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Abstract

This paper formulates the Tokamak Magneto-Hydrodynamics (TMHD), initially outlined by X. Li and L.E. Zakharov [Plasma Science and Technology, accepted, ID:2013-257 (2013)] for proper simulations of macroscopic plasma dynamics. The simplest set of magneto-hydrodynamics equations, sufficient for disruption modeling and extendable to more refined physics, is explained in detail. First, the TMHD introduces to 3-D simulations the Reference Magnetic Coordinates (RMC), which are aligned with the magnetic field in the best possible way. The numerical implementation of RMC is adaptive grids. Being consistent with the high anisotropy of the tokamak plasma, RMC allow simulations at realistic, very high plasma electric conductivity. Second, the TMHD splits the equation of motion into an equilibrium equation and the plasma advancing equation. This resolves the 4 decade old problem of Courant limitations of the time step in existing, plasma inertia driven numerical codes. The splitting allows disruption simulations on a relatively slow time scale in comparison with the fast time of ideal MHD instabilities. A new, efficient numerical scheme is proposed for TMHD.

Contents

1 Introduction *(to ToC)*

This paper formulates a new approach, called Tokamak Magneto-Hydrodynamics (TMHD), for addressing the needs in numerical simulation of macroscopic plasma dynamics in tokamaks. Briefly it was described in Ref. [1]. Here TMHD is presented in its complete form.

TMHD is a special version of one-fluid magneto-hydrodynamics (MHD). Consistent with tokamak plasma properties, it removes long standing issues with numerical simulations of macroscopic plasma dynamics. TMHD relies on a new representation of pure MHD equations (no unnecessary "extended" MHD), where the equation of motion is split onto two equations: equilibrium and plasma advancing. For proper reflection of the high plasma anisotropy TMHD introduces adaptive grids based on so-called Reference Magnetic Coordinates (RMC). RMC are suitable for the high-temperature plasma in tokamaks (as well as other toroidal confinement systems stellarators, reverse field pinches) which is very anisotropic with respect to the direction of the magnetic field. With the progress toward the burning fusion plasma [2] the plasma becomes less collisional and anisotropy becomes even stronger: e.g., the typical electron and ion velocities $V_e \simeq 10^7\sqrt{T_{e,kev}},\ V_i \simeq 2\cdot 10^5\sqrt{T_{i,keV}}$ m/s are very large ($T_{e,keV}, T_{i,keV}$ are the particle temperatures in keV Units). The plasma velocity transverse to the magnetic surfaces $V_\perp < 100$ m/s is relatively small. RMC allow to reproduce the plasma anisotropy numerically and have a very fast algorithm for their alignment with the magnetic field.

Plasma anisotropy is very important for tokamak stability. Without its extreme anisotropy the plasma in tokamak devices would never be stable with respect to MHD instabilities. Because of anisotropy, with exception of the thin layers at the resonant magnetic surfaces, the non-ideal tokamak plasma behaves like an ideally conducting fluid. At the same time, the high plasma anisotropy, which is absent in conventional fluid or gas media, represents a substantial challenge for numerical simulations.

The tokamak disruption simulations require a realistic model of the conducting structures around the plasma, referred here for simplicity as a "wall". A proper representation of its 3-D structure (ribs, limiters, penetrations, gaps) is absolutely essential. First, the real geometry of the wall separates the physical position of the plasma facing surface from the position of an electromagnetic equivalent of the wall, typically used in theory and simulations for determination of the wall response. Second, the 3-D wall structure determines the position of the electric contact of the plasma with the wall during disruption, which leads to the current sharing between the plasma and the wall.

The numerical scheme of TMHD uses Hermite finite elements for MHD variables and a thin wall model with triangle representation of conducting surfaces. Remarkably, all TMHD finite element (FE) equations (equilibrium, plasma advancing, Faraday law in the plasma, circuit equations in the wall, and the current sharing between plasma and the wall) have their own energy principle and are reduced to solving matrix equations with positively defined symmetric matrices. Moreover, for the plasma MHD equations these matrices are block tri-diagonal and well suitable for GPU computing.

Historically, adaptive grids for MHD simulations were used in the early 1970s for first simulations of nonlinear kink instability [3] in tokamaks [4, 5]. But soon after this the experimental discovery of reconnection phenomenon in the plasma core motivated the development of numerical codes based on laboratory numerical grids. Created initially for simulation of internal reconnection in the plasma core, these codes have gradually expanded their applications to simulations of the entire plasma and its disruptions. The vacuum region between the plasma and the wall in the discharge chamber was represented by an artificial rare plasma with low electric conductivity, thus, allowing the use of the same MHD model for the entire simulation region.

At present, all magneto-hydrodynamic (MHD) numerical codes (RSF [6], CTD [7], M3D [8], NIMROD [9], JOREK [10] are just few USA and Europe examples,, see also [11, 12] for earlier examples) use this model and are based on different kinds of fixed laboratory grids. In fact, all these codes essentially represent a version of hydro-dynamics (fluid) codes modified by the Lorentz force. The common deficiency of these codes is the boundary condition $V_{normal} = 0$ at the wall, preventing the plasma flow into the wall. Taken from hydrodynamics, this condition is wrong for the high temperature plasma which annihilates at the wall surface (the plasma ions are converted into neutral atoms which do not participate in MHD).

In 1991, the importance of the electric contact of the plasma with conducting structures of the discharge chamber during disruption instabilities was understood during vertical displacement events (VDE) on DIII-D [13] where the currents to the plasma facing tiles were measured with a special diagnostics. The tile currents were interpreted (in our opinion erroneously) as the so-called "halo" currents from the region with the open field lines between the plasma and the side walls. Then, toroidal asymmetry of tile currents were found [14, 15, 16, 17] on different tokamaks.

The key breakthrough in understanding the effect of the current sharing is related to the direct measurements of the current sharing between the plasma and the wall and of toroidal asymmetry of the plasma current on JET [18, 19, 20, 21, 22, 23]. In 2007, motivated by the needs of International Tokamak Experimental Reactor (ITER), the theory of the wall touching kink mode (WTKM) [24] was created and predicted large currents, called "Hiro currents", generated by the plasma motion into the wall. They explain the sideways forces and toroidal asymmetry in JET disruptions and, in particular, have the same direction as experimental observations. Moreover, the theory assessment of the Hiro current reproduced remarkably well the waveforms of the signals. The comparison with the large disruption data base of the JET tokamak, which contains about 5000 disruption cases [25, 26, 27] has unambiguously confirmed the prediction of an unexpected sign of the currents in the wall, generated by the disruptive instability.

At the same time, the halo current interpretation of the toroidal asymmetry and previous theoretical explanation, which is based on halo currents [28] failed in all of these cases even in prediction of the direction of the wall currents. In reality the current in the wall is consistent with the Hiro current theory and always has an opposite direction with respect to the plasma current.

The Hiro currents not only were missed in all 2- and 3-D simulations but cannot be reproduced by the existing codes. The laboratory grid, inconsistent with the plasma anisotropy and high Lundquist number (the ratio of the resistive penetration time of the magnetic field to the plasma inertial time, which is $> 10^7$), the need of an artificial plasma outside the plasma core, the need of a boundary condition for plasma velocity $V_{normal} = 0$, which is wrong in the tokamak plasma, the inertia driven numerical schemes, which are capable of simulating only fast instabilities at the ideal MHD time scale, the simplistic unrealistic geometry of the wall (in some codes ideally conducting) and the overall complexity of the model (extended MHD) prevent meaningful disruption simulations with existing 3-D numerical codes. The so-called extended MHD model, which mixes micro- and macroscopic scale lengths, is used essentially to hide the inability of numerical grids to make the proper scale separation when the non-ideal effects are localized in the vicinity of the resonant surfaces and in the plasma edge, while the major plasma can be described by the simple one-fluid equations. In addition to these numerous inconsistencies with reality, all numerical codes suffer from severe Courant limitations on the time step, determined by the Alfven or magneto-sonic oscillations. They exist in MHD equations but being stable play no role in macroscopic plasma dynamics.

TMHD is free from all of these deficiencies and has no Courant condition for time advancing its numerical model. It considers disruptions as a fast equilibrium evolution with magnetic flux conservation and generation of sheet currents. This model properly captures the MHD part of disruption physics and neglects inessential non-MHD effects. At the same time TMHD is suitable for proper physics scale separation and efficient incorporation of the non-ideal physics.

Sect. 2 outlines the problems of disruption simulations. Sect. 3 presents the Reference Magnetic Coordinates and an efficient Newton method for advancing RMC. Sect. 4 describes the TMHD model and its equations, Sect. 5 presents the 3-D equilibrium equation, which is the backbone of TMHD. Sect. 6 introduces the wall model and two types of wall currents: eddy currents and currents shared with the plasma. Sect. 7 introduces the energy functionals for all TMHD equations (including wall currents) for generation of the finite element numerical matrix representation. Sect. 8 describes the global algorithm of solving TMHD equations and the Summary 9 summarized the property and importance of TMHD.

Throughout the paper, m, m/s T, MA, MPa, MN, $Vsec$ are adopted as Units for lengths, velocities, magnetic field strength, currents, pressure, forces, and magnetic fluxes. Accordingly, the magnetic permeability $\mu_0 = 0.4\pi$. The rationalized variables for the current density $\mathbf{j}$, plasma pressure p and poloidal and toroidal magnetic fluxes Ψ, Φ have a bar in notations

$$\bar{\mathbf{j}} \equiv \mu_0 \mathbf{j}, \quad \bar{p} \equiv \mu_0 p, \quad \bar{\Psi} \equiv \frac{\Psi}{2\pi}, \quad \bar{\Phi} \equiv \frac{\Phi}{2\pi}. \tag{1.1}$$

2 Disruptive instability in tokamaks *(to ToC)*

The violation of macroscopic stability leads to a fast termination of the plasma discharge in the form of disruptive instability. It was discovered in 1963 [29]. The specific property of this instability is that in the very beginning of the disruption before discharge termination, the plasma current exhibits a very sharp enhancement in its value, known as the current spike. A very high negative spike in voltage is generated by the current spike. Up to now, this feature is not yet explained.

Associated with the kink modes, this kind of disruption (referred here as a conventional disruption) phenomenologically haa two phases: (a) a thermal quench, and (b) the current quench. During the thermal quench the plasma loses a substantial part of its thermal energy and the electron temperature falls from the 1 to 10 keV range to the low level of 10 to 50 eV. This is the fastest phase of disruption. E.g., on JET (the biggest tokamak) it lasts for about 3 ms. Because of a significant rise of the plasma electric resistivity, the toroidal electric field (loop voltage) increases by orders of magnitude. At certain conditions this can accelerate the runaway electrons up to the energy of tens of MeV. The current quench follows the thermal quench and lasts for the resistive decay time. On JET the current quench is of the order of tens or even hundreds of ms, depending on the plasma parameters and wall conditions.

The another type of disruption is related to the vertical instability and is called the Vertical Displacement Event (VDE). It happens when the stabilization of the vertical motion of an elongated plasma is lost for some reason. In contrast to conventional disruptions, VDE has no thermal quench. Instead, the plasma starts to move vertically and loses gradually its temperature due to contact with the wall structure. But at some point in time, a secondary instability can be excited as a kink mode. As it was found on JET, this is the $m/n = 1/1$ kink mode. Recently, VDE became a top issue due to big forces expected in the next step tokamaks [30], and specifically in ITER [31], and TMHD is the most suitable model for addressing the issue.

There are two kinds of forces on the vacuum vessel and its plasma facing components due to VDE. The first one is associated with the vertical instability itself. This force F_z is directed vertically in the direction of the plasma motion

$$F_z \propto 2\pi R I_{pl} B_{ext} \propto R I_{pl}^2. \tag{2.1}$$

Here, R is the major radius of the plasma, I_{pl} is the plasma current and B_{ext} is the quadrupole component of the external field, which provides the elongated shape of the plasma. With respect to the existing JET tokamaks ($R = 3$ m, $I_{pl} = 3$ MA) the VDE forces in ITER ($R = 6$ m, $I_{pl} = 15$ MA) are expected to be 50 times larger.

The second kind of force, called sideways forces F_x, are directed horizontally. They are generated by the secondary instability, i.e. the $m/n = 1/1$ kink mode. When the plasma moves vertically and touches the internal structures of the vessel, its cross-section shrinks. This reduces the edge q_a value below 1 and excites the Shafranov's kink mode $m/n = 1/1$. The electro-magnetic interaction of the kink mode with the vessel creates the sideways force [18, 19, 20, 21, 22, 23].

The sideways force F_x is created by interaction of the plasma current with the toroidal magnetic field during the $m/n = 1/1$ plasma deformation

$$F_x \propto a_{pl} I_{pl} B_{tor}, \quad F_x^{ITER} \simeq 20 F_x^{JET}. \tag{2.2}$$

(The toroidal fields in JET and ITER are 3 and 6 T, correspondingly).

The major concern with the sideways force on the vacuum vessel is that it has to be withstood by an external structure. This is difficult to arrange. Another concern is the mode rotation and its possible resonance with the mechanical structure of the vessel.

Big forces and other undesirable phenomena (e.g., possible generation of runaway electrons due to high voltage during disruptions) put the disruption studies, and VDE specifically, into the list of topics of the highest priority in the plasma physics and fusion research.

In the beginning of the 1970s, when tokamaks had a plasma with a circular cross-section and safety factor $q_a > 2.5$, the disruptive instability was associated with the free boundary kink modes, and especially with the mode $m/n = 2/1$. In 1973 the simplified Reduced MHD model, based on the relation $B_{tor} \gg B_{pol}$, was formulated by Kadomtsev and Pogutse [3], who predicted a formation of the "vacuum bubbles" in the kink mode development. Their paper stimulated nonlinear simulations of the kink mode, and the formation of bubbles has been confirmed [4, 5]. These two simulations were the first made with the free boundary plasma and a vacuum gap between the plasma and the wall. Unfortunately, they were the last free boundary simulations for more than 3 decades.

In 1974, the discovery [32] of periodic relaxations of the central plasma temperature (called sawtooth oscillations) in tokamaks, explained by Kadomtsev as an internal reconnection event [33], stimulated the plasma MHD modeling of internal reconnection with the central value of the safety factor $q(0) < 1$, which is necessary for the instability. For these studies the plasma boundary condition was not essential and it was considered as fixed with the plasma velocity to the wall $V_{normal} = 0$ as the boundary condition. Moreover, the wall itself was considered as ideally conducting.

