n})$ is given by

$$\langle\!\langle \Delta V \rangle\!\rangle = u\langle (\tilde{n}/\bar{n})^2 \rangle_x = \langle \tilde{v}^2 \rangle_x/u. \tag{B7}$$

Since, in the assumed model, one has $\epsilon(\omega, k) = 1 - \omega_p^2/(\omega^2 - 3k^2 v_T^2)$, where $3v_T^2 \doteq \partial_{\bar{n}}\Pi(\bar{n})/m$ [42, Chap. 3], Eq. (18) is then easily recovered.

APPENDIX C: APPROXIMATIONS FOR INTEGRALS OVER SMOOTH DISTRIBUTIONS

Here, we show how to approximate integrals like

$$\mathcal{G} \doteq \int_0^\infty GF(J)\,dJ, \tag{C1}$$

where F is smooth compared to G, and $G \equiv G(r(j)) \equiv G(j)$ is either G_1 or G_2, so it has asymptotics [cf. Eqs. (51) and (52)]

$$G(j) = \frac{c_2}{j^2} + \frac{3c_4}{j^4} + \frac{5c_6}{j^6} + \ldots, \tag{C2}$$

where c_q are independent of j. (The dependence on z is implied but will not be emphasized in this appendix.) The asymptotics are derived as follows. First of all, let us introduce an auxiliary function

$$Y(J) \doteq -\int_0^J G\,d\tilde{J}. \tag{C3}$$

It is convenient to express it also as $Y = \hat{J}\sqrt{a}\,\Psi$, where

$$\Psi \doteq -\int_0^j G(\tilde{j})\,d\tilde{j} \tag{C4}$$

is a dimensionless function with the following properties:

$$\Psi(j \to 0) = 0, \quad \Psi(j \to \infty) = 0. \tag{C5}$$

The former equality is obvious from the definition, and the latter equality is understood as follows. Notice that, with G being either of G_i ($i = 1, 2$), the function $\Psi(j)$ is proportional to $\mathcal{G}_i$ calculated on the waterbag (flat) distribution with thermal speed $\bar{v} \propto j$. At $\bar{v} \to \infty$, the boundaries of this distribution are infinitely far from the resonance, so the trajectories on these boundaries are unaffected by the wave. Thus, $\langle\!\langle \mathcal{E} \rangle\!\rangle$ is insensitive to the wave amplitude, and thus both $\mathcal{G}_1$ and $\mathcal{G}_2$ are zero. [For $z = 0$, this is also shown in Ref. [36] via a straightforward calculation.] Thus, in the limit of large j, the function $\Psi(j)$ must vanish, which proves the second part of Eq. (C5) and also leads to the following asymptotics at large j:

$$\Psi(j) = \frac{c_2}{j} + \frac{c_4}{j^3} + \frac{c_6}{j^5} + \ldots \tag{C6}$$

Hence one can rewrite $\mathcal{G}$ as follows,

$$\mathcal{G} = -\int_0^\infty Y'(J)F(J)\,dJ = \int_0^\infty Y(J)F'(J)\,dJ. \quad \text{(C7)}$$

Using the fact that F' is smooth compared to Y, we can approximate the integrand, $Q(J) \doteq Y(J)F'(J)$, as

$$Q(J) = Q_1(J) + Q_2(J) - Q_3(J), \quad \text{(C8)}$$

where the indexes denote the small-J, large-J, and intermediate-J asymptotics, correspondingly. At small J, Q_3 cancels Q_2, so $Q(J) \approx Q_1(J)$. At large J, Q_3 cancels Q_1, so $Q(J) \approx Q_2(J)$. Thus, Eq. (C8) approximates the true function Q accurately at all J.

To construct the specific expressions for $Q_{1,3}$, notice that the derivatives of F at $J = 0$, $F_0^{(q)} \doteq F^{(q)}(0)$, are yielded by Eq. (14) in the form [54]

$$F_0^{(q)} = 2f_0^{(q)}(u)\left(\frac{k}{m}\right)^{q+1} \times \begin{cases} 0, & q \text{ is odd}, \\ 1, & q \text{ is even}. \end{cases} \quad \text{(C9)}$$

