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Relativistic Guiding Center Equations

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Abstract

In toroidal fusion devices it is relatively easy that electrons achieve relativistic velocities, so to simulate runaway electrons and other high energy phenomena a nonrelativistic guiding center formalism is not sufficient. Relativistic guiding center equations including flute mode time dependent field perturbations are derived. The same variables as used in a previous nonrelativistic guiding center code are adopted, so that a straightforward modifications of those equations can produce a relativistic version.

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I. INTRODUCTION

Ions in toroidal fusion devices are rarely to be found at relativistic velocities, but this can easily happen for electrons. Thus it is useful to have a relativistic guiding center formalism for energetic electron simulation. Previous derivations insisted on the use of exact canonical coordinates[1–5], making the equations unnecessarily complicated, or used coordinate systems significantly different from those used in a previously published[9] nonrelativistic version used in the code Orbit. In this work we derive relativistic guiding center equations which are a straightforward modification of those equations, requiring a simple modification of the code Orbit. It is shown that a non explicit periodic modification of the toroidal particle motion puts the equations into canonical Hamiltonian form.

Introducing units of time given by ω_0^{-1}, where $\omega_0 = eB/(mc)$ is the on axis gyro frequency, and units of distance given by the major radius R, the basic unit of energy becomes $m\omega_0^2 R^2$, which can also be written as $(mv^2/2)(2R^2/\rho^2)$, the gyro radius is $\rho = v/B \ll 1$, and the magnetic moment $\mu = mv_\perp^2/(2B)$ is of order ρ^2. Particle motion both along and across the field lines is of order ρ, but to leading order the cross field motion is the gyro motion, and cross field drift is of order ρ^2.

The Lagrangian for guiding center motion was derived by Littlejohn using Lie algebra, capable of producing the correct expression to all orders in the gyro radius. The method consists of an order by order expansion in gyro radius, at each step adding exact time derivatives to the Lagrangian in order to produce simplification at that order.

Begin with the Lagrangian for a charged relativistic particle[6]

$$L = (\vec{A} + Q\rho_\parallel \gamma \vec{B}) \cdot \vec{v} + \mu\dot{\xi} - H \tag{1}$$

with $\gamma = 1/\sqrt{1 - v^2/c^2}$ and the Hamiltonian

$$H = mc^2 \left(1 + \frac{v^2\gamma^2}{c^2}\right)^{1/2} + \Phi \tag{2}$$

with $mv^2\gamma^2 = m\rho_\parallel^2 B^2\gamma^2 + 2\mu B$, and Φ the electric potential, m the particle rest mass, $\rho_\parallel = v_\parallel/B$, with $v_\parallel = \vec{B}\cdot\vec{v}/B$ the parallel velocity, μ the magnetic moment, ξ the gyrophase. