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Generation of large-scale magnetic fields by small-scale dynamo in shear flows

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Abstract

We propose a new mechanism for turbulent mean-field dynamo in which the magnetic fluctuations resulting from a small-scale dynamo drive the generation of large-scale magnetic fields. This is in stark contrast to the common idea that small-scale magnetic fields should be harmful to large-scale dynamo action. These dynamos occur in the presence of large-scale velocity shear and do not require net helicity, resulting from off-diagonal components of the turbulent resistivity tensor as the magnetic analogue of the "shear-current" effect. Given the inevitable existence of non-helical small-scale magnetic fields in turbulent plasmas, as well as the generic nature of velocity shear, the suggested mechanism may help to explain generation of large-scale magnetic fields across a wide range of astrophysical objects.

Astrophysical magnetic fields are observed to be well-correlated over length and time scales far exceeding that of the underlying fluid motions. Beautiful in its regularity, the 22-year solar cycle is the most well-known example of this behavior [1]. Such large-scale structure is puzzling given that strong magnetic fields are expected to emerge through the stretching and twisting of field lines by smaller scale turbulence. As the primary theoretical framework to study such behavior, mean-field dynamo theory examines how large-scale magnetic fields develop due to these small scale turbulent motions. This splitting between scales is captured by the mean-field average; the average of a small-scale quantity vanishes by definition ($\langle \boldsymbol{b} \rangle = 0$), while the average of a large-scale field is itself ($\langle \boldsymbol{B} \rangle = \boldsymbol{B}$). An average of the induction equation, which governs evolution of the magnetic field within magnetohydrodynamics (MHD), leads to [2]

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) + \nabla \times \mathcal{E} + \frac{1}{\mathrm{Rm}} \nabla^2 \boldsymbol{B}, \tag{1}$$

where Rm is the magnetic Reynolds number, a dimensionless measure of the plasma resistivity, and $\boldsymbol{U}$ and $\boldsymbol{B}$ are the large-scale velocity and magnetic field. The *electromotive force*, $\mathcal{E} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$, is the average of the small-scale fields ($\boldsymbol{u}$ and $\boldsymbol{b}$) and responsible dynamo action. In the early phases of a dynamo, the mean-fields can be considered a small perturbation to the underlying turbulence. Combined with an assumption of scale-separation between small-scale and mean fields, this allows a Taylor expansion [3, 4] of $\mathcal{E}$ in terms of $\boldsymbol{B}$,

$$\mathcal{E} = \alpha \circ \boldsymbol{B} + \beta \circ \nabla \boldsymbol{B} + \ldots, \tag{2}$$

where α, β are the tensorial transport coefficients, calculated from the small-scale fields [5]. Since these depend on the large-scale fields, a solution to Eq. (1) requires knowledge of how $\mathcal{E}$ changes with $\boldsymbol{B}$ (and possibly $\boldsymbol{U}$), essentially a statistical closure for inhomogenous MHD.

Historically, much work has focused on kinematic dynamo theory, in which $\boldsymbol{u}$ is uninfluenced by the magnetic field [3]. Kinematic theory predicts large-scale dynamo instability when the fluid motions possess helicity, $\int \boldsymbol{u} \cdot \nabla \times \boldsymbol{u} \, d\boldsymbol{x} \neq 0$ [2]. However, the applicability of such predictions has been called into question by a number of authors [6, 7]. In particular, above modest Reynolds numbers in both helical and non-helical flows, the small-scale dynamo [8] causes $\boldsymbol{b}$ to grow and saturate much more rapidly [9] than $\boldsymbol{B}$. This violates the kinematic assumption, both because $\boldsymbol{u}$ is altered before $\boldsymbol{B}$ grows significantly and because a dynamically important $\boldsymbol{b}$ exists independently of $\boldsymbol{B}$. The buildup of small-scale fields is the origin of "α-quenching", in which the mean-field saturates well before reaching amplitudes consistent with observation [10–13] due to the adverse influence of $\boldsymbol{b}$. Here we show that in turbulence with large-scale velocity shear, it is possible and realizable to have the small-scale dynamo *enhance* the growth of the large-scale dynamo. We demonstrate this both with statistical simulation [14], in which the effect is very clear but applies rigorously only at low Reynolds numbers, and through calculation of transport coefficients from direct numerical simulations (DNS). In addition, analytic results obtained using the second-order correlation approximation (not presented here) agree with τ-approximation analyses [15], in contrast to previous kinematic studies [16–18]