Then, much like in Ref. [36], one can take

$$Q_1(J) = Y(J)\left(F_0^{(2)}J + F_0^{(4)}\frac{J^3}{3!} + F_0^{(6)}\frac{J^5}{5!} + \dots\right), \quad \text{(C10)}$$

$$Q_2(J) = \hat{J}^2 a\left(\frac{c_2}{J} + \frac{c_4}{J^3}\,\hat{J}^2 a + \frac{c_6}{J^5}\,\hat{J}^4 a^2 + \dots\right)F'(J), \quad \text{(C11)}$$

$$Q_3(J) = \hat{J}^2 a\left(\frac{c_2}{J} + \frac{c_4}{J^3}\,\hat{J}^2 a + \frac{c_6}{J^5}\,\hat{J}^4 a^2 + \dots\right)\left(F_0^{(2)}J + F_0^{(4)}\frac{J^3}{3!} + F_0^{(6)}\frac{J^5}{5!} + \dots\right). \quad \text{(C12)}$$

After substituting this into Eq. (C8) and rearranging the terms, we can express Q as

$$\begin{aligned} Q = \hat{J}^2 a \left[F'(J)\left(\frac{c_2}{J} + \frac{c_4}{J^3}\,\hat{J}^2 a + \frac{c_6}{J^5}\,\hat{J}^4 a^2\right) - F_0^{(2)}J\left(\frac{c_4}{J^3}\,\hat{J}^2 a + \frac{c_6}{J^5}\,\hat{J}^4 a^2\right) - F_0^{(4)}\frac{J^3}{3!}\left(\frac{c_6}{J^5}\,\hat{J}^4 a^2\right)\right] \\ + F_0^{(2)}(\hat{J}^2 a)\,[\Psi(j)j - c_2] + F_0^{(4)}(\hat{J}^2 a)^2\,\frac{1}{3!}\,[\Psi(j)j^3 - c_2 j^2 - c_4] + F_0^{(6)}(\hat{J}^2 a)^3\,\frac{1}{5!}\,[\Psi(j)j^5 - c_2 j^4 - c_4 j^2 - c_6] + \dots \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \mathcal{G} = \big[(\hat{J}^2 a)c_2\chi^{(2)} + (\hat{J}^2 a)^2 c_4\chi^{(4)} + (\hat{J}^2 a)^3 c_6\chi^{(6)} + \dots\big] \\ + \big[(\hat{J}^2 a)^{3/2}F_0^{(2)}\eta^{(2)} + (\hat{J}^2 a)^{5/2}F_0^{(4)}\eta^{(4)} + (\hat{J}^2 a)^{7/2}F_0^{(6)}\eta^{(6)} + \dots\big], \end{aligned} \quad \text{(C13)}$$

where the coefficients are defined as the following manifestly converging integrals:

$$\chi^{(2)} \doteq \int_0^\infty J^{-1}F'(J)\,dJ, \quad \text{(C14)}$$

$$\chi^{(4)} \doteq \int_0^\infty J^{-3}[F'(J) - F_0^{(2)}J]\,dJ, \quad \text{(C15)}$$

$$\chi^{(6)} \doteq \int_0^\infty J^{-5}[F'(J) - F_0^{(2)}J - F_0^{(4)}J^3/3!]\,dJ, \quad \text{(C16)}$$

$$\eta^{(2)} \doteq \int_0^\infty [\Psi(j)j - c_2]\,dj, \quad \text{(C17)}$$

$$\eta^{(4)} \doteq \frac{1}{3!}\int_0^\infty [\Psi(j)j^3 - c_2 j^2 - c_4]\,dj, \quad \text{(C18)}$$

$$\eta^{(6)} \doteq \frac{1}{5!}\int_0^\infty [\Psi(j)j^5 - c_2 j^4 - c_4 j^2 - c_6]\,dj. \quad \text{(C19)}$$

It is convenient to express $\chi^{(2)}$ in terms of the dielectric function given by Eq. (20). This is done as follows:

$$\begin{aligned}
\chi^{(2)} &= \frac{k^2}{m^2}\int_0^\infty \left[f_0'(u+kJ/m) - f_0'(u-kJ/m)\right]\frac{dJ}{J}\\
&= \frac{k^2}{m^2}\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \left[f_0'(u+w) - f_0'(u-w)\right]\frac{dw}{w}\\
&= \frac{k^2}{m^2}\,\mathsf{P}\int_{-\infty}^\infty \frac{f_0'(v)}{v-u}\,dw\\
&= \frac{k^4}{m^2\omega_p^2}\left[1-\epsilon(\omega,k)\right].
\end{aligned} \tag{C20}$$