For these reconnection studies the laboratory grid was the most suitable. Although the codes were not very accurate in reproducing the singular layers, they confirmed the reconnection pattern described by Kadomtsev. Later on, the fast reconnection in the experiments was explained theoretically [34, 35, 36, 37] based on the Hall effect of two-fluid MHD. Following this theoretical explanation, the numerical codes have extended their reconnection model from one- to two-fluid MHD. Still the triggering mechanism of internal reconnection was not revealed either by theory or by simulations.

In the 1980s, the same, fixed boundary numerical schemes used for internal dynamics were applied for disruption studies [6, 7]. One of the first MHD codes, the CTD (IFS, UT) [38], was intensively used for MHD studies in formulations with fixed boundary conditions. The code works in the toroidal coordinate system with a conforming mapping to the circular one. To some degree this mimics the alignment of coordinates to the magnetic field. The author was a pioneer in simulations of two-fluid effects in collisionless magnetic reconnection and resistive wall mode simulations. The claims were made that the codes reproduce the current spike but no physical mechanism was identified.

Extensive work has been done with CTD in simulation of VDEs. CTD [39, 7], has been modified to include a "vacuum region" and a resistive wall to be able to address, not only the three dimensional VDE problem that will be considered here, but nonlinear external kink modes in general. The CTD code was capable of reproducing the currents, generated by the plasma motion and currents to the wall. Still, it used the same zero velocity boundary condition, mentioned earlier. The model of low temperature, post thermal quench, plasma missed the physics of the wall touching kink and vertical modes. As a result, the plasma current asymmetry, observed in JET disruptions [18, 19, 20, 25, 27] was not reproduced.

In the 1990s the M3D code [40, 41, 42, 43, 44, 45, 46] became the major player in disruption simulations. The accent was made on extension of the MHD model in order to reproduce the fast reconnection rate in the plasma core. The major success of M3D was modeling [42] of the coupling between the internal kink mode in the plasma center and ballooning modes in TFTR (Tokamak Fusion Test Reactor) in PPPL, when the plasma pressure approaches the stability limit for small-scale MHD instabilities, called ballooning modes. This coupling sometimes triggered disruptions in TFTR.

In the mid 1990s a new MHD code NIMROD was created [9] for modeling of macroscopic dynamics of toroidal devices. Used for studies of some effects, associated with disruptions (internal reconections, stochastization of magnetic structure, generation and losses of runaway electrons) this code so far has the ideal wall boundary conditions with both $B_{normal} = 0$, $V_{normal} = 0$ and is not able to simulate the wall touching MHD modes driving the disruptions.

The present situation with disruption simulations by present hydro-dynamic codes is illustrated by a failure of application of M3D code [8, 47] to simulations of sideways forces for ITER commented in Refs. [48, 49]. The authors neglected to simulate the well diagnosed disruptions on JET, where the sideways forces have been discovered in 1996 [18, 19] and which has the most complete data base of the effect [26, 27]. Instead the M3D simulations pretend to consider the ITER reference configuration. But in fact, in simulations [8] the plasma current in this stable configuration was artificially enhanced by a factor 1.6 in order to make a typically benign internal $m = 1, n = 1$ kink mode highly unstable and disruptive.

These "simulations" with the hidden 24 MA current, were presented as modeling sideways forces in ITER with design current of 15 MA. In the following paper [47], the ITER plasma current was corrected but a statement, totally contradicting the JET measurements and theory [24, 25], was made that the sideways forces are produced by the $m = 2, n = 1$ kink mode.

The plasma current substitution by M3D, using the "current enhancement factor" 1.6 was overlooked for 3 years and noticed only in 2013. It was made public recently in Ref. [50] where the key inconsistencies of the simulations with the tokamak physics were outlined and the boundary condition for the plasma flow was derived instead of a number of erroneous ones [51, 52] used in M3D and other MHD codes.

In fact, the numerical scheme of M3D is driven by plasma inertia, which makes the code limited to simulations of fast global instabilities, non-existing in tokamaks. For realistic slower dynamics the boundary conditions for the plasma velocity plays an important role and what was used is erroneous. As a result, despite the long history of M3D and other code development, at present, there is no single code capable of reproducing the basic experimental data on disruptions.

3 Reference magnetic coordinates (RMC) *(to ToC)*

This section introduces the Reference Magnetic Coordinates (RMC), which represent the proper generalization of flux coordinates for 3D ergodic magnetic fields when the flux surfaces do not exist. Being best fitted to the structure of the perturbed confinement magnetic field, RMC can find a broad range of applications, including tokamak plasma perturbations, stellarator equilibria, simulations of magneto-hydrodynamic instabilities, etc. A simple Newton algorithm for construction and advancing of RMC is presented as well. RMC is the basis for adaptive grid generation on TMHD, but their importance extends far beyond the TMHD.

The plasma anisotropy and fast expansion along the magnetic field requires closed magnetic surfaces for good confinement. For axisymmetric configuration (tokamak, reverse field pinch) this can be provided by an appropriate equilibrium field (see, e.g., Ref. [53]). In 3-dimensional stellarator systems good magnetic surfaces can be obtained only in exceptional, although realistic, cases (pointed out by Hamada in 1961 [54]). Only in these exceptional cases the toroidal coordinates a, θ, ζ, called flux coordinates, can be created with $a = const$ corresponding to the toroidal magnetic surfaces, while poloidal and toroidal angles $0 \leq \theta, \zeta \leq 2\pi$ can be arbitrary.

The cases of nested magnetic surfaces represent only a minor part of realities in fusion research. The stellarator configuration is prone to destruction by imperfectness in the magnetic coil system or by evolving plasma profiles. Moreover, the design approach for stellarator configurations consists in gradual elimination of natural perturbations of the nested topology of magnetic field by adjusting the equilibrium coil system [55].

In tokamaks, some intrinsic instabilities (like neo-classical tearing modes) disturb the nestedness of magnetic configuration. Triggering of major MHD instabilities (like disruptions) starts with destruction of magnetic topology, followed by complete loss of confinement and plasma termination. In the reverse field pinches magnetic turbulence, which is always present, makes nested surfaces non-existing in the rigorous sense.

In all of these cases, there is a jump in topology of magnetic field from a simply nested geometry to magnetic islands and stochasticity. As a result, the very convenient and efficient descriptions of magnetic field based on flux coordinates fails even for small perturbations. At the same time, the plasma anisotropy still requires the appropriate description for perturbed configurations. Otherwise, the crucial aspect related, for example, to triggering the macroscopic instabilities of sheet currents generated in macroscopic instabilities could be missed.

The importance of the efficient description of ergodic confinement magnetic fields have been well understood in stellarator equilibrium studies long time ago (see, e.g., Refs. [56, 57, 58, 59, 60]). In fact, the discussion of ergodic fields has its roots in the 1950s (see, e.g., Refs.[61, 62] and bibliography in Ref.[63]).

Still, despite many suggestions of nested coordinate systems, which would simplify the representation of ergodic fields, a practical approach did not emerge. All suggestions in the existing literature include the line tracing as a necessary element, i.e., solving the equations for the magnetic field lines

$$d\mathbf{r} = \frac{\mathbf{B}}{|\mathbf{B}|} dl \tag{3.1}$$

and then processing the Poincare plots (which is the discrete set of points of intersection of the field lines with the plane $\zeta =$const. This technique is not only very computationally expensive it becomes more difficult for small resonant perturbations. In other words, with the line tracing there is no smooth transition to pure flux coordinates with gradually diminishing islands.

The RMC suggested here does not involve the line tracing and complicated numerical approximation techniques. RMC are based on a rigorous theory, which gives an efficient and practical algorithm for constructing and advancing the coordinate system. The adaptive grids for TMHD numerical codes are based on RMC.

The next Sect. 3.1 introduces notations for nested toroidal coordinates. Sect. 3.2 provides the simplest representation of confinement magnetic field without assumption on existence of nested magnetic surfaces. Sect. 3.3 gives the representation of magnetic field and Ampere law in toroidal coordinates. Sect. 3.4 explains the algorithm of construction of flux coordinate surfaces for the simplest case of 2-dimensional toroidal configurations. Sect. 3.4 introduces the definition of Reference Magnetic Coordinates and the fast Newton algorithm for RMC construction. Sect. 3.5 describes the island geometry expressed in a compact way using RMC.

3.1 Curvilinear toroidal coordinates and metric tensor *(to ToC)*

In the paper we use a, θ, ζ as toroidal curvilinear coordinates

$$\mathbf{r} = \mathbf{r}(a, \theta, \zeta), \tag{3.2}$$

where $\mathbf{r}$ is the radius-vector, a is the radial coordinate, which determines the shape of toroidal surface $a =$const, and θ, ζ are poloidal and toroidal angles correspondingly. Their relation to the cylindrical coordinates r, φ, z can be specified as

$$r = r(a, \theta, \zeta), \quad \varphi = \varphi(a, \theta, \zeta), \quad z = z(a, \theta, \zeta). \tag{3.3}$$

For the purpose of TMHD and in most of other cases the toroidal angle ζ is the same as the azimuth φ.

If coordinates Eq. (3.3) are specified, e.g., by spline representation, the element of the length

$$dl^2 = g_{aa}da^2 + 2g_{a\theta}dad\theta + 2g_{a\zeta}dad\zeta + g_{\theta\theta}d\theta^2 + 2g_{\theta\theta}d\theta d\zeta + g_{\zeta\zeta}d\zeta^2 \quad (3.4)$$

allows to calculate the metric tensor

$$g_{aa} = r_a'^2 + z_a'^2 + r^2\varphi_a', \quad g_{a\theta} = r_a'r_\theta' + z_a'z_\theta' + r^2\varphi_a\varphi_\zeta', \quad g_{a\zeta} = r_a'r_\zeta' + z_a'z_\zeta' + r^2\varphi_a\varphi_\zeta', \quad (3.5)$$

$$g_{\theta\theta} = r_\theta'^2 + z_\theta'^2 + r^2\varphi_\theta', \quad g_{\theta\zeta} = r_\theta'r_\zeta' + z_\theta'z_\zeta' + r^2\varphi_\theta\varphi_\zeta', \quad g_{\zeta\zeta} = r_\zeta'^2 + z_\zeta'^2 + r^2\varphi_\zeta'. \quad (3.6)$$

The Jacobian $J = \sqrt{g}$ of the metric tensor can be calculated in a straightforward manner

$$J = \sqrt{g} \equiv \frac{D(r,\varphi,z)}{D(a,\theta,\zeta)}. \quad (3.7)$$

In terms of gradients of curvilinear coordinates

$$\frac{1}{J} = (\nabla a \cdot (\nabla\theta \times \nabla\zeta)). \quad (3.8)$$

Both notations J and $\sqrt{g}$ will be used below (as in the literature). For the case when $\zeta = \varphi$ Jacobian has a form, similar to the two dimensional case

$$\sqrt{g} = rD, \quad D \equiv -\frac{D(r,z)}{D(a,\theta)}. \quad (3.9)$$

The same formulas (3.2,3.3) specify the covariant

$$\mathbf{e}_a \equiv \mathbf{r}_a' = J(\nabla\theta \times \nabla\zeta), \quad \mathbf{e}_\theta \equiv \mathbf{r}_\theta' = J(\nabla\zeta \times \nabla a), \quad \mathbf{e}_\zeta \equiv \mathbf{r}_\zeta' = J(\nabla a \times \nabla\theta) \quad (3.10)$$

and contravariant basis vectors

$$\mathbf{e}^a \equiv \nabla a, \quad \mathbf{e}^\theta \equiv \nabla\theta, \quad \mathbf{e}^\zeta \equiv \nabla\zeta, \quad (3.11)$$

which are mutually orthogonal

$$(\mathbf{e}_i \cdot \mathbf{e}^j) = \delta_i^j. \quad (3.12)$$

Here, 'i', 'j' stand for one of a, θ, ζ and δ_i^j is the Kronecker symbol.

The covariant A_a, A_θ, A_ζ and contravariant A^a, A^θ, A^ζ components of any vector $\mathbf{A}$ are defined by

$$A_a \equiv (\mathbf{A}\cdot\mathbf{e}_a), \quad A_\theta \equiv (\mathbf{A}\cdot\mathbf{e}_\theta), \quad A_\zeta \equiv (\mathbf{A}\cdot\mathbf{e}_\zeta), \quad (3.13)$$

$$A^a \equiv (\mathbf{A}\cdot\mathbf{e}^a), \quad A^\theta \equiv (\mathbf{A}\cdot\mathbf{e}^\theta), \quad A^\zeta \equiv (\mathbf{A}\cdot\mathbf{e}^\zeta). \quad (3.14)$$

They relate to each other by

$$A_a = g_{aa}A^a + g_{a\theta}A^\theta + g_{a\zeta}A^\zeta, \quad A_\theta = g_{\theta a}A^a + g_{\theta\theta}A^\theta + g_{\theta\zeta}A^\zeta, \quad A_\zeta = g_{\zeta a}A^a + g_{\zeta\theta}A^\theta + g_{\zeta\zeta}A^\zeta. \quad (3.15)$$

For the future use we introduce the notations for important combinations of the metric tensor entering to the energy principle for plasma equilibrium

$$M \equiv \frac{g_{aa}}{J}, \quad N \equiv \frac{g_{a\theta}}{J}, \quad K \equiv \frac{g_{\theta\theta}}{J}, \quad \tilde{M} \equiv \frac{g_{a\zeta}}{J}, \quad \tilde{N} \equiv \frac{g_{\theta\zeta}}{J}, \quad Q \equiv \frac{g_{\zeta\zeta}}{J}. \quad (3.16)$$

3.2 The simplest form of confinement magnetic fields *(to ToC)*

The design of the magnetic field for plasma confinement is based on MHD equilibrium equation

$$\nabla p = (\mathbf{j} \times \mathbf{B}), \quad (3.17)$$

where p is the plasma pressure, $\mathbf{j}, \mathbf{B}$ are the current density and the magnetic field in the plasma

$$(\nabla \cdot \mathbf{B}) = 0, \quad \mathbf{j} = (\nabla \times \mathbf{B}). \quad (3.18)$$

Because of the relation

$$(\mathbf{B} \cdot \nabla p) = 0 \tag{3.19}$$

the magnetic field has a toroidal geometry and two components

$$\mathbf{B} = \mathbf{B}_{tor} + \mathbf{B}_p, \tag{3.20}$$

toroidal $\mathbf{B}_{tor}$ and poloidal $\mathbf{B}_p$.

Despite its simplicity, the equilibrium equation (3.17) is the most reliable plasma physics equation for the toroidal plasmas. Its is valid in the presence of the plasma and magnetic field fluctuations as well as it is not sensitive to details of distribution functions of the plasma particles. Moreover, for tokamaks the equilibrium equations extend their applicability to TMHD of disruptions, which is the topic of the paper.

From the property of $\nabla \cdot \mathbf{B} = 0$, the magnetic field can be expressed in terms of a vector potential $\mathbf{A}$

$$\mathbf{B} = (\nabla \times \mathbf{A}). \tag{3.21}$$

Its covariant representation in a curvilinear coordinates a, θ, ζ

$$\mathbf{A} = A_a \nabla a + A_\theta \nabla \theta + A_\zeta \nabla \zeta + \nabla u \tag{3.22}$$

is the most useful. The scalar u is an arbitrary function, not affecting the magnetic field. The angle dependence of 3-dimensional periodic functions can be expressed in Fourier space as

$$\begin{array}{lllll} u & \equiv & u_{00}(a) & +u_{\sim 0}(a,\theta) & +u_{0\sim}(a,\zeta) & +u_{\sim\sim}(a,\theta,\zeta) \\ & = & u_{00}(a) & +\sum_{m\neq 0} u_{m0}(a) e^{im\theta} & +\sum_{n\neq 0} u_{0n}(a) e^{-in\zeta} & +\sum_{m\neq 0}\sum_{n\neq 0} u_{mn}(a) e^{im\theta - in\zeta} \end{array}, \tag{3.23}$$

where u_{00} is the average part while others are oscillatory functions of angles

$$u_{00}(a) \equiv \frac{1}{4\pi^2} \oint u(a,\theta,\zeta) d\theta d\zeta, \quad u_{\sim 0}(a,\theta) \equiv \frac{1}{2\pi} \oint (u - u_{00}) d\zeta, \quad u_{0\sim}(a,\zeta) \equiv \frac{1}{2\pi} \oint (u - u_{00}) d\theta, \tag{3.24}$$

and

$$\oint u_{\sim\sim} d\theta = \oint u_{\sim\sim} d\zeta = 0. \tag{3.25}$$

The θ-dependence of in the poloidal component A_θ can be eliminated by using an appropriate form of $u_{\sim 0} + u_{\sim\sim}$

$$\frac{\partial (u_{\sim 0} + u_{\sim\sim})}{\partial \theta} = -(A_{\theta,\sim 0} + A_{\theta,\sim\sim}). \tag{3.26}$$

As a result the remaining part of the θ-component of $\mathbf{A}$ can be represented as

$$A_\theta = A_{\theta,00} + A_{\theta,0\sim} = \bar{\Phi}(a) + \phi(a,\zeta). \tag{3.27}$$

Similarly, the $A_{\zeta,0\sim}$ dependence can be eliminated by using the remaining freedom in $u_{0\sim}$, thus reducing the A_ζ component to

$$A_\zeta = \bar{\Psi}(a) + \psi, \quad \psi = \psi_{\sim 0}(a,\theta) + \psi_{\sim\sim}(a,\theta,\zeta). \tag{3.28}$$

These steps give the most compact form of the vector potential $\mathbf{A}$ in a given coordinate system

$$\mathbf{A} = -\bar{\Phi}' \eta \nabla a + (\bar{\Phi} + \phi) \nabla \theta + (\bar{\Psi} + \psi) \nabla \zeta. \tag{3.29}$$

It is easy to show that the function $2\pi(\bar{\Phi} + \phi)$ represents the toroidal magnetic flux through the contour a=const, ζ=const

$$2\pi(\bar{\Phi} + \phi) = \oint A_\theta d\theta = \int J B^\varphi da d\theta. \tag{3.30}$$

Similarly, $2\pi(\bar{\Psi}+\psi)$ represents the poloidal magnetic flux through the contour a=const, θ=const

$$2\pi(\bar{\Psi}+\psi) = \oint A_\zeta d\zeta = \int JB^\theta da d\zeta. \tag{3.31}$$

The radial component η in $\mathbf{A}$ is determined by the choice of poloidal and toroidal angles. It can be eliminated by massaging the angles. For example, the substitution

$$\theta = \bar{\theta} + \frac{\bar{\Phi}'}{\bar{\Phi}' + \phi'_a}\eta, \quad \zeta = \bar{\zeta}, \tag{3.32}$$

leads to following representation of the vector potential

$$\mathbf{A} = \nabla\left(\frac{\bar{\Phi}'\phi + \bar{\Phi}\phi'_a}{\bar{\Phi}' + \phi'_a}\eta\right) + (\bar{\Phi}+\phi)\nabla\bar{\theta} + \left(\bar{\Psi} + \psi - \frac{\bar{\Phi}'\phi'_{\bar{\zeta}}}{\bar{\Phi}' + \phi'_a}\eta\right)\nabla\bar{\zeta}. \tag{3.33}$$

The first gradient term here is not important. The toroidal coordinates $a, \bar{\theta}, \bar{\zeta}$ marked by a bar, for which the radial component of the vector potential is absent, are called "straight field line coordinates" as a generalization of the conventional notion to ergodic fields. In the absence of the ϕ, ψ terms the magnetic field lines in these coordinates are straight. Eqs. (3.32-3.33) give a generalization of straight field line coordinates for the case when this property can be implemented only approximately.

3.3 Magnetic field and Ampere law in curvilinear coordinates *(to ToC)*

Based on the definition of vector potential Eq. (3.21) the magnetic field has the following form

$$\mathbf{B} \equiv \nabla\times\mathbf{A} = -\bar{\Phi}'\nabla\eta\times\nabla a + (\nabla\bar{\Phi}+\nabla\phi)\times\nabla\theta + (\nabla\bar{\Psi}+\nabla\psi)\times\nabla\zeta, \tag{3.34}$$

which is a contravariant representation of $\mathbf{B}$

$$\mathbf{B} = B^a\mathbf{e}_a + B^\theta\mathbf{e}_\zeta + B^\zeta\mathbf{e}_\zeta. \tag{3.35}$$

Explicitly, the contravariant components of $\mathbf{B}$ are given by

$$B^a \equiv \mathbf{B}\cdot\nabla a = \frac{\psi'_\theta - \phi'_\zeta}{J}, \tag{3.36}$$

$$B^\theta \equiv \mathbf{B}\cdot\nabla\theta = -\frac{\bar{\Psi}' + \psi'_a + \bar{\Phi}'\eta'_\zeta}{J}, \tag{3.37}$$

$$B^\zeta \equiv \mathbf{B}\cdot\nabla\zeta = \frac{\bar{\Phi}'(1+\eta'_\theta) + \phi'_a}{J}. \tag{3.38}$$

The calculations of the current density $\bar{\mathbf{j}} = \mu_0\mathbf{j}$

$$\bar{\mathbf{j}} = (\nabla\times\mathbf{B}) \tag{3.39}$$

requires the covariant representation of the magnetic field

$$\mathbf{B} \equiv B_a\nabla a + B_\theta\nabla\theta + B_\zeta\nabla\zeta \equiv -\nu\nabla a + \hat{J}\nabla\theta + \hat{F}\nabla\zeta + \nabla\sigma. \tag{3.40}$$

Here, the gradient term σ is extracted from the B_θ component in order to absorb the θ dependence in toroidal current $\hat{J} = \hat{J}(a,\zeta)$. In the poloidal current $\hat{F}$ the gradient term absorbs all Fourier harmonics $m = 0, n \neq 0$. The radial component ν remains the general functions of angles $\nu = \nu(a,\theta,\zeta)$.

Covariant components can be expressed in terms of components of vector potential using the general relationships Eq. (3.15)

$$B_a = M(\psi'_\theta - \phi'_\zeta) - N(\bar{\Psi}' + \psi'_a + \bar{\Phi}'\eta'_\zeta) + \tilde{M}(\bar{\Phi}'(1+\eta'_\theta) + \phi'_a), \tag{3.41}$$

$$B_\theta = N(\psi'_\theta - \phi'_\zeta) - K(\bar{\Psi}' + \psi'_a + \bar{\Phi}'\eta'_\zeta) + \tilde{N}(\bar{\Phi}'(1+\eta'_\theta) + \phi'_a), \tag{3.42}$$

$$B_\zeta = \tilde{M}(\psi'_\theta - \phi'_\zeta) - \tilde{N}(\bar{\Psi}' + \psi'_a + \bar{\Phi}'\eta'_\zeta) + Q(\bar{\Phi}'(1+\eta'_\theta) + \phi'_a). \tag{3.43}$$

Then the contravariant components of the current density $\bar{\mathbf{j}}$ can be calculated as

$$\bar{\jmath}^a = \frac{1}{J}\left(\frac{\partial B_\zeta}{\partial\theta} - \frac{\partial B_\theta}{\partial\zeta}\right), \quad \bar{\jmath}^\theta = \frac{1}{J}\left(\frac{\partial B_a}{\partial\zeta} - \frac{\partial B_\zeta}{\partial a}\right), \quad \bar{\jmath}^\zeta = \frac{1}{J}\left(\frac{\partial B_\theta}{\partial a} - \frac{\partial B_a}{\partial\theta}\right), \tag{3.44}$$

$$\bar{\jmath}^a = \frac{\hat{F}'_\theta - \hat{J}'_\zeta}{J}, \quad \bar{\jmath}^\theta = -\frac{\hat{F}'_a + \nu'_\zeta}{J}, \quad \bar{\jmath}^\zeta = \frac{\hat{J}'_a + \nu'_\theta}{J}\ , \tag{3.45}$$

which is the representation of Ampere's law

$$(\nabla\times(\nabla\times\mathbf{A})) = \bar{\mathbf{j}} \tag{3.46}$$

in curvilinear coordinates.

The explicit form of these equations follows from two equations (3.45) for $\bar{\jmath}^\theta, \bar{\jmath}^\zeta$

$$\frac{\partial}{\partial a}\left[\tilde{M}(\psi'_\theta - \phi'_\zeta) - \tilde{N}(\bar{\Psi}' + \psi'_a + \bar{\Phi}'\eta'_\zeta) + Q(\bar{\Phi}'(1+\eta'_\theta) + \phi'_a)\right] \tag{3.47}$$

$$-\frac{\partial}{\partial\zeta}\left[M(\psi'_\theta - \phi'_\zeta) - N(\bar{\Psi}' + \psi'_a + \bar{\Phi}'\eta'_\zeta) + \tilde{M}(\bar{\Phi}'(1+\eta'_\theta) + \phi'_a)\right] = \hat{F}'_a + \nu'_\zeta, \tag{3.48}$$

$$\frac{\partial}{\partial a}\left[N(\psi'_\theta - \phi'_\zeta) - K(\bar{\Psi}' + \psi'_a + \bar{\Phi}'\eta'_\zeta) + \tilde{N}(\bar{\Phi}'(1+\eta'_\theta) + \phi'_a)\right] \tag{3.49}$$

$$-\frac{\partial}{\partial\theta}\left[M(\psi'_\theta - \phi'_\zeta) - N(\bar{\Psi}' + \psi'_a + \bar{\Phi}'\eta'_\zeta) + \tilde{M}(\bar{\Phi}'(1+\eta'_\theta) + \phi'_a)\right] = \hat{J}'_a + \nu'_\theta, \tag{3.50}$$

which determine the combinations of the unknowns $\bar{\Psi}(a) + \psi(a,\theta,\zeta)$,$\bar{\Phi}(a) + \phi(a,\zeta)$ and $\eta(a,\theta,\zeta)$ given the right hand side $\hat{J}(a,\zeta), \hat{F}(a,\theta,\zeta), \nu(a,\theta,\zeta)$.

Simple configurations with the axial symmetry

$$\zeta \equiv \varphi, \quad \frac{\partial}{\partial\varphi} = 0, \quad \phi = 0 \tag{3.51}$$

are of special importance as they represent the background field of the tokamaks. The general representation of the 2-dimensional magnetic field has the form

$$\mathbf{B} = (\nabla\hat{\Psi}\times\nabla\varphi) + \hat{F}\nabla\varphi, \quad \hat{\Psi}(a,\theta) \equiv \bar{\Psi}(a) + \psi(a,\theta), \quad \hat{F}(a,\theta) \equiv \bar{F}(a) + \sigma(a,\theta), \tag{3.52}$$

where $\bar{F}$, as it will be seen below, is the poloidal current which determines the magnetic field B_{tor} (physical component).

$$B_{tor} = \frac{\hat{F}}{r}. \tag{3.53}$$

The function $\hat{\Psi}$ determines the magnetic surfaces of the poloidal magnetic field

$$\hat{\Psi}(a,\theta) = \text{const}, \quad (\mathbf{B}\cdot\nabla\hat{\Psi}) = 0, \tag{3.54}$$

which have no normal component of the magnetic field.

In curvilnear coordinates a, θ, φ the representations of the vector potential $\mathbf{A}$ has the form

$$\mathbf{A} = -\bar{\Phi}'\eta\nabla a + \bar{\Phi}\nabla\theta + \hat{\Psi}\nabla\varphi \tag{3.55}$$

and the contravariant components of are

$$B^a = \frac{\psi'_\theta}{J}, \quad B^\theta = -\frac{\hat{\Psi}'_a}{J}, \quad B^\varphi = \frac{\bar{\Phi}'(1+\eta'_\theta)}{J}. \tag{3.56}$$

In particular, the toroidal flux is related to the poloidal current $\hat{F}$ by

$$\bar{\Phi}'(1+\eta'_\theta) = \frac{J}{r^2}\hat{F}, \quad \bar{\Phi}' = \left(\frac{J}{r^2}\hat{F}\right)_0, \tag{3.57}$$

where $()_0$ is the averaged over angle component of the function.