Our calculations are carried out within the shearing box formalism, with homogenous Cartesian geometry and periodic boundary conditions in the shearing frame. A large-scale velocity shear $\boldsymbol{U}_0 = -Sx\hat{\boldsymbol{y}}$ is imposed across the domain. This commonly-used setup is designed to represent a small "patch" of turbulent fluid in large-scale velocity shear. Using the Snoopy code [19] for DNS, we non-helically force the velocity at small scales and study the generation of larger scale magnetic fields, in a similar way to previous authors [20, 21]. The mean-field average is defined as an average over the radial (x) and azimuthal (y) directions, such that the mean magnetic fields $\boldsymbol{B}$ depend only on z. We also include the effect of system rotation through a mean Coriolis force in DNS studies, since shear typically arises due to differential rotation in astrophysical objects. The rotation $\boldsymbol{\Omega}$ is aligned with $\hat{\boldsymbol{z}}$ (antiparallel to $\nabla \times \boldsymbol{U}_0$), perpendicular to the flow.

For the chosen horizontal average, inserting Eq. (2) into (1) gives

$$\begin{aligned} \partial_t B_x &= -\alpha_{yx}\partial_z B_x - \alpha_{yy}\partial_z B_y - \eta_{yx}\partial_z^2 B_y + \eta_{ty}\partial_z^2 B_x \\ \partial_t B_y &= -S B_x + \alpha_{xx}\partial_z B_x + \alpha_{xy}\partial_z B_y - \eta_{xy}\partial_z^2 B_x + \eta_{tx}\partial_z^2 B_y, \end{aligned} \tag{3}$$

using velocity shear $\boldsymbol{U}_0$ but neglecting other mean velocities, and defining $\eta_{it} \equiv \eta_{ii} + \mathrm{Rm}^{-1}$. Here α_{ij} and η_{ij} are the α-effect and turbulent resistivity tensors respectively, with the 4 components of η_{ij} relatable to the β_{ij3} elements of the full tensor [Eq. 2]. Due to homogeneity and reflectional symmetry (vanishing net helicity), α_{ij} must vanish when averaged over a suitably large time or number of realizations [3]. There is no such constraint on η_{ij}, and η_{yx} is very important throughout this work due to its coupling with the shear. In particular, neglecting fluctuations in α and assuming diagonal resistivities are equal ($\eta_{tx} = \eta_{ty} = \eta_t$), the least stable eigenmode of Eq. (3) for a mode of

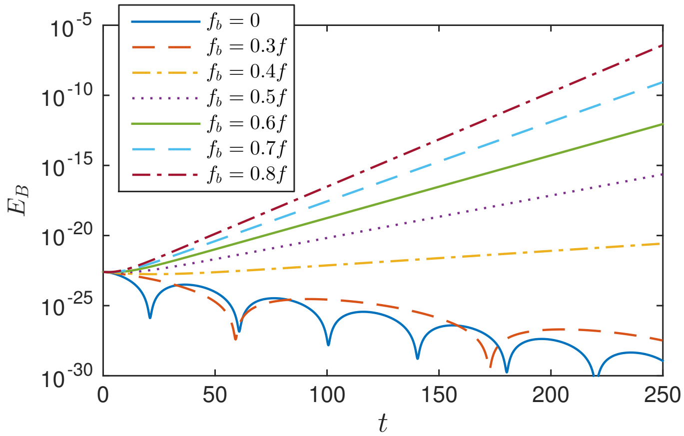

FIG. 1. Time development of the mean-field energy, $E_B = \int dz\, B^2/2$, in quasi-linear statistical simulation. Small-scale fields are forced at $k_f = 6\pi$, $\mathrm{Rm} = u_{rms}/\eta k_f \approx 5$ (here η is the resistivity, $\mathrm{Pm} = \mathrm{Rm}/\mathrm{Re} = 1$), $S = 2$ and the box has dimensions $\left(L_x, L_y, L_z\right) = (1, 1, 4)$ with resolution $(28, 28, 128)$. As well as forcing the momentum equation, the induction equation is forced to excite homogenous magnetic fluctuations so as to emulate a small-scale dynamo. Total forcing energy is kept constant in each simulation, but the proportion of magnetic forcing is increased from 0 to 0.8. As this is done, the growth rate of the mean-field increases enormously due to the magnetically driven dynamo.

vertical wavenumber k grows at

$$\gamma = k\sqrt{\eta_{yx}\left(-S + k^2\eta_{xy}\right)} - k^2\eta_t. \tag{4}$$

Since $S \gg \eta_{ij}$, dynamo action is possible without an α effect if $\eta_{yx} < 0$.