Using the formulas derived in Appendix D [namely, Eqs. (D7) and (D12)], we also obtain

$$\begin{aligned}
\chi^{(4)} &= \frac{k^2}{m^2}\int_0^\infty \left[f_0'(u+kJ/m) - f_0'(u-kJ/m) - 2f_0^{(2)}(u)\,(kJ/m)\right]\frac{dJ}{J^3}\\
&= \frac{k^4}{m^4}\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \left[f_0'(u+w) - f_0'(u-w) - 2f_0^{(2)}(u)w\right]\frac{dw}{w^3}\\
&= \frac{k^4}{m^4}\,\mathsf{P}\int_{-\infty}^\infty \left[f_0'(u+w) - f_0^{(2)}(u)w\right]\frac{dw}{w^3}\\
&= \frac{k^4}{m^4}\,\frac{1}{2}\,\frac{\partial^2}{\partial u^2}\,\mathsf{P}\int_{-\infty}^\infty \frac{f_0'(v)}{v-u}\,dv\\
&= -\frac{1}{2}\,\frac{k^8}{m^4\omega_p^2}\,\frac{\partial^2\epsilon(\omega,k)}{\partial\omega^2},
\end{aligned} \tag{C21}$$

$$\begin{aligned}
\chi^{(6)} &= \frac{k^2}{m^2}\int_0^\infty \left[f_0'(u+kJ/m) - f_0'(u-kJ/m) - 2f_0^{(2)}(u)\,kJ/m - 2f_0^{(4)}(u)\,\frac{(kJ/m)^3}{3!}\right]\frac{dJ}{J^5}\\
&= \frac{k^6}{m^6}\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \left[f_0'(u+w) - f_0'(u-w) - 2f_0^{(2)}(u)w - 2f_0^{(4)}(u)\,\frac{w^3}{3!}\right]\frac{dw}{w^5}\\
&= \frac{k^6}{m^6}\,\mathsf{P}\int_{-\infty}^\infty \left[f_0'(u+w) - f_0^{(2)}(u)w - f_0^{(4)}(u)\,\frac{w^3}{3!}\right]\frac{dw}{w^5}\\
&= \frac{k^6}{m^6}\,\frac{1}{4!}\,\frac{\partial^4}{\partial u^4}\,\mathsf{P}\int_{-\infty}^\infty \frac{f_0'(v)}{v-u}\,dv\\
&= -\frac{1}{4!}\,\frac{k^{12}}{m^6\omega_p^2}\,\frac{\partial^4\epsilon(\omega,k)}{\partial\omega^4}.
\end{aligned} \tag{C22}$$

Then one can rewrite Eq. (C13) as

$$\begin{aligned}
\mathcal{G} = \frac{a\omega^2}{\omega_p^2}\left\{[1-\epsilon(\omega,k)]c_2 - \frac{\omega^2}{2!}\,\frac{\partial^2\epsilon(\omega,k)}{\partial\omega^2}\,c_4 a - \frac{\omega^4}{4!}\,\frac{\partial^4\epsilon(\omega,k)}{\partial\omega^4}\,c_6 a^2 + \ldots\right\}\\
+ 2a\left\{\eta^{(2)}a^{1/2}u^3 f_0^{(2)}(u) + \eta^{(4)}a^{3/2}u^5 f_0^{(4)}(u) + \eta^{(6)}a^{5/2}u^7 f_0^{(6)}(u) + \ldots\right\}.
\end{aligned} \tag{C23}$$

Finally, notice the following. In Eq. (C23), the term in the first curly brackets, which we call the fluid term, contains an expansion in integer powers of a. If c_q with neighboring q are about the same, the ratio of the neighboring terms is of the order of a. The term in the second curly brackets, which we call the kinetic term, contains an expansion in half-integer powers of a. If $\eta^{(q)}$ with neighboring q are about the same, the ratio of the neighboring terms is about au^4/v_T^4 (assuming, e.g., the Maxwellian f_0), where v_T is the thermal velocity. Hence, the applicability conditions of Eq. (C23) are

$$a \ll 1, \quad au^4/v_T^4 \ll 1. \tag{C24}$$

That said, the applicability conditions can be relaxed due to the fact that the fluid term can be much larger than the kinetic term, in which case the latter does not need to be evaluated with high accuracy (and vice versa). The applicability conditions can be relaxed even further if one accounts also for the numerical coefficients $O(1)$, which we ignored in Eq. (C24); see Sec. IV.