The covariant components of $\mathbf{B}$ (3.40) are given by

$$B_a \equiv -\nu + \sigma'_a = M\psi'_\theta - N\hat{\Psi}'_a, \quad B_\theta \equiv \bar{J} + \sigma'_\theta = N\psi'_\theta - K\hat{\Psi}'_a, \quad B_\varphi \equiv \hat{F}. \tag{3.58}$$

Similarly to $\mathbf{B}$, the current density can be represented in the form

$$\bar{\mathbf{j}} = (\nabla\hat{F} \times \nabla\varphi) + \bar{\jmath}_\varphi \nabla\varphi, \quad \bar{\jmath}_{tor} = \frac{\bar{\jmath}_\varphi(a,\theta)}{r}. \tag{3.59}$$

Here j_φ is the covariant component of the current density, while $\bar{\jmath}_{tor}$ is the real toroidal current density. The contravariant components are given by

$$\bar{\jmath}^a = \frac{\hat{F}'_\theta}{J}, \quad \bar{\jmath}^\theta = -\frac{\hat{F}'_a}{J}, \quad \bar{\jmath}^\varphi = \frac{\bar{\jmath}_\varphi}{r^2} = \frac{\hat{J}'_a + \nu'_\theta}{J}, \tag{3.60}$$

$$\bar{\jmath}^a = \frac{1}{J}\frac{\partial B_\varphi}{\partial\theta}, \quad \bar{\jmath}^\theta = -\frac{1}{J}\frac{\partial B_\varphi}{\partial a}, \quad \bar{\jmath}^\varphi = \frac{1}{J}\left(\frac{\partial B_\theta}{\partial a} - \frac{\partial B_a}{\partial\theta}\right). \tag{3.61}$$

The explicit form of these equations for magnetic fluxes reduces to a single differential equation for the poloidal flux and an explicit formula for the toroidal flux:

$$\left[K(\bar{\Psi}' + \psi'_a) - N\psi'_\theta\right]'_a + \left[M\psi'_\theta - N(\bar{\Psi}' + \psi'_a)\right]'_\theta = -\bar{J}'_a - \nu'_\theta, \tag{3.62}$$

$$\bar{\Phi}'(1 + \eta'_\theta) = \frac{\sqrt{g}}{r^2}\hat{F}, \tag{3.63}$$

This equations determine the unknowns the combinations of the unknowns $\bar{\Psi}(a) + \psi(a,\theta,\zeta)$,$\bar{\Phi}(a)$ and $\bar{\Psi}'\eta(a,\theta,\zeta)$ given the right hand side $\bar{J}(a), \hat{F}(a,\theta), \nu(a,\theta)$.

3.4 3-dimensional RMC and the algorithm for their generation *(to ToC)*

In the 2-dimensional axisymmetric case the poloidal flux function $\hat{\Psi}$ specifies the shape of magnetic surfaces (3.54). In flux coordinates in which the radial coordinate $\bar{a}$ is chosen as a label of magnetic surfaces, ψ and the normal component of the magnetic field are zero $\hat{\Psi} = \bar{\Psi}$. Accordingly, the vector potential and magnetic field have simple representations

$$\mathbf{A} = -\bar{\Phi}'\eta\nabla a + \bar{\Phi}'\nabla\theta + \bar{\Psi}'\nabla\varphi, \quad \mathbf{B} = -\bar{\Psi}'\mathbf{e}_\theta + \bar{\Phi}'\mathbf{e}_\varphi = -\bar{\Psi}'\mathbf{e}_\theta + \hat{F}\nabla\varphi. \tag{3.64}$$

They can be additionally simplified in the straight field coordinates with $\eta = 0$

$$\mathbf{A} = \bar{\Phi}'\nabla\bar{\theta} + \bar{\Psi}'\nabla\varphi, \quad \mathbf{B} = -\bar{\Psi}'\mathbf{e}_{\bar{\theta}} + \bar{\Phi}'\mathbf{e}_\varphi = -\bar{\Psi}'\mathbf{e}_{\bar{\theta}} + \hat{F}\nabla\varphi. \tag{3.65}$$

Starting from non-flux coordinates, flux coordinates can be generated by solving the equation $\bar{\Psi} + \psi =$const using, e.g., fast Newton iterative method

$$\tilde{\xi} = -\frac{\psi^n}{\bar{\Psi}'^n}, \quad a^{n+1} = a^n + \xi, \quad r^{n+1} = r^n + r'^n_a\tilde{\xi} + r'^n_\theta\sigma, \quad z^{n+1} = z^n + z'^n_a\tilde{\xi} + z'^n_\theta\sigma, \quad a^n \to \bar{a}. \tag{3.66}$$

The perturbation σ, introduced here, does not change the shape of the coordinate surfaces but allows control of the distribution of poloidal angle. This method was suggested in Ref. [64] and implemented in the Equilibrium and Stability Code (ESC).

In the case without axisymmetry, there are no function $\hat{\Psi}$ to generate coordinate surfaces. In this situation the goal is to generate the coordinates which would be best aligned with the magnetic field.

This goal can be understood in two ways. The coordinate surfaces a =const should be adjusted either to minimize the normal component of the magnetic field or to create the simplest possible representation of the magnetic field.

The idea is to eliminate the terms ϕ, ψ in the general form of the magnetic vector potential Eq. (3.29) by considering these terms small relative to $\bar{\Phi}, \bar{\Psi}$

$$\mathbf{A} = -\eta\bar{\Phi}'\nabla a + (\bar{\Phi} + \phi)\nabla\theta + (\bar{\Psi} + \psi)\nabla\zeta, \quad \phi \ll \bar{\Phi}, \quad \psi \ll \bar{\Psi}, \tag{3.67}$$

$$\mathbf{B} = (\psi'_\theta - \phi'_\zeta)\mathbf{e}_a - (\bar{\Psi}' + \psi + \bar{\Phi}'\eta'_\zeta)\mathbf{e}_\theta + (\bar{\Phi}' + \phi'_a + \bar{\Phi}'\eta'_\theta)\mathbf{e}_\varphi. \tag{3.68}$$

As in 2-dimensional case we massage the coordinate surfaces $a = const$ by a perturbation ξ

$$a^{n+1} = a^n + \xi. \tag{3.69}$$

Elimination of the normal component of the magnetic field B^a by linearizing with respect to ξ is given by

$$(\mathbf{B}\cdot\nabla(a+\xi))=0, \quad (\mathbf{B}\cdot\nabla\xi)^{n+1}=-B^{a,n}. \tag{3.70}$$

The ϕ,ψ terms in $\mathbf{B}$ in the left hand side are neglected as the higher order corrections. As a result, the equation for ξ is reduced to the so-called magnetic differential equation (MDE) for ξ

$$J(\mathbf{B}\cdot\nabla\xi)=\bar\Phi'(1+\eta'_\theta)\xi'_\zeta-(\bar\Psi'+\bar\Phi'\eta'_\zeta)\xi'_\theta=\phi'_\zeta-\psi'_\theta. \tag{3.71}$$

MDE equation can be easily solved in Fourier space using the following representation with a modified poloidal coordinate $\bar\theta$

$$\bar\theta\equiv\theta+\eta, \tag{3.72}$$

$$\xi=\sum\xi_{mn}(a)e^{im\bar\theta-in\zeta}, \tag{3.73}$$

$$\psi=\sum\psi_{mn}(a)e^{im\bar\theta-in\zeta}, \tag{3.74}$$

$$\phi=\sum\phi_n(a)e^{-in\zeta}. \tag{3.75}$$

This gives

$$\xi'_\theta=\sum\xi_{mn}e^{im\bar\theta-in\zeta}im(1+\eta'_\theta), \tag{3.76}$$

$$\xi'_\zeta=\sum\xi_{mn}e^{im\bar\theta-in\zeta}(im\eta'_\zeta-in) \tag{3.77}$$

which after substitution into the equation (3.71) leads to a simple relation

$$(m\bar\Psi'+n\bar\Phi')\xi_{mn}=m\psi_{mn}-\delta^0_m n\phi_n, \tag{3.78}$$

with Kronecker delta δ^0_m. This equation for ξ_{mn} can be resolved for all non-resonant harmonics m',n' for which the factor in front of ξ_{mn} is not zero. We mark the resonant term by a superscript '*', while the non-resonant wave numbers by the apostrophe ''''.

The non-resonant components in ξ have the explicit form

$$\xi=\sum_{m'n'}\xi_{m'n'}(a)e^{im'\bar\theta-in'\phi}, \quad \xi_{m'n'}=\frac{m'\psi_{m'n'}-\delta^0_{m'}n'\phi_{n''}}{m'\bar\Psi'+n'\bar\Phi'}, \tag{3.79}$$

while the resonant $\xi_{m^*n^*}=0$.

The RMC are generated by advancing the coordinate system using exclusively non-resonant components of ξ and ignoring the resonant terms.

As a result of successive application of this Newton algorithm, the coordinate system is deformed $a\to\bar a$ in a such way, that the vector potential acquires the simplest representation, achievable without massaging the angles of coordinates.

$$\mathbf{A}=-\bar\Phi'\eta\nabla\bar a+\bar\Phi(\bar a)\nabla\theta+\hat\Psi^*\nabla\zeta, \quad \hat\Psi^*\equiv\bar\Psi+\psi^*, \quad \psi^*=\sum_{m^*n^*}\psi_{m^*n^*}(\bar a)e^{im^*\bar\theta-in^*\zeta}, \tag{3.80}$$

where ψ^* contains only resonant terms.

We denote the RMC as $\bar a,\theta,\zeta$ and the ζ-component of the vector potential as $\hat\Psi^*$. In the case of good magnetic surfaces RMC are the same as flux coordinates and $\hat\Psi^*=\bar\Psi(\bar a)$ and are their substitution for ergodic magnetic fields.

Note, that by changing the definition of one of the angles, e.g., θ the η term in the vector potential can be eliminated as well, thus, making the representation of $\mathbf{A}$ similar to the 2-D straight field coordinates form. Typically this procedure leads to a highly non-uniform distribution of θ =const lines, which makes the straight field line coordinates impractical for simulations.

The contravariant components of $\mathbf{B}$ in RMC

$$B^a=\frac{\psi'^*_\theta}{J}, \quad B^\theta=-\frac{\bar\Psi'+\psi'^*_a+\bar\Phi'\eta'_\zeta}{J}, \quad B^\zeta=\frac{\bar\Phi'(1+\eta'_\theta)}{J} \tag{3.81}$$

and the covariant components of the magnetic field

$$\mathbf{B}\equiv B_a\nabla a+B_\theta\nabla\theta+B_\zeta\nabla\zeta\equiv-\nu\nabla a+\hat J\nabla\theta+\hat F\nabla\zeta+\nabla\sigma. \tag{3.82}$$

Covariant components can be expressed in terms of components of the vector potential using the general relationships Eq. (3.15)

$$B_a = M\psi_\theta'^* - N(\bar\Psi' + \psi_a'^* + \bar\Phi'\eta_\zeta') + \tilde M\bar\Phi'(1+\eta_\theta'), \tag{3.83}$$

$$B_\theta = N\psi_\theta'^* - K(\bar\Psi' + \psi_a'^* + \bar\Phi'\eta_\zeta') + \tilde N\bar\Phi'(1+\eta_\theta'), \tag{3.84}$$

$$B_\zeta = \tilde M\psi_\theta'^* - \tilde N(\bar\Psi' + \psi_a'^* + \bar\Phi'\eta_\zeta') + Q\bar\Phi'(1+\eta_\theta'). \tag{3.85}$$

3.5 Magnetic islands at the resonant surfaces *(to ToC)*

The resonant terms in ψ^* produce magnetic islands (in the leading approximation on their amplitude), whose topology cannot be reproduced by simple nested toroidal coordinates.

Nevertheless, the magnetic topology can be easily reconstructed by the perturbation method, using the island size as an expansion parameter. In the simple case, when there is no radial overlapping of the islands, it is possible to introduce the local flux function $\hat\Psi^*_{mn}$ which determines the nested surfaces around the resonant magnetic field lines:

$$\hat\Psi^*_{nm} \equiv \bar\Psi^*_{mn} + \left(\psi^*_{mn}e^{im\bar\theta - in\zeta} + c.c.\right) = \text{const}, \quad \bar\Psi^*_{mn} \equiv \bar\Psi + \frac{n}{m}\bar\Phi. \tag{3.86}$$

It can be written also as

$$\hat\Psi^*_{nm} = \bar\Psi^*_{mn} + 2|\psi^*_{mn}|\cos(m\bar\theta - n\zeta + \alpha), \quad \psi^*_{mn} \equiv |\psi^*_{mn}|e^{i\alpha}. \tag{3.87}$$

It is straightforward to show that

$$J(\mathbf{B}\cdot\nabla\hat\Psi^*_{mn}) = JB^a\left(\bar\Psi' + \frac{n}{m}\bar\Phi'\right) + \left(\bar\Phi'\frac{\partial}{\partial\zeta} - \bar\Psi'\frac{\partial}{\partial\theta}\right)2|\psi^*_{mn}|\cos(m\bar\theta - n\zeta + \alpha) = 0 \tag{3.88}$$

and, thus, there is no normal component of the magnetic field to the surfaces $\bar\Psi^*_{n,m}$ =const.

The expansion of $\hat\Psi^*_{mn}$ near the resonant point $\bar a_{mn}$, where $\bar\Psi'^*_{mn} = 0$,

$$x \equiv \bar a - \bar a_{mn}, \quad \hat\Psi^*_{nm} = \bar\Psi^*_{mn} + \frac{1}{2}\bar\Psi''^*_{mn}x^2 + 2|\psi^*_{mn}|\cos m\beta, \quad \beta \equiv \theta - \frac{n}{m}\zeta + \alpha, \tag{3.89}$$

determines the island geometry

$$x^2 = \frac{2(\hat\Psi^*_{nm} - \bar\Psi^*_{mn})}{\bar\Psi''^*_{mn}} - \left|\frac{4\psi^*_{mn}}{\bar\Psi''^*_{mn}}\right| + \left|\frac{8\psi^*_{mn}}{\bar\Psi''^*_{mn}}\right| \left\{ \begin{array}{ll} \sin^2\frac{m\beta}{2}, & \text{for}\bar\Psi''^*_{mn} > 0 \\ \cos^2\frac{m\beta}{2}, & \text{for}\bar\Psi''^*_{mn} < 0 \end{array} \right. . \tag{3.90}$$

The factor in front of $\cos^2\frac{m\beta}{2}$ term represents the half island width squared, thus, giving the following value for the island width W_{mn}

$$W_{mn} = 2w_{mn} = 4\sqrt{\left|\frac{2\psi^*_{mn}}{\bar\Psi''^*_{mn}}\right|}, \quad \bar\Psi^*_{mn} = \bar\Psi + \frac{n}{m}\bar\Phi. \tag{3.91}$$

The size of the resulting islands relative to the minor radius $\bar a$ gives a self-contained condition of applicability of the perturbation theory. It is possible to extend the perturbation theory in order to include the higher order terms.