Subsequent to early analytic work [16], it was found that kinematically $\eta_{yx} > 0$ (at least at low Rm), and several authors have concluded that this coherent shear dynamo cannot explain observed field generation [17, 18, 21]. Instead, a popular theory is that temporal fluctuations in α_{ij} cause an *incoherent* mean-field dynamo. Importantly, in such a dynamo, $\boldsymbol{B}(z,t)$ cannot have a constant phase in time as it grows, since the average of $\boldsymbol{B}$ over an ensemble of realizations vanishes, implying $\boldsymbol{B}$ must be uncorrelated with itself after $t \gtrsim \left(k^2\eta_t\right)^{-1}$ [22]. While such incoherent dynamos are certainly possible in a variety of situations, here we argue for a different situation—magnetic fluctuations act to substantially decrease η_{yx}, causing the onset of a coherent large-scale dynamo that overwhelms the incoherent dynamo in some situations.

Our first method illustrating this effect is quasi-linear statistical simulation. By linearizing the equations for the small-scale fields, one can directly write down the equation for the small-scale *statistics* as a function of the large-scale fields [14, 23, 24]. This yields $\mathcal{E}$, which can be fed directly into Eq. (1) resulting in a closed system of equations. Importantly, since the statistics are

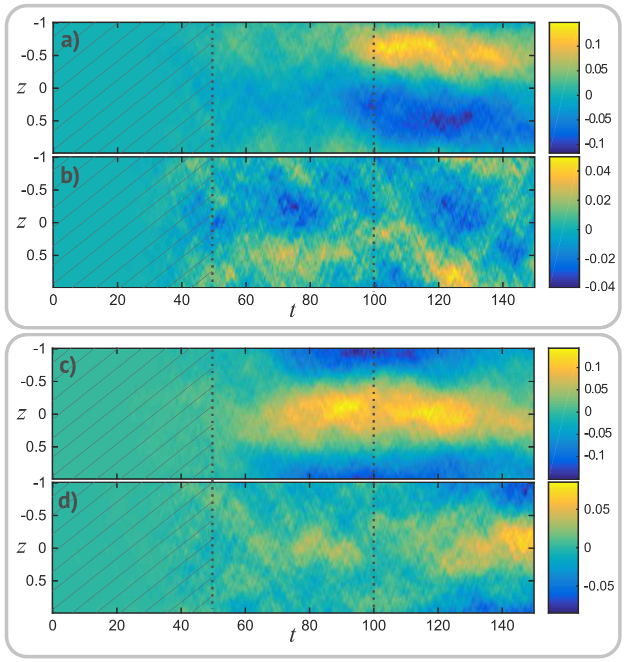

FIG. 2. Example spatiotemporal B_y evolutions for non-rotating (a-b) and Keplerian rotating (c-d) turbulence at $\mathrm{Rm} = u_{rms}/\eta k_f \approx 15$ ($k_f = 6\pi$, $\eta = 1/2000$, $\mathrm{Pm} = 8$), $S = 1$, in a box of dimension $(1, 4, 2)$ with resolution $(64, 128, 128)$. The first examples in each case [(a) and (c)] show B_y when a coherent dynamo develops, while the second examples [(b) and (d)] illustrate the case when it is more incoherent. The main factors in distinguishing these are the coherency in phase of B_y over some time-period and the amplitude at saturation, which is larger in the coherent cases. In general the rotating simulations are substantially more coherent. The hatched area illustrates the region of small-scale dynamo growth. The fitting method used to compute transport coefficients (see Fig. 3) is applied between the dashed lines ($t = 50 \to 100$).

calculated directly [18], there is no possibility of an incoherent dynamo, and the method offers a direct probe of the coherent effect. Due to the lack of a small-scale dynamo in this approach, we drive homogenous small-scale magnetic fluctuations by forcing the induction equation. Results are illustrated in Fig. 1, successively increasing the proportion of magnetic forcing from pure velocity forcing, with the total energy injection kept constant. The presence of the magnetically driven dynamo is evident, becoming slightly unstable when magnetic forcing accounts for 0.4 of the total and increasing the growth rate thereafter. This sustained period of exponential growth due to magnetic fluctuations is not possible to see in DNS, since the mean-field will immediately come into approximate equipartition with the small-scale field due to the finite size of the system.