APPENDIX D: DERIVATIVES OF PRINCIPAL VALUE INTEGRALS

In this appendix, we report some useful identities involving derivatives of principal-value integrals. It is very likely that these formulas can be found in the existing literature, but it is instructive to rederive them here. Specifically, we will consider integrals of the type

$$\mathsf{P}\int_{-\infty}^{\infty} \frac{h(v)}{v-u}\, dv \equiv \lim_{\varepsilon\to 0} \int_{\varepsilon} \frac{h(v)}{v-u}\, dv, \tag{D1}$$

where h is some function, and where the following notation is introduced,

$$\int_{\varepsilon} (\ldots)\, dv \doteq \int_{-\infty}^{u-\varepsilon} (\ldots)\, dv + \int_{u+\varepsilon}^{\infty} (\ldots)\, dv. \tag{D2}$$

Note one special type of such integrals,

$$(2\ell - 1) \int_{\varepsilon} \frac{1}{(v-u)^{2\ell}}\, dv = \frac{2}{\varepsilon^{2\ell-1}}, \quad \ell \in \mathbb{N}, \tag{D3}$$

which is a convenient formula to be used below.

1. Second derivative

Consider an expression of the form

$$\begin{aligned} P_2[h] &\doteq \frac{1}{2}\,\frac{\partial^2}{\partial u^2}\,\mathsf{P}\int_{-\infty}^{\infty} \frac{h(v)}{v-u}\, dv \\ &= \frac{1}{2}\,\frac{\partial^2}{\partial u^2}\,\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{u-\varepsilon} \frac{h(v)}{v-u}\, dv + \int_{u+\varepsilon}^{\infty} \frac{h(v)}{v-u}\, dv\right] \\ &= \frac{1}{2}\,\frac{\partial}{\partial u}\,\lim_{\varepsilon\to 0}\left[\frac{h(u-\varepsilon)}{-\varepsilon} + \int_{-\infty}^{u-\varepsilon} \frac{h(v)}{(v-u)^2}\, dv + \int_{u+\varepsilon}^{\infty} \frac{h(v)}{(v-u)^2}\, dv - \frac{h(u+\varepsilon)}{\varepsilon}\right] \\ &= \lim_{\varepsilon\to 0}\left[\frac{h'(u-\varepsilon)}{-2\varepsilon} + \frac{h(u-\varepsilon)}{2\varepsilon^2} + \int_{\varepsilon} \frac{h(v)}{(v-u)^3}\, dv - \frac{h(u+\varepsilon)}{2\varepsilon^2} - \frac{h'(u+\varepsilon)}{2\varepsilon}\right] \\ &= \lim_{\varepsilon\to 0}\left[\int_{\varepsilon} \frac{h(v)}{(v-u)^3} dv + \Delta P_2\right]. \end{aligned} \tag{D4}$$

Here we introduced

$$\Delta P_2 \doteq -\frac{h'(u-\varepsilon)}{2\varepsilon} + \frac{h(u-\varepsilon)}{2\varepsilon^2} - \frac{h(u+\varepsilon)}{2\varepsilon^2} - \frac{h'(u+\varepsilon)}{2\varepsilon} = -\frac{2h'(u)}{\varepsilon} + o(1) = -\int_{\varepsilon} \frac{h'(u)}{(v-u)^2}\, dv + o(1), \tag{D5}$$

where Eq. (D3) was substituted. Hence,

$$P_2[h] = \lim_{\varepsilon\to 0}\left[\int_{\varepsilon} \frac{h(v)}{(v-u)^3}\, dv - \int_{\varepsilon} \frac{h'(u)}{(v-u)^2}\, dv\right] = \mathsf{P}\int_{-\infty}^{\infty}\left[\frac{h(v)}{(v-u)^3} - \frac{h'(u)}{(v-u)^2}\right] dv. \tag{D6}$$

This leads to the main result of this subsection, which is that

$$\frac{1}{2!}\,\frac{\partial^2}{\partial u^2}\,\mathsf{P}\int_{-\infty}^{\infty} \frac{h(v)}{v-u}\, dv = \mathsf{P}\int_{-\infty}^{\infty} \frac{h(v) - h'(u)(v-u)}{(v-u)^3}\, v. \tag{D7}$$