Having the estimate for the island width the RMC can be further optimized in an obvious way by keeping the resonant harmonics in ψ^* localized radially within their resonant regions slightly wider that the island size W_{mn}.

The described coordinate advancing scheme, which leads to RMC, is analogous to the 2-D scheme of ESC for fast and explicit advancing of the numerical mesh. It is remarkable that the generation of field aligned RMC for ergodic magnetic fields has such a simple and fast Newton scheme (not discovered earlier despite numerous attempts).

4 Tokamak MHD) *(to ToC)*

Modeling of disruptions needs a special formulation of MHD equations, which would reflect the experimental measurements and specific requirements on numerical implementation.

The **first** important characteristic of disruptions is that their time scale τ_{TMHD} is much shorter than the magnetic field penetration time $\tau_{resistive}$ into the plasma

$$\tau_{resistive} \simeq \mu_0 \sigma_{\|} a^2. \tag{4.1}$$

Its numerical value can be estimated as

$$\sigma_{\|} = 41 \cdot 10^6 \cdot T_{keV}^{3/2} \frac{14.8}{Z \ln \Lambda} \ [\Omega^{-1} \cdot m^{-1}], \quad \ln \Lambda = 14.8 + \ln \frac{T_{e,keV}}{n_{e,20}^{1/2}}. \tag{4.2}$$

Here $\sigma_{\|}$ is Spitzer parallel conductivity, $\ln \Lambda$ is Coulomb logarithm, $n_{e,20}$ electron density in $10^{20}/m^3$ units, $T_{e,keV}$ is the electron temperature in keV [65], a is the plasma minor radius. (For comparison, the electric conductivity of copper is $60 \cdot 10^6$ at 25^oC).

For ITER $T_{e,keV} \simeq 10$, $a = 2$

$$\tau_{resistive}^{ITER} \simeq 50 \cdot T_{keV}^{3/2} a^2 \simeq 6400 \cdot T_{keV}^{3/2}, \tag{4.3}$$

and, as for most other tokamaks, $\tau_{resistive}$ is longer than the discharge time.

The slow magnetic field penetration implies that the plasma dynamics preserves magnetic fluxes and as a result excites sharp localized currents. There are two types of these currents: surfaces currents at the plasma boundary, and the sheet currents at the resonant magnetic surfaces.

The **second** property of disruptions is that they are relatively slow with respect to the plasma inertial time τ_{MHD}

$$\tau_{MHD} \equiv \frac{R}{V_A}, \quad V_A = 1.54 \cdot 10^6 \frac{B_{tor}}{\sqrt{n_{i,20}}} \ [m/s], \tag{4.4}$$

where R is the plasma major radius, V_A is the Alfven speed, and n_{20} is the ion density in $10^{20}/m^3$ Units.

For the ITER device the plasma inertial time can be estimated as

$$R^{ITER} = 6, \quad B_{tor} = 6, \quad n_{i,20} = 1, \quad V_A = 9.25 \cdot 10^6 \ [m/s], \quad \tau_{MHD} \simeq 0.64 \cdot 10^{-6}, \tag{4.5}$$

and is approximately the same for JET and other tokamaks.

In comparison to this time scale, the thermal quench (the loss of plasma thermal energy) in non-VDE disruptions, which is the fastest event in disruptions, lasts more than 1 ms in large machines. This implies that the plasma inertia, which is the driving term in numerical simulations, plays a minor role, while the force balance is much more important.

The **third** characteristic of plasma disruptions in tokamaks is that the plasma starts to interact with the wall at the very beginning of a disruption. Hard X-ray emission during this phase indicates this unambiguously. Also the thermal quench itself, in which the electron plasma temperature drops from the level of several kilo electron volts to tens of volts is an evidence of a strong plasma-wall contact and interaction.

This fact means that the boundary condition at the wall surface is important for the plasma dynamics. In contrast to the hydro-dynamic numerical codes, which use fluid like boundary conditions, like $V_{normal} = 0$, the tokamak plasma has no restrictions to its flow to the wall: its ions pick up an electron and become neutral, not participating in the plasma dynamics. The new effect, specific for disruptions is the galvanic contact of the plasma with the conducting surfaces and the current sharing between plasma and the wall.

Another aspect of the same characteristic is that the real structure of the wall (3-D ribs, gaps, ports, penetrations) is important for disruption simulations. In reality there is no place for simplistic representation of the wall as a continuous toroidally symmetric structure: with strong interaction with the wall the 3-D wall structure determines the time behavior of the plasma.

The TMHD model reflects the above mentioned properties. The characteristic time scale of TMHD τ_{TMHD} is intermediate between the inertial and resistive penetration times

$$\tau_{MHD} \simeq \underbrace{R/V_A}_{<1\ \mu s} \quad \ll \quad \tau_{TMHD} \quad < \quad \underbrace{\tau_{transport}}_{0.01-1\ s} \quad \ll \quad \underbrace{\tau_{resistive}}_{1-1000\ s}. \tag{4.6}$$

Here, the time $\tau_{transport}$ of transport processes is mentioned. They determine the evolution of the plasma pressure. The value of $\tau_{transport}$ strongly depends on the plasma regime but except for the thermal quench, which cannot be described by MHD model alone, is typically larger than the disruption time and shorter that the resistive time. VDE disruptions typically have no thermal quench.

The simplest form of macroscopic TMHD equations is presented by the following set of equations

$$\nabla p = (\mathbf{j} \times \mathbf{B}), \quad \bar{\Psi}(\bar{\Phi}) = \text{const}, \tag{4.7}$$

$$\lambda\vec{\xi} = -\frac{\bar{F}}{r^2}\nabla\tilde{F}, \quad \left(\nabla \cdot \frac{\bar{F}^2}{r^4}\nabla\tilde{F}\right) = 0, \tag{4.8}$$

$$-\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi^E + (\mathbf{V} \times \mathbf{B}) = \frac{\mathbf{j}^{pl}}{\sigma^{pl}}, \quad \mathbf{V} \equiv \frac{d\vec{\xi}}{dt}, \tag{4.9}$$

$$\sigma = \sigma(\bar{\Phi}), \quad (\mathbf{B} \cdot \nabla) \simeq 0, \tag{4.10}$$

$$\mathbf{E}^{pl}_{\parallel} = \mathbf{E}^{wall}_{\parallel} = \frac{\mathbf{j}^{pl}}{\sigma^{pl}} - (\mathbf{V} \times \mathbf{B}) = \frac{\mathbf{j}^{wall}}{\sigma^{wall}}. \tag{4.11}$$

At the free plasma surface the boundary condition expresses the force balance in presence of the surface currents

$$\left(p + \frac{|\mathbf{B}|^2}{2\mu_0} + \frac{\bar{F}\tilde{F}}{r^2\mu_0}\right)_i = \left(\frac{|\mathbf{B}|^2}{2\mu_0}\right)_e, \tag{4.12}$$

where subscripts 'i, e' specify the inner and outer sides of the plasma surface.

The TMHD system of equations (4.7-4.12) describes the macroscopic dynamics of tokamak plasma as a fast equilibrium evolution with flux conservation, excitation of sheet currents or creation of islands at the resonant surfaces, and the surface currents at the plasma boundary and at the wall plasma facing surface. The equations for the currents in the wall are described in the following section.

The first two equations Eq. (4.7,4.8) represent the TMHD replacement of the ordinary equation of motion

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla p + (\mathbf{j} \times \mathbf{B}), \tag{4.13}$$

where ρ is the plasma density $m_i n_i$, and $\mathbf{V}$ is the plasma velocity. The Eq. (4.7) describes a sequence of flux conserving equilibrium configurations, with $\bar{\Psi}$ understood as the averaged poloidal flux in RMC (3.80). It gives the main representation of the evolving magnetic configuration.

The second equation (4.8) is the approximate remaining part of the equation of motion. The inertial term $\rho\frac{d\mathbf{V}}{dt}$ is replaced by $\rho\gamma\mathbf{V}$, which is a friction force $\propto -\mathbf{V}$ if moved to the right hand side. The plasma displacement $\vec{\xi}$ is used instead of velocity $\frac{\partial\vec{\xi}}{\partial t} = \mathbf{V}$ but because time does not affect the solution of first two equations of TMHD. For numerical simulations the factor λ is a relaxation parameter which determines the displacement to the next equilibrium. The corresponding time interval is determined by Faraday's law.

The equilibrium equation (4.7) is coupled to the equation of motion (4.8) because the flux conserving sequence of equilibria generates the surface currents at the plasma boundary. This current depends exclusively on the plasma deformation. There is an unbalanced force acting on the surface current, determined by the equilibrium solution. As a response to this force the plasma generates small poloidal currents $\delta\mathbf{j}$ in the core. These currents enter the plasma edge as perpendicular to the plasma surface currents $(\delta\mathbf{j} \cdot \nabla a)$, which generates an additional, poloidal component of the surface current. As a result, the total surface currents becomes force-free [66, 67, 68] and evolving as a finite width skin currents in accordance with the plasma resistivity. In the core $(\delta\mathbf{j} \times \mathbf{B})$ creates a force driving the plasma motion.

In TMHD, the equation of motion (4.8) is treated in an approximate way, assuming small plasma deformation δa with respect to the major radius r

$$\left(\frac{\delta a}{r}\right)^2 \ll 1, \quad \left(\frac{\partial}{\partial\zeta}\right) \ll \left(\frac{\partial}{\partial\theta}\right), \quad \left(\frac{\partial}{\partial\zeta}\right) \ll a\left(\frac{\partial}{\partial a}\right). \tag{4.14}$$

The angle $\zeta \equiv \phi$ is the cylindrical azimuth. This approximation may slightly affect the shape of the plasma but keeps all physics effects related to disruptions.

The perturbation of the magnetic field $\delta\mathbf{B}$ associated with the plasma displacement can be taken in the form

$$\delta\mathbf{B} = (\nabla a \times \nabla\tilde{A}) = \tilde{A}'_\theta(\nabla a \times \nabla\theta) - \tilde{A}'_\zeta(\nabla\zeta \times \nabla a), \tag{4.15}$$

where $\delta\tilde{A}$ is a perturbation of the vector potential. The field $\delta\mathbf{B}$ is normal to magnetic surfaces and is divergence free. The dominant term in $\delta\mathbf{B}$ is the first one. The second term is $a\delta a/r^2$ times smaller. Here the radial coordinate a is considered as a minor radius of the plasma, and δa is the amplitude of the toroidal deformation of the magnetic surfaces. Because there is no need in precise simulations of the inertia term, the following approximation

$$\delta\mathbf{B} \simeq \tilde{A}'_\theta(\nabla a \times \nabla\theta), \quad \delta B^\theta = 0, \quad \delta B^\zeta = \frac{\tilde{A}'_\theta}{J} \tag{4.16}$$

is accepted for the TMHD model. In the same approximation the perturbation of the magnetic field can be presented as

$$, \tag{4.17}$$

$$B_a = 0, \quad B_\theta = 0, \quad B_\zeta = \frac{g_{\zeta\zeta}}{J}\tilde{A}'_\theta, \quad \delta\mathbf{B} \simeq \tilde{F}\nabla\zeta, \quad \tilde{F} \equiv g_{\zeta\zeta}\delta B^\zeta \simeq r^2\delta B^\zeta, \tag{4.18}$$

where the perturbation of the poloidal current $\tilde{F}$ is introduced,

This gives the following expression for the perturbed current density

$$\delta\mathbf{j} = (\nabla\tilde{F} \times \nabla\zeta). \tag{4.19}$$

The Lorentz force in Eq. (4.8) of interaction of the current $\delta\mathbf{j}$ with the equilibrium toroidal magnetic field in the core is then given by

$$(\delta\mathbf{j} \times \mathbf{B}) \simeq \Big((\nabla\tilde{F} \times \nabla\zeta) \times (JB^\zeta(\nabla a \times \nabla\theta))\Big) \simeq -B^\zeta\nabla\tilde{F} \simeq -\frac{\bar{F}}{r^2}\nabla\tilde{F}. \tag{4.20}$$

The perturbation of $\tilde{F}$ is determined by the condition that the virtual plasma displacement $\vec{\xi}$ does not perturb the toroidal magnetic field at the level larger than it was considered for the Lorentz force. The curl of the Faraday law gives the time dependence of the magnetic field

$$-\frac{\partial\mathbf{B}}{\partial t} + (\nabla \times (\mathbf{V} \times \mathbf{B})) = -\left(\nabla \times \frac{\mathbf{j}}{\sigma}\right). \tag{4.21}$$

The poloidal currents in the right hand side of the toroidal component of this equation are negligible because of the dominance of the toroidal field in tokamaks. The substitution $\mathbf{V} = \frac{\partial\vec{\xi}}{\partial t}$ determines the perturbation of toroidal magnetic field $\tilde{B}^\zeta$ in terms of $\vec{\xi}$

$$\tilde{B}^\zeta = (\nabla \times (\vec{\xi} \times \mathbf{B}))^\zeta. \tag{4.22}$$

Neglecting $\tilde{B}^\zeta$ and using the main toroidal field $JB^\zeta(\nabla a \times \nabla\theta)$ in the right hand side leads to the following restriction on $\vec{\xi}$

$$(\nabla \times (\vec{\xi} \times \mathbf{B}))^\zeta = \frac{1}{J}\begin{vmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial a} & \frac{\partial}{\partial\theta} & 0 \\ B^\zeta J\xi^\theta & -B^\zeta J\xi^a & 0 \end{vmatrix} = -\left(\nabla \cdot (B^\zeta\vec{\xi})\right) = 0 \tag{4.23}$$

or

$$\left(\nabla \cdot \frac{\bar{F}}{r^2}\vec{\xi}\right) = -\left(\nabla \cdot \frac{\bar{F}^2}{r^4}\nabla\tilde{F}\right) = 0 \tag{4.24}$$

The approximations here are used exclusively for describing the inertial term, which in reality is small and does not need excessive precision. It simply determines a relaxation process (one of them) for finding the final equilibrium which evolves at intermediate resistive-inertial time scale.