The formal applicability of statistical simulation is limited to low Rm due to the quasi-linear approximation. Our second method utilizes DNS to show that small-scale fields arising consistently *through the small-scale dynamo* can drive a coherent large-scale dynamo. To this end, we directly calculate transport coefficients from nonlinear simulation before and after the saturation of the small-scale dynamo. We choose moderate Reynolds numbers [20], small enough such that there is no transition to self-sustaining turbulence (although effects may be similar even when this occurs [25]), and run ensembles of 100 simulations both with and without Keplerian rotation. At these parameters, the prevalence of the coherent large-scale dynamo depends on the realization (see Fig. 2), and it appears that the coherent effect cannot always overcome fluctuations in $\mathcal{E}$ immediately after small-scale saturation, although the dynamo always develops after a sufficiently long time [e.g., Fig. 2(d) near $t = 150$]. This behavior seems generic when the coherent dynamo is close to its threshold for excitation and we have observed similar structures when the induction equation is driven directly at lower Rm. Notwithstanding this variability in the dynamo's qualitative behavior, measurement of the transport coefficients illustrates that the η_{yx} coefficient decreases after the magnetic fluctuations reach approximate equipartition with velocity fluctuations at small scales.

At low times, we use the test-field method to measure the kinematic α and η, fixing the mean field and calculating $\mathcal{E}$, with no Lorentz force [3, 21]. Since the small-scale dynamo grows quickly, test-fields are reset every $t = 5$. After small-scale saturation, standard test-field methods are inapplicable [26]. Instead, we extract $\boldsymbol{B}$ and $\mathcal{E}$ simulation data and calculate $\left(\alpha_{ij}, \eta_{ij}\right)$ directly from Eq. (2) by computing $\int dz\, \mathcal{E}_i B$ for each of $B = \left(B_x, B_y, \partial_z B_x, \partial_z B_y\right)$. This method is very similar to that presented in Ref. [27]; however, we impose the constraints $\eta_{yy} = \eta_{xx}$, $\alpha_{xx} = \alpha_{yy}$ and $\alpha_{yx} = \eta_{xy} = 0$, and solve the resulting equations in the least-squares sense. While these changes may appear to make the method less accurate, they in fact achieve the opposite by reducing the influence of B_x, which is both very noisy and strongly correlated with B_y (through $-S B_x$) and $\mathcal{E}_y$ (through $\partial_t B_x = -\partial_z \mathcal{E}_y + \ldots$). (These correlations are very harmful to the quality of the fit, for instance causing unphysical negative values for η_{yy} [27]). It is straightforward to show that the systematic errors caused by our constraints on the transport coefficients should be less than 1% for the shear dynamos studied here, so long as $\eta_{xx} \approx \eta_{yy}$. We have verified the method agrees with the test-field method through application to low Rm shear dynamos [28], where the rotation can be used to test nonhelical dynamos over a range of η_{yx}. Due to the short time-window, measurements of the transport coefficients after small-scale saturation vary significantly between realization, as should be expected from Fig. 2. Nonetheless, an average over the ensemble illustrates a statistically significant change in η_{yx} that is consistent with observed behavior.

Results are illustrated in Fig. 3. In the kinematic phase without rotation, we see $\eta_{yx} = (4.1 \pm 1.6) \times 10^{-4}$, in qualitative agreement with previous studies [21]. With rotation, we find