2. Fourth derivative

The fourth derivative is treated similarly:

$$
\begin{aligned}
P_4[h] &\doteq \frac{1}{4!}\,\frac{\partial^4}{\partial u^4}\,\mathsf{P}\int_{-\infty}^{\infty}\frac{h(v)}{v-u}\,dv \\
&= \frac{2}{4!}\,\frac{\partial^2}{\partial u^2}\,P_2[h] \\
&= \frac{1}{12}\,\frac{\partial^2}{\partial u^2}\,\mathsf{P}\int_{-\infty}^{\infty}\left[\frac{h(v)}{(v-u)^3}-\frac{h'(u)}{(v-u)^2}\right]dv, \\
&= \frac{1}{12}\,\frac{\partial}{\partial u}\,\lim_{\varepsilon\to 0}\left\{\int_{\varepsilon}\left[\frac{3h(v)}{(v-u)^4}-\frac{2h'(u)}{(v-u)^3}-\frac{h''(u)}{(v-u)^2}\right]dv+\Delta P_4\right\} \\
&= \frac{1}{12}\,\lim_{\varepsilon\to 0}\left\{\frac{\partial}{\partial u}\int_{\varepsilon}\left[\frac{3h(v)}{(v-u)^4}-\frac{h''(u)}{(v-u)^2}\right]dv+\frac{\partial}{\partial u}\,\Delta P_4\right\}
\end{aligned}
\tag{D8}
$$

[the term $2h'(u)(v-u)^{-3}$ was dropped because the principal value integral over it is zero], where

$$
\Delta P_4 \doteq \left[\frac{h(v)}{(v-u)^3}-\frac{h'(u)}{(v-u)^2}\right]_{v=u-\varepsilon}-\left[\frac{h(v)}{(v-u)^3}-\frac{h'(u)}{(v-u)^2}\right]_{v=u+\varepsilon}=-\frac{1}{\varepsilon^3}\,[h(u-\varepsilon)+h(u+\varepsilon)]. \tag{D9}
$$

Then,

$$
P_4[h]=\frac{1}{12}\,\lim_{\varepsilon\to 0}\left\{\frac{\partial}{\partial u}\int_{\varepsilon}\left[\frac{3h(v)}{(v-u)^4}-\frac{h''(u)}{(v-u)^2}\right]dv+\frac{\partial}{\partial u}\,\Delta P_4\right\}=\lim_{\varepsilon\to 0}\left\{\int_{\varepsilon}\left[\frac{h(v)}{(v-u)^5}-\frac{h'''(u)}{12(v-u)^2}\right]dv+\frac{\Delta R_4}{12}\right\},
$$

where we introduced

$$
\Delta R_4 \doteq \left[\frac{3h(v)}{(v-u)^4}-\frac{h''(u)}{(v-u)^2}\right]_{v=u-\varepsilon}-\left[\frac{3h(v)}{(v-u)^4}-\frac{h''(u)}{(v-u)^2}\right]_{v=u+\varepsilon}+\frac{\partial}{\partial u}\,\Delta P_4. \tag{D10}
$$

It is easy to see that

$$
\Delta R_4=\frac{3}{\varepsilon^4}\,[h(u-\varepsilon)-h(u+\varepsilon)]-\frac{1}{\varepsilon^3}\,[h'(u-\varepsilon)+h'(u+\varepsilon)]=-\frac{8h'(u)}{\varepsilon^3}-\frac{2h'''(u)}{\varepsilon}=\int_{\varepsilon}\left[-\frac{12h'(u)}{(v-u)^4}-\frac{h'''(u)}{(v-u)^2}\right]dv,
$$

where we substituted Eq. (D3). Reverting to P_2, we then obtain

$$
P_4[h]=\mathsf{P}\int_{-\infty}^{\infty}\left[\frac{h(v)}{(v-u)^5}-\frac{h'(u)}{(v-u)^4}-\frac{h'''(u)}{6(v-u)^2}\right]dv. \tag{D11}
$$

This leads to the main result of this subsection, which is that

$$
\frac{1}{4!}\,\frac{\partial^4}{\partial u^4}\,\mathsf{P}\int_{-\infty}^{\infty}\frac{h(v)}{v-u}\,dv=\mathsf{P}\int_{-\infty}^{\infty}\frac{1}{(v-u)^5}\left[h(v)-h'(u)(v-u)-h'''(u)\,\frac{(v-u)^3}{3!}\right]dv. \tag{D12}
$$

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