The proper splitting of equation of motion to a macroscopic force balance and to a the configuration advancing equation removes the Courant limitation for the time step, $\Delta t < h/V_A$.

In existing numerical codes the Courant condition requires the tiny time step Δt almost in the nanosecond range and remains an unsolved problem in MHD simulations for 4 decades. The TMHD split of the equation of motion eliminates any fast propagating Alfven or magneto-sonic waves. In fact, the extraction of the main force balance equation allows reintroduction of the conventional inertia term instead of friction in Eq. (4.8) of TMHD.

$$\lambda\vec{\xi} \to \rho\frac{\partial \mathbf{V}}{\partial t}, \tag{4.25}$$

if it would be necessary, without introducing the fast time scales and associated Courant limitations.

The $\lambda\vec{\xi}$ term, replacing the plasma inertia, corresponds to the simplest relaxation process. The absence of plasma velocity in the equation of motion automatically eliminates the need in a special boundary condition for the plasma flow to the wall.

Faraday's law (4.9) is the only equation in TMHD which contains the time derivatives and the plasma velocity $\mathbf{V}$ and determines the time evolution depending on the resistivities of the plasma σ^{pl} and the wall σ^{wall}.

For fast, $\Delta t \ll \tau_{TMHD}$ time evolution, which is not present in reality, but is typical for numerical MHD simulation, which start from an unstable configuration, Faraday's law is not essential. The flux conservation by Eqs. (4.7, 4.8) is sufficient for determine the unique sequence of plasma evolving configurations. It is remarkable, that when the plasma touches the wall and then reaches the equilibrium maintained by the currents in the wall [24, 25], the equilibrium sequence remains unique. The currents in the wall decay resistively. As a result the plasma moves into the wall and the plasma cross-section shrinks. Still the sequence of equilibrium configurations remains unique under flux conservation. The pressure term has only a minor effect on this sequence.

Given the sequence of equilibria, Faraday's law determines the plasma velocity and the time associated with each configuration. Thus, in TMHD Faraday's law has a reversed meaning with respect to electrodynamics, in which the time dependence and media velocity determine the current density.

In TMHD, the plasma instability acts as a current generator, and the currents determine the plasma velocity and voltage based on resistivity, not vice versa.

The physics of plasma-wall interaction in disruptions goes far beyond MHD. Its realistic model could be developed only in close collaboration between theory, experiment and numerical simulations in order to determine the equivalent of "resistivity" at the plasma edge interacting with the wall.

TMHD does not contain the equation of state for the plasma pressure p. It is determined by the transport processes whose time scale in the perturbed configuration can be comparable with the TMHD time scale. Because of this there is no universal equation of state for p evolution. The simplest choice is a prescribed $p = p(\bar{\Phi})$. Potentially, p can be provided to Eq. (4.7) either based on experimental interpretations or from a transport modeling. In many situations the disruption dynamics is not sensitive to the plasma pressure.

The Eq. (4.10) in TMHD reflects the plasma anisotropy. In a high temperature tokamak plasma, electrons move very fast along the magnetic field lines ($V_e \simeq 1.34 \cdot 10^7\sqrt{T_{e,keV}}$), making the electron temperature T_e almost constant on the magnetic surfaces. Since the electron conductivity is a function of electron temperature $\sigma = \sigma(T_e)$, it is constant along the field lines, $(\mathbf{B} \cdot \nabla\sigma) = 0$. This condition, difficult for conventional codes, can be explicitly reproduced by adaptive grids as $\sigma = \sigma(\bar{\Phi})$.

The last equation Eq. (4.11) is the electro-magnetic boundary condition at a wall corresponding to continuity of the electric field parallel to the wall surface

$$\mathbf{E}^{pl} = -\frac{\partial \mathbf{A}^{pl}}{\partial t} - \nabla\varphi^{E,pl} = \mathbf{E}^{wall} = -\frac{\partial \mathbf{A}^{wall}}{\partial t} - \nabla\varphi^{E,wall}. \tag{4.26}$$

In particular, this boundary condition determines the plasma velocity into the wall.

Despite its simplicity in describing the macroscopic dynamics, the TMHD model is instrumental in providing the scale separation for interfacing MHD for more complicated physics at the in vicinity of resonant surfaces and at the plasma edge. The adaptive grids, which are required for implementation of TMHD, can represent the resonant surfaces in the best possible fashion, while the TMHD solution between the resonant surfaces can provide the boundary conditions for more complicated physics at the resonant layer and near the plasma edge.

5 Equilibrium equations in RMC for TMHD *(to ToC)*

The 2-dimensional equilibrium plays an important role in tokamaks. First, all magnetic systems of tokamaks are designed using based on equilibrium solutions for different plasma regimes. Second, in operational plasmas the equilibrium equations are used in the equilibrium reconstruction mode for obtaining information on the magnetic configuration and current density distribution. This information serves as a basis for interpretation of data and results from numerous diagnostics on the machines.

5.1 Grad-Shafranov equation *(to ToC)*

The equilibrium equation Eq. (3.17)

$$\nabla\bar{p} = (\bar{\mathbf{j}} \times \mathbf{B}), \quad (\mathbf{B}\cdot\nabla p) = 0, \quad (\bar{\mathbf{j}}\cdot\nabla\bar{p}) = 0 \tag{5.1}$$

together with representation of the magnetic field in axisymmetrical configurations Eq. (3.52)

$$\mathbf{B} = (\nabla\hat{\Psi} \times \nabla\varphi) + \hat{F}\nabla\varphi, \quad \bar{\mathbf{j}} = (\nabla\hat{F} \times \nabla\varphi) + \bar{\jmath}_\varphi\nabla\varphi \tag{5.2}$$

suggests that

$$p = p(\hat{\Psi}) = p(\hat{F}), \quad \hat{F} = \hat{F}(\hat{\Psi}). \tag{5.3}$$

We recall that $\bar{\jmath}_\varphi$ is the covariant component of the current density $j_\varphi \equiv r\bar{\jmath}_{tor}$. The representation Eq. (5.2), if substituted into Eq. (5.1), shows that only the $\nabla\hat{\Psi}$ projection of the equilibrium equation is significant. It gives the expression for $\bar{\jmath}_\varphi$

$$\bar{\jmath}_\varphi = r\bar{\jmath}_{tor} = r^2P + T, \quad P(\hat{\Psi}) \equiv \frac{d\bar{p}}{d\hat{\Psi}}, \quad T(\hat{\Psi}) \equiv \hat{F}\frac{d\hat{F}}{d\hat{\Psi}}. \tag{5.4}$$

The toroidal component of the Ampere law in cylindrical coordinates relates the $\hat{\Psi}$ function with the toroidal current density

$$((\nabla \times (\nabla \times \mathbf{A}))\cdot\nabla\varphi) = -(\bar{\mathbf{j}}\cdot\nabla\varphi) \tag{5.5}$$

and represent the Grad-Shafranov (GSh) equation

$$\Delta^*\hat{\Psi} \equiv \frac{\partial^2\hat{\Psi}}{\partial r^2} - \frac{1}{r}\frac{\partial\hat{\Psi}}{\partial r} + \frac{\partial^2\hat{\Psi}}{\partial z^2} = -\bar{\jmath}_\varphi = -r\bar{\jmath}_{tor} = -r^2P(\hat{\Psi}) - T(\hat{\Psi}). \tag{5.6}$$

One of the remarkable property of this equation is that its differential operator is linear, while non-linearity is located only in the right hand side. Because of this property, the perturbation of this equation is very simple

$$\hat{\Psi} = \bar{\Psi} + \psi, \quad \psi \ll \bar{\Psi}, \quad \Delta^*(\bar{\Psi} + \psi) = -r^2P(\bar{\Psi}) - T(\bar{\Psi}) - r^2\frac{dP}{d\bar{\Psi}}\psi - \frac{dT}{d\bar{\Psi}}\psi - r^2\delta P - \delta T. \tag{5.7}$$

Here $\delta P, \delta T$ are the perturbations of the right hand side.

One of application of GSh equation is the equilibrium reconstruction which uses the signals from magnetic and other diagnostics in order to reconstruct the right hand side of the GSh equation. The degree of freedom in variations of the right hand side is very limited in order to provide the convergence of reconstruction. For equilibrium reconstruction the ability of easy implementation of the solution of the linearized GSh equation is very essential for variance (or sensitivity) analysis [69] which gives the information on what kind of perturbations are well reconstructed and which are not.

In curvilinear coordinates a, θ, φ the form of the GSh equation can be obtained using the expression for $\bar{\jmath}^\varphi$ component of the current density Eq. (3.61) and covariant components of the magnetic field Eq. (3.58)

$$\bar{\jmath}^\varphi = r^2\bar{\jmath}_\varphi = \frac{1}{J}\left(\frac{\partial B_\theta}{\partial a} - \frac{\partial B_a}{\partial\theta}\right), \quad B_a \equiv -\nu + \sigma'_a = M\hat{\Psi}'_\theta - N\hat{\Psi}'_a, \quad B_\theta \equiv \bar{J} + \sigma'_\theta = N\hat{\Psi}'_\theta - K\hat{\Psi}'_a. \tag{5.8}$$

This gives the following form of the GSh equation in toroidal coordinates

$$\mathcal{L}\hat{\Psi} \equiv (K\hat{\Psi}'_a - N\hat{\Psi}'_\theta)'_a + (M\hat{\Psi}'_\theta - N\hat{\Psi}'_a)'_\theta = -r^2JP(a) - JT(a). \tag{5.9}$$

The linearized version of the GSh equation has the form

$$\hat{\Psi} = \bar{\Psi}(a) + \psi, \quad \mathcal{L}(\bar{\Psi} + \psi) + r^2J\frac{P'(a)}{\bar{\Psi}'(a)}\psi + J\frac{T'(a)}{\bar{\Psi}'(a)}\psi = -r^2J(P + \delta P) - J(T + \delta T). \tag{5.10}$$

The functions $\delta P(a), \delta T(a)$ in the right hand side allow for solving the GSh equation not only in the classical formulation with the given P, T (when $\delta P = \delta T = 0$), but also for given $\bar{p}(a)$ profile, $q(a)$ (for stability studies), $\bar{\Psi}(a)$ (for interfacing with the transport analysis codes), or any other representative profiles.

Together with the fast algorithm for advancing the coordinate surfaces Eq. (3.66) and the linearized GSh equation (5.10) constitutes a very efficient Newton scheme for solving equilibrium problems for in all possible formulations, including the variance analysis of equilibrium reconstruction.

5.2 3-dimensional equilibrium equations *(to ToC)*

The goal of this section is to formulate the equilibrium equations in a form suitable for 3-dimensional TMHD. As a first step we consider the simplest case when the 3-dimensional magnetic field has magnetic surfaces

$$\psi * (\bar{a}, \theta, \zeta) = 0. \tag{5.11}$$

Accordingly RMC $\bar{a}, \theta, \zeta$ are the flux coordinates for this case.

In equilibrium the plasma pressure should be constant at the magnetic surfaces, $\bar{p} = \bar{p}(\bar{a})$. Also in equilibrium $(\bar{\mathbf{j}} \cdot \nabla \bar{p}) = 0$. Then, the equilibrium condition for the radial contravariant component $\bar{\jmath}^a = 0$ and the Eq. (3.45) gives

$$\hat{F}'_\theta - \hat{J}'_\zeta = 0, \quad \hat{F} = \bar{F}(\bar{a}) + \lambda'_\zeta, \quad \hat{J} = \bar{J}(\bar{a}) + \lambda'_\theta. \tag{5.12}$$

Because of definitions of $\hat{J} = \hat{J}(a, \zeta)$ in Eq. (3.40), the term $\lambda = 0$ and in equilibrium

$$\hat{F} = \bar{F}(\bar{a}), \quad \hat{J} = \bar{J}(\bar{a}). \tag{5.13}$$

With this representation, the equilibrium equation is fulfilled automatically along the magnetic surface. The radial force balance is reduced to

$$\bar{p}'\sqrt{g} = (\bar{J}' + \nu'_\theta)(\bar{\Psi}' + \bar{\Phi}'\eta'_\zeta) - (\bar{F}' + \nu'_\zeta)\bar{\Phi}'(1 + \eta'_\theta) = \bar{J}'\bar{\Psi}' - \bar{F}'\bar{\Phi}' + \bar{\Psi}'\nu'_\theta - \bar{\Phi}'\nu'_\zeta. \tag{5.14}$$

We use here $\sqrt{g}$ as a notation for Jacobian J in order to avoid its potential confusion with the toroidal current $\bar{J}$.

This equation can be split into the angle averaged $(\sqrt{g})_0$ and oscillatory parts $\sqrt{g} - (\sqrt{g})_0$

$$\bar{p}'(\sqrt{g})_0 \bar{J}'\bar{\Psi}' - \bar{F}'\bar{\Phi}', \quad \bar{\Psi}'\nu'_\theta - \bar{\Phi}'\nu'_\zeta = \bar{p}'[\sqrt{g} - (\sqrt{g})_0]. \tag{5.15}$$

The second equation is the MDE equation in (5.15) for the oscillatory component ν of the current density. Given Fourier representation for the Jacobian

$$\sqrt{g} = \sum_{mn} J_{mn} e^{im\theta - n\zeta} \tag{5.16}$$

the solution for non-resonant Fourier harmonics $\nu_{m'n'}$ can be found

$$\nu_{m'n'} = -i\bar{p}' \frac{J_{m'n'}}{m'\bar{\Psi}' + n'\bar{\Phi}'}. \tag{5.17}$$

The resonant harmonics are singular and inconsistent with the equilibrium equation. The exception would be only if the Jacobian does not contain the resonant harmonics or if the pressure gradient is zero. This statement constitutes the Hamada principle [54].

This example illustrates the problem in calculation of equilibria in 3-dimensional configurations. One approach is to provide in calculations the absence of the resonant harmonics in the Jacobian. This is possible but requires a special boundary conditions. The plasma dynamics is not such a case.