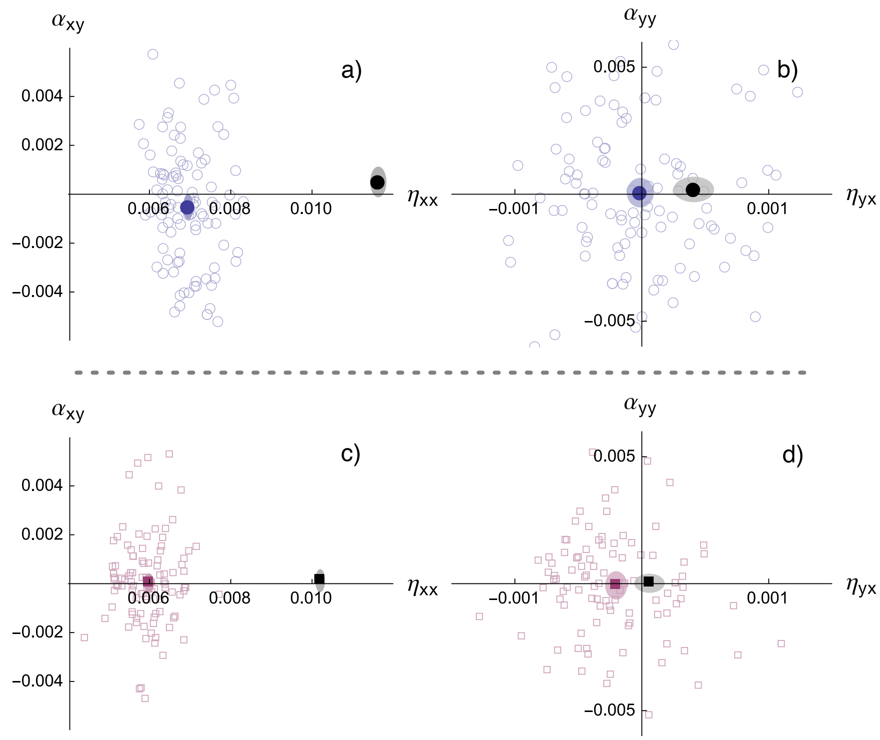

FIG. 3. Measurements of the turbulent transport coefficients for 100 realizations of the simulations at the same parameters as those in Fig. 2; (a) η_{xx} coefficients, no rotation, (b) η_{yx} coefficients, no rotation, (c) η_{xx} coefficients, rotating, (b) η_{yx} coefficients, rotating. Unfilled markers in each plot (circles and squares for non-rotating and rotating runs respectively) show coefficients measured from each of the individual realizations, with mean values displayed with solid markers and the shaded regions indicating error in the mean (2 standard deviations). Black markers illustrate the kinematic transport coefficients, with grey shaded regions indicating the error. After saturation of the small-scale dynamo, we calculate η_{ij} by solving Eq. (2) approximately at each time-point (see text), taking the mean from $t = 50$ to $t = 100$. This limited time-window is chosen to avoid capturing the saturation phase of the large-scale dynamo, since η_{ij} is presumably modified in this phase. In both methods used to compute transport coefficients, we have also calculated the corresponding α coefficients. In all cases these are zero to within error as expected, and the scatter between simulations is of a similar magnitude to that of η_{ij} if their different units are accounted for (it is necessary to divide α by a characteristic k value).

$\eta_{yx} = (0.6 \pm 1.2) \times 10^{-4}$, consistent with a reduction in η_{yx} due to the $\boldsymbol{\Omega} \times \boldsymbol{J}$ effect [5]. After saturation of the small-scale dynamo, $\eta_{yx} = (-0.1 \pm 1.0) \times 10^{-4}$ for the non-rotating case, while $\eta_{yx} \approx -(2.0 \pm 0.8) \times 10^{-4}$ in the rotating case—the same reduction in each to within error. Values for the diagonal resistivity are smaller after saturation, as expected since the velocity fluctuation energy decreases (by a factor ~ 1.4). The values of $\left(\eta_{xx}, \eta_{yx}\right)$ show that the dynamo is slightly stable on average in the non-rotating case and marginal in the rotating case. However, the coefficients vary significantly between realizations, sometimes yielding larger growth rates, and measurements match observed growth of the mean-field for individual realizations. We illustrate this in Fig. 4,

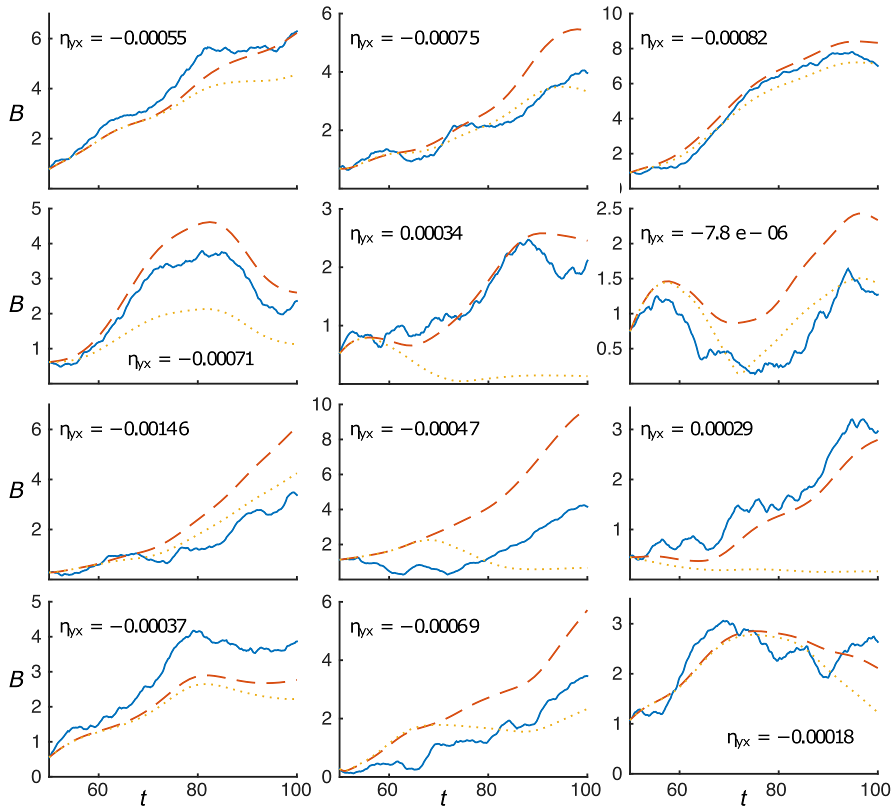

FIG. 4. Evolution of the mean-field magnitude for a sample of the ensemble of rotating simulations (Figs. 2-3). Here B, the mean-field magnitude, is $\sqrt{|\hat{B}_x^1|^2 + |\hat{B}_y^1|^2}$ where $\hat{B}_i^1$ is the largest scale Fourier mode of B_i. In each plot the solid blue curve shows data taken from the simulation. The dashed red curve shows the corresponding expected evolution, using the calculated values of the transport coefficients, smoothed in time using a Gaussian filter of width 5. Finally, the dotted yellow curve illustrates the expected evolution with all α coefficients artificially set to zero. We list the measured mean of η_{yx} in each plot to show that lower values do generally lead to substantially more growth of the mean field as expected for a coherent dynamo. For reference, at the measured $\eta_{xx} \approx 0.006$, the coherent dynamo is unstable below $\eta_{yx} = -0.00036$.

which demonstrates consistency between the measured transport coefficients and mean-field evolution by solving Eq. 3 directly, for a sample of the rotating simulations. In addition, by artificially removing α_{ij} coefficients, we illustrate that cases with more negative η_{yx} are driven primarily by this, rather than a stochastic-α effect. We thus conclude that small-scale magnetic fluctuations act to *decrease* η_{yx}, and that in some realizations (or after a sufficiently long time period) a coherent large-scale dynamo develops as a result.

To summarize, in this letter we have demonstrated that small-scale magnetic fluctuations, excited by small-scale dynamo action, can drive large-scale magnetic field generation. The mechanism is a magnetic analogue of the "shear-current" effect [15, 16], arising through the off-diagonal turbulent resistivity in the presence of large-scale shear flow. We have demonstrated its existence numerically using both direct numerical simulation, with measurements of mean-field transport coefficients before and after small-scale dynamo saturation, and through quasi-linear statistical simulation.

More work is needed to precisely assess regimes in which the magnetically driven dynamo might dominate, as well as its behavior at higher Reynolds numbers where self-sustained turbulence is possible [25]. Another interesting question regards whether a magnetic dynamo can remain influential in the presence of net helicity and an α-effect, particularly as small-scale dynamo may be suppressed by shear [29]. While such questions may be difficult to answer definitively, the generic presence of magnetic fluctuations in plasma turbulence gives us some confidence that the proposed mechanism could cause large-scale dynamo growth in the wide variety of astrophysical systems with velocity shear.

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