The RMC are consistent with the presence of stochastization of the magnetic field and allow to formulate the equilibrium equations in for the purpose of TMHD.

The contravariant and covariant components of $\mathbf{B}$ in RMC differ from the flux coordinates only by the resonant term ψ^*

$$B^a = \frac{\psi'^*_\theta}{J}, \quad B^\theta = -\frac{\bar{\Psi}' + \psi'^*_a + \bar{\Phi}'\eta'_\zeta}{J}, \quad B^\zeta = \frac{\bar{\Phi}'(1 + \eta'_\theta)}{J}, \tag{5.18}$$

$$B_a = M\psi'^*_\theta - N(\bar{\Psi}' + \psi'^*_a + \bar{\Phi}'\eta'_\zeta) + \tilde{M}\bar{\Phi}', \tag{5.19}$$

$$B_\theta = N\psi'^*_\theta - K(\bar{\Psi}' + \psi'^*_a + \bar{\Phi}'\eta'_\zeta) + \tilde{N}\bar{\Phi}'(1 + \eta'_\theta), \tag{5.20}$$

$$B_\zeta = \tilde{M}\psi'^*_\theta - \tilde{N}(\bar{\Psi}' + \psi'^*_a + \bar{\Phi}'\eta'_\zeta) + Q\bar{\Phi}'(1 + \eta'_\theta). \tag{5.21}$$

Moreover, the resonant terms can be localized near their resonant surfaces.

First we describe the equations outside the islands at the resonant surfaces. There the radial force balance has the form

$$\bar{p}'\sqrt{g} = (\bar{J}' + \nu'_\theta)(\bar{\Psi}' + \psi'^*_a + \bar{\Phi}'\eta'_\zeta) - (\bar{F}' + \nu'_\zeta)\bar{\Phi}'(1 + \eta'_\theta). \tag{5.22}$$

In RMC the term ψ^* can be made small and we neglect it in the force balance equation. In order to solve it we introduce straight field line coordinates $\bar\theta, \bar\zeta$ and a new unknown function $\bar\nu$

$$\bar\theta \equiv \theta + \eta, \quad \bar\zeta \equiv \zeta, \quad \bar\nu \equiv \nu - \bar J'\eta, \quad \nu'_\theta = \bar\nu_{\bar\theta}(1+\eta'_\theta) + \bar J'\eta'_\theta, \quad \nu'_\zeta = \bar\nu_{\bar\zeta} + \bar\nu_{\bar\theta}\eta'_\zeta + \bar J'\eta'_\zeta. \tag{5.23}$$

This reduces the force balance equation to its form in the straight field line coordinates

$$\bar p'\sqrt{g} = (\bar J' + \bar\nu_{\bar\theta}(1+\eta'_\theta) + \bar J'\eta'_\theta)(\bar\Psi' + \bar\Phi'\eta'_\zeta) - (\bar F' + \bar\nu_{\bar\zeta} + \bar\nu_{\bar\theta}\eta'_\zeta + \bar J'\eta'_\zeta)\bar\Phi'(1+\eta'_\theta), \tag{5.24}$$

$$\bar p'\frac{\sqrt{g}}{1+\eta'_\theta} = \bar J'\bar\Psi' - \bar F'\bar\Phi' + \bar\Psi'\bar\nu_{\bar\theta} - \bar\Phi'\bar\nu_{\bar\zeta}. \tag{5.25}$$

It can be split into the averaged

$$\bar J'\bar\Psi' - \bar F'\bar\Phi' = \bar p'\left(\frac{\sqrt{g}}{1+\eta'_\theta}\right)_0 \tag{5.26}$$

and the oscillatory MDE equation for $\bar\nu$

$$\bar\Psi'\bar\nu_{\bar\theta} - \bar\Phi'\bar\nu_{\bar\zeta} = \bar p'\frac{\sqrt{g}}{1+\eta'_\theta} - \bar p'\left(\frac{\sqrt{g}}{1+\eta'_\theta}\right)_0, \tag{5.27}$$

which can be solved in Fourier space as it was described earlier in Eqs. (3.75-3.78)

$$\bar\nu \equiv \sum_{m,n}\nu_{mn}e^{im\bar\theta - in\bar\zeta}, \quad \frac{\sqrt{g}}{1+\eta'_\theta} - \left(\frac{\sqrt{g}}{1+\eta'_\theta}\right)_0 \equiv \sum_{m,n}J_{mn}e^{im\bar\theta - in\bar\zeta}, \tag{5.28}$$

$$\bar\nu_{mn} = -i\bar p'\frac{J_{mn}}{m\bar\Psi' + n\bar\Phi'}. \tag{5.29}$$

Outside the islands the MDE for the oscillatory current density $\bar\nu$ is not singular and is well defined. It can be solved in the same way as the similar equation (5.15) for flux coordinates.

The partial differential part of the set of equilibrium equations in RMC has the form

$$\frac{\partial}{\partial a}\left[\tilde M\psi'^*_\theta - \tilde N(\bar\Psi' + \psi'^*_a + \bar\Phi'\eta'_\zeta) + Q(\bar\Phi'(1+\eta'_\theta))\right] \tag{5.30}$$

$$-\frac{\partial}{\partial\zeta}\left[M\psi'^*_\theta - N(\bar\Psi' + \psi'^*_a + \bar\Phi'\eta'_\zeta) + \tilde M\bar\Phi'(1+\eta'_\theta)\right] = \bar F'_a + \nu'_\zeta, \tag{5.31}$$

$$\frac{\partial}{\partial a}\left[N\psi'^*_\theta - K(\bar\Psi' + \psi'^*_a + \bar\Phi'\eta'_\zeta) + \tilde N(\bar\Phi'(1+\eta'_\theta))\right] \tag{5.32}$$

$$-\frac{\partial}{\partial\theta}\left[M\psi'^*_\theta - N(\bar\Psi' + \psi'_a + \bar\Phi'\eta'_\zeta) + \tilde M\bar\Phi'(1+\eta'_\theta)\right] = \bar J'_a + \nu'_\theta. \tag{5.33}$$

The radial force balance determines the function ν and the relationship between the plasma pressure gradient, poloidal and toroidal currents as functions of $\bar a$. Given $\bar J, \bar F, \nu$ the differential equations determine fluxes $\bar\Psi, \psi, \bar\Phi$ and function η.

In the vicinity of the island m, n where the shape of magnetic surfaces is described by function $\hat\Psi^*_{mn}(\bar a, \bar\theta, \zeta)$ as it was explained in Sect. 3.5 in Eqs. (3.86-3.91). In order to simplify the explanation, we assume that the RMC represent the straight field line coordinates $\bar a, \bar\theta, \bar\zeta$ with $\eta = 0$ in Eq. (3.81,3.85).

Here we introduce a local flux coordinate χ inside the island $\hat\Psi^*_{mn} = \hat\Psi^*_{mn}(\chi)$

$$\chi^2 = \frac{2(\hat\Psi^*_{nm} - \bar\Psi^*_{mn})}{\bar\Psi''^*_{mn}} + \frac{w^2}{2}, \quad w^2 = \left|\frac{8\psi^*_{mn}}{\bar\Psi''^*_{mn}}\right|, \tag{5.34}$$

which determines the geometry of the flux surfaces $x \equiv \bar a - \bar a^*$ by

$$x^2 = \chi^2 - w^2 + w^2\left\{\begin{array}{ll}\sin^2\frac{m\beta}{2}, & \text{for}\bar\Psi''^*_{mn} > 0,\\ \cos^2\frac{m\beta}{2}, & \text{for}\bar\Psi''^*_{mn} < 0\end{array}\right. . \tag{5.35}$$

The variable $\chi = w$ at the center of magnetic island (O-point), and $\chi = 0$ along the separatrix of the island.

In order to describe the equilibrium current density in the vicinity of the island it is necessary to introduce a local poloidal coordinate

$$\hat{\theta} \equiv \theta - \frac{n}{m}\zeta, \quad \left(\bar{\Psi}'\frac{\partial}{\partial\theta} - \bar{\Phi}'\frac{\partial}{\partial\zeta}\right)_{\bar{a}=\bar{a}^*}\hat{\theta} = 0. \tag{5.36}$$

The equilibrium current density near the island has no normal component to the island magnetic surface $\hat{\Psi}^*_{mn}$ =const. Accordingly, it can be represented as

$$\bar{\mathbf{j}} = (\nabla\hat{F}^* \times \nabla\zeta) + (\nabla\hat{J}^* \times \nabla\hat{\theta}), \quad \hat{J}^* = \hat{J}^*(\chi,\hat{\theta}) = \bar{J}^*(\chi) + \nu^*(\chi,\hat{\theta}), \quad \bar{F}^* = \bar{F}^*(\chi,\bar{\zeta}). \tag{5.37}$$

The plasma pressure near the island is a function of χ: $\bar{p} = \bar{p}(\chi)$.

The set of equilibrium equations described in this section represents a generalization of the 2-dimensional GSh equation and will be referred as 3D Grad-Shafranov equation. Its important property is that the two differential equations for magnetic fluxes are linear. The non-linearity of the physics problem is localized in the determination of the right hand side.

Moreover, as the 2-dimensional GGh equation can be solved using the simplest r, z grid, in a similar way the 3D GSh equation can be solved in cylindrical coordinates r, φ, z. The differential operator represents simply the components of Ampere's law (3.46). The transition to the RMC coordinates is necessary only for generating the right current density in the right hand side. This kind of codes, although not very accurate in describing the core plasma, have the advantage in being able to reproduce a complicated magnetic fields with large stochastic regions, which is not possible with the flux coordinates.

The use of RMC can combine the properties of the flux coordinate based grids with laboratory grids: in the regions with high stochasticity the RMC grid can frozen and serve as a laboratory grid.

6 Electromagnetic model of a conducting wall for TMHD *(to ToC)*

As already mentioned in the Introduction, TMHD requires a realistic representation of the wall, which has little in common to simplistic toroidally symmetric shells in existing simulations like in Ref. [8]. The universal and practical representation of the wall surface can be achieved using triangles with uniform currents as an element of electrical circuits. A "thin" wall model can be considered as a reasonable first step with wall currents represented by the surface currents.

For the purpose of TMHD modeling the surface current density $h\mathbf{j}$ in the conducting shell (h is the thickness of the current distribution) can be split into two components: (a) one is a divergence free surface current $\mathbf{i}$ and (b) the second one is a current $\propto -\nabla\phi^S$ with potentially finite divergence in order to describe the plasma sink/source of the wall current:

$$h\mathbf{j} = \mathbf{i} - \bar{\sigma}\nabla\phi^S, \quad \mathbf{i} \equiv \nabla I \times \mathbf{n}, \quad (\nabla\cdot\mathbf{i}) = 0, \quad \bar{\sigma} \equiv h\sigma. \tag{6.1}$$

Here, I is the stream function of the divergence free component and $\mathbf{n}$ the unit normal vector to the wall. For simplicity of notations we drop in this Section the index "wall".

The second term containing the gradient of the surface function ϕ^S, which we call here the plasma source potential, is the surface current originated from the shearing of the electric current between the plasma and the wall. It is determined from the following equation

$$(\nabla\cdot(h\mathbf{j})) = -(\nabla\cdot(\bar{\sigma}\nabla\phi^S)) = -j_\perp, \tag{6.2}$$

where $j_\perp$ is the density of the current coming from/to the plasma and acting as a galvanic source for the surface currents on the wall. Some of $j_\perp$ can be associated with the Hiro currents, which are inductive effect and are not sensitive to the details of plasma-wall interaction. Another kind of $j_\perp$ is associated with the Evans currents [25] and represent the "resistive" effect determined by the plasma-wall interaction. A reasonable prediction would be that they are limited by the ion saturation current at the plasma edge.

In terms of components of the surface current Faraday's law can be written as

$$-\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi^E = \bar{\eta}(\nabla I \times \mathbf{n}) - \nabla\phi^S, \quad \bar{\eta} \equiv \frac{1}{\bar{\sigma}} \tag{6.3}$$

where the effective resistance $\bar{\eta}$ is introduced for convenience. The normal component of the curl from this equation gives the equation for the stream function I [70, 71, 72, 73, 74, 75, 76, 77, 78]

$$(\nabla\cdot(\bar{\eta}\nabla I)) = \frac{\partial B_\perp}{\partial t} = \frac{\partial(B_\perp^{pl} + B_\perp^{coil} + B_\perp^I + B_\perp^S)}{\partial t}. \tag{6.4}$$

In the right hand side the normal component of the magnetic field is represented by separate contributions from the plasma, external coils, and from two components of the surface current.

Two equations (6.2,6.4) describe the current distribution in the thin wall given the sources $j_\perp, B_\perp^{pl}, B^{coil}$ specified as functions of space and time. In addition to Eqs. (6.2,6.4) the expression for the magnetic field from the surface current (e.g., Bio-Savar formulas) has to specified in order to close the system of equations.

The important property of this system of equation is that the equation for ϕ^S is separated from the second equation and can be solve independently. After this, the contribution of ϕ^S can be taken into account in calculations of the right hand side of Eq. (6.4). These equations are suitable for the finite difference method used in Ref. [78] for simulation of the thin wall response to the perturbations from a rotating kink mode.

The triangle electromagnetic model representation of the thin wall is based on the following expressions for the vector potential $\mathbf{A}$ and magnetic field $\mathbf{B}$ of the uniform surface current $\mathbf{i}$=const inside the triangle

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= h\mathbf{j}\phi(\mathbf{r}) = [(\nabla I \times \mathbf{n}) - \bar{\sigma}\nabla\phi^S]\phi(\mathbf{r}), \quad \phi(\mathbf{r}) \equiv \int \frac{dS'}{X}, & (6.5)\\
\mathbf{B}(\mathbf{r}) &= (h\mathbf{j}\times\mathbf{e}) = \left\{[(\nabla I \times \mathbf{n}) - \bar{\sigma}\nabla\phi^S]\times\mathbf{e}\right\}, \quad \mathbf{e}(\mathbf{r}) \equiv \int \frac{\mathbf{X}}{X^3}dS', & (6.6)\\
\mathbf{X} &\equiv \mathbf{r}-\mathbf{r}', \quad X \equiv |\mathbf{r}-\mathbf{r}'|, & (6.7)
\end{aligned}$$

where ϕ is the electric potential of a triangle with a uniform unit charge, $\mathbf{e}$ is its electric field, $\mathbf{r}, \mathbf{r}'$ are the coordinates of the observation and the source points, and the integrals are taken over the surface of the triangle. Both ϕ and $\mathbf{e}$ have analytical expressions in terms of elementary functions.

In 2008 the triangle based electro-dynamic model of the thin wall was implemented into the numerical Shell simulation code (`Cbshl`), mentioned in Ref. [25] and used for simulations of eddy currents in the Lithium Tokamak Experiment (LTX) [79]. Earlier, the triangle based wall model was implemented in STARWALL code of P. Merkel [80, 81, 76] for resistive wall mode studies.

The current density inside the each triangle, covering the wall surface, is given by

$$h\mathbf{j} = \nabla I \times \mathbf{n} - \bar{\sigma}\nabla\phi^S = -\frac{I_a\mathbf{r}_{cb} + I_b\mathbf{r}_{ac} + I_c\mathbf{r}_{ba}}{2S} + \bar{\sigma}\left(\frac{\phi_a^S\mathbf{r}_{cb} + \phi_b^S\mathbf{r}_{ac} + \phi_c^S\mathbf{r}_{ba}}{2S}\times\mathbf{n}\right). \qquad (6.8)$$

Here, $I_a, I_b, I_c, \phi_a^S, \phi_b^S, \phi_c^S$ are values of I and η^S in three vertexes and $\mathbf{n}$ is the unit normal vector.

7 Energy functionals for finite element representation of TMHD *(to ToC)*

It is remarkable that all TMHD equations have their own energy principle leading to positively defined symmetric matrices if expressed in terms of finite elements.

1. The 3-D Ampere law equation (3.50) can be obtained from the energy principle

$$\frac{1}{\mu_0}\int\left\{\mathbf{B}^2 + (\mathbf{A}\cdot\mathbf{j})\right\}d^3r, \qquad (7.1)$$

   where $\mathbf{j}$ is considered as a given function and $\mathbf{A}$ is the test function. The substitution of magnetic field in terms of unknown functions $\bar{\Psi}(a)+\psi(a,\theta,\zeta), \bar{\Phi}(a)+\phi(a,\zeta), \bar{\Phi}'\eta(a,\theta,\zeta)$ (3.38) and metric tensor coefficients (3.16) is straightforward

$$\begin{aligned}
W^{\mathbf{j}\times\mathbf{B}} \equiv \frac{1}{2\mu_0}\int & \left\{K(\bar{\Psi}'+\psi_a'+\bar{\Phi}'\eta_\zeta')^2 - 2N(\bar{\Psi}'+\psi_a'+\bar{\Phi}'\eta_\zeta')(\psi_\theta'-\phi_\zeta') + M(\psi_\theta'-\phi_\zeta')^2\right.\\
&+Q(\bar{\Phi}'+\phi_a'+\bar{\Phi}'\eta_\theta')^2 - 2\tilde{N}(\bar{\Psi}'+\psi_a'+\bar{\Phi}'\eta_\zeta')(\bar{\Phi}'+\phi_a'+\bar{\Phi}'\eta_\theta') \qquad (7.2)\\
&\left.+2\tilde{M}(\psi_\theta'-\phi_\zeta')(\bar{\Phi}'+\phi_a'+\bar{\Phi}'\eta_\theta') - (\bar{\Phi}+\phi)\hat{F}_a' + (\bar{\Psi}+\psi)(\hat{J}_a'+\nu_\theta')\right\}dad\theta d\zeta.
\end{aligned}$$

   The current density components in last two terms are known at each iteration of solving the equilibrium equation.

2. The equation for $\bar{F}$ for plasma advancing (4.8) can be obtained from minimization of the following energy functional

$$\begin{aligned}
W^F &\equiv \frac{1}{2}\int\frac{\bar{F}^2}{r^4}|\nabla\tilde{F}|^2 d^3r & (7.3)\\
&= \frac{1}{2}\int\bar{F}^2\frac{g^{aa}\tilde{F}_a'^2 + 2g^{a\theta}\tilde{F}_a'\tilde{F}_\theta' + g^{\theta\theta}\tilde{F}_\theta'^2 + 2g^{a\zeta}\tilde{F}_a'\tilde{F}_\zeta' + 2g^{\theta\zeta}\tilde{F}_\theta'\tilde{F}_\zeta' + g^{\zeta\zeta}\tilde{F}_\zeta'^2}{r^4}Jdad\theta d\zeta, & (7.4)
\end{aligned}$$

there contravariant metric coefficients are elements of the inverted matrix of covariant metric coefficients

$$\begin{vmatrix} g^{aa} & g^{a\theta} & g^{a\zeta} \\ g^{a\theta} & g^{\theta\theta} & g^{\theta\zeta} \\ g^{a\zeta} & g^{\theta\zeta} & g^{\zeta\zeta} \end{vmatrix} \cdot \begin{vmatrix} g_{aa} & g_{a\theta} & g_{a\zeta} \\ g_{a\theta} & g_{\theta\theta} & g_{\theta\zeta} \\ g_{a\zeta} & g_{\theta\zeta} & g_{\zeta\zeta} \end{vmatrix} = \mathsf{I}. \tag{7.5}$$

3. The perpendicular components of Faraday's law determines plasma velocity as $\mathbf{V} = \frac{d\vec{\xi}}{dt}$, while the parallel component determines the rate of plasma evolution. The parallel component of the curl of Faraday's law (4.9)

$$-\frac{\partial \mathbf{B}}{\partial t} + (\nabla \times (\mathbf{V} \times \mathbf{B})) = (\nabla \times (\eta^{pl} \mathbf{j}^{pl})), \quad \eta \equiv \frac{1}{\sigma^{pl}} \tag{7.6}$$

can be obtained from the following energy functional

$$W^t \int \left\{ \frac{1}{2} \frac{\partial |\mathbf{B}|^2}{\partial t} + (\mathbf{V} \cdot (\mathbf{j} \times \mathbf{B})) + \frac{1}{2} \eta^{pl} |\mathbf{j}^{pl}|^2) \right\} d^3 r = \int \left\{ \frac{1}{2} \frac{\partial |\mathbf{B}|^2}{\partial t} + (\mathbf{V} \cdot \nabla p) + \frac{1}{2} \eta^{pl} |\mathbf{j}^{pl}|^2) \right\} d^3 r, \tag{7.7}$$

where the term with the plasma velocity does not affect the minimization. In RMC the energy functional for Faraday's law has the following representation

$$W^t = \frac{1}{2} \int \left\{ \frac{\partial}{\partial t} \left(K B^\theta B^\theta + 2\tilde{M} B^\theta B^\zeta + Q B^\zeta B^\zeta \right) + \eta^{pl} \left(K j^\theta j^\theta + 2\tilde{M} j^\theta j^\zeta + Q j^\zeta j^\zeta \right) \right\} d^3 r. \tag{7.8}$$

Here the radial contravariant components B^a, j^a were dropped as vanishing in the equilibrium configurations and not essential for the parallel part of Faraday's law in TMHD.

4. The equation (6.2) for ϕ^S, which determines the current in the wall due to current sharing with the plasma can be obtained by minimizing the functional W^S

$$W^S = \int \left\{ \frac{\bar{\sigma}(\nabla \phi^S)^2}{2} + j_\perp \phi^S \right\} dS - \frac{1}{2} \oint \phi^S \bar{\sigma} [(\mathbf{n} \times \nabla \phi^S) \cdot d\vec{l}]. \tag{7.9}$$

Here the surface integral dS is taken along the wall surface, while the contour integral $d\vec{l}$ is taken along the edges of the conducting surfaces with the integrand representing the surface current normal to the edges. In the typical case when there is no sink or sources at the edges the contour integral vanishes. The substitution of ϕ^S as a set of plane triangles (6.8) is straightforward and leads to the finite element representation of W^s.

5. The equation (6.3) for the divergence free part of the surface current $\mathbf{i} = (\nabla I \times \mathbf{n})$ can be obtained from the following energy functional

$$W^I \equiv \frac{1}{2} \int \left\{ \frac{\partial (\mathbf{i} \cdot \mathbf{A}^I)}{\partial t} + \bar{\eta} |\nabla I|^2 + 2 \left(\mathbf{i} \cdot \frac{\partial \mathbf{A}^{ext}}{\partial t} \right) \right\} dS - \oint (\phi^E - \phi^S) \frac{\partial I}{\partial l} dl. \tag{7.10}$$

The first, inductive term in the surface integral represents the change of magnetic energy of the current $\mathbf{i}$, which in triangle wall model is given by Eq. (6.8). Its vector potential $\mathbf{A}^I$ (6.5) can be expressed in terms of I using explicit formulas for triangle representation of the wall. The second term describes resistive losses, and the third one represents excitation of the current by the other sources

$$\mathbf{A}^{ext} \equiv \mathbf{A}^{pl} + \mathbf{A}^{coil} + \mathbf{A}^S. \tag{7.11}$$

Here, $\mathbf{A}^S$ is generated by the current $\bar{\sigma} \nabla \phi^S$, $\mathbf{A}^{coil}$ comes from the external coils and $\mathbf{A}^{pl}$ from the plasma.

The use of 3-D Hermit elements for plasma variables in energy functionals leads to block tri-diagonal symmetric matrices and opportunities for very efficient solution of equations with use of GPU. The equations for ϕ^S on the wall is explicitly block tri-diagonal. The circuit equations, resulted from minimization of W^I, contains the full matrix of mutual inductances. But because this matrix is stationary, it can be Cholesky decomposed a priory into a product of two triangular matrices, thus, making circuit equations solvable very efficiently using GPU.

8 Two steps in solving the MHD equations *(to ToC)*

The general procedure of solving TMHD equations can be outlined as a two step procedure. As the first step the 3-D the equilibrium equation

$$\nabla p = (\mathbf{j} \times \mathbf{B}) \tag{8.1}$$

has to be solved for the plasma core with a perturbed boundary in the environment of the external magnetic fields which include t. Under flux conservation this step generates the sheet currents or islands at the resonant surfaces. Also the surface current at the plasma boundary is generated as a reflection of a macroscopic instability. The sheet currents at the resonant surfaces are in equilibrium in the direction perpendicular to the resonant surfaces. At the same time there is no equilibrium along the surfaces leading to a reconnection process and creation of magnetic islands. The islands can saturate and then evolve at the resistive time scale. As it was mentioned earlier, the plasma model near the resonant surfaces can be extended to a non-MHD physics without perturbing the global TMHD model.

The second step is related to the surface current at the plasma boundary, which is not in equilibrium even in the normal direction to the plasma surface. The plasma response in TMHD is described by Eqs. (4.8) which determines the plasma virtual displacement and the plasma boundary for the next step configurations.

Two TMHD equations (4.7, 4.8) determine uniquely the sequence of magnetic configurations. Until plasma touches the wall, the total toroidal magnetic flux is preserved. It also can be considered as preserved when the plasma touches the tile surface but does not reach the MHD equilibrium. The evolution is still fast until the plasma reaches an equilibrium maintained by the Hiro or eddy currents.

The toroidal flux and the plasma cross-section start to shrink when the Hiro or eddy currents decay and the plasma moves into material surfaces in order to maintain the level of these currents necessary for equilibrium. Finally, this self-sustained equilibrium evolution exhausts the entire plasma.

The important property of the tokamak plasma is that even during contact, the plasma shape is conformal to the wetting zone on material surface and the normal magnetic field essentially absent.

It is remarkable that the TMHD equations(4.7, 4.8) determine a unique sequence of flux conserving magnetic configurations from the beginning of a disruption until the complete termination of the plasma.

In TMHD the timing is attached to the sequence of equilibria as a “post-processing” using the Faraday law (4.9). This is another remarkable property of TMHD that, given the pre-calculated sequence of plasma configurations different plasma physics models for plasma non-ideal properties and plasma-wall interactions can be tested against experimental data without recalculating the plasma dynamics.

9 Summary *(to ToC)*

In 2007 theory made a significant advance [24] in understanding disruptions. The key effect of Hiro currents, resulting from magnetic flux conservation in plasma dynamics, was discovered. It explained the toroidal asymmetry of the plasma current measurements in JET and the current sharing between the plasma and the wall. This progress was not matched by numerical simulations, which fell short in addressing the practical needs of the next step fusion device ITER in resolving the disruption problem. Now it became clear that it is not possible to move forward based on hydro-dynamic approaches of the present numerical codes.

TMHD puts numerical simulations into consistency with theoretical understanding and experimental observations. It addresses the problem of numerical consistency with high anisotropy of the tokamak plasma by introduction of Reference Magnetic Coordinates, which resolve the long standing problem of practical coordinates for stochastic and ergodic magnetic fields.

TMHD makes proper consideration of the equation of motion for plasma confinement configurations when the inertia plays a minor role, while the effect of force balance is dominant. This resolves the long standing problem in MHD codes related to Courant limitation of the time step.

The TMHD model contains a reasonably realistic wall model and properly describes the excitation of currents by the inductive effects and by the direct current sharing with the plasma.

All partial differential equations of TMHD have energy functionals, which give a straightforward way for finite element implementation into the numerical scheme. The positiveness of resulting symmetric matrices guarantees stability and efficient processing, including the use of GPU. Moreover, with 3-D Hermite finite elements for the plasma core functions, the matrices are simply block tri-diagonal.

The theory and TMHD predict that during disruptions because of magnetic flux conservation the plasma follows a sequence of equilibria, which is not sensitive to the plasma resistivity and interaction with the wall.

The non-ideal effects determine only the rate of evolution. This property would significantly simplify the comparison of TMHD simulations with experiments and help to reveal the physics of the plasma edge and plasma-wall interactions.

This paper does not describe in detail the important topic of resonant layers and matching conditions. This will be done elsewhere in future.

At present, TMHD was implemented into an operational 2-D Vertical Disruption Code (VDE). The finite elements and corresponding solvers for the plasma equations were created and tested at the level of 2-D. The Hiro and eddy currents were reproduced as predicted by theory, thus, confirming the basic consistency of TMHD with theory and observation.

The details of the VDE code and its results will be published elsewhere. It gives the start for development of a 3-D disruption simulation code based on TMHD.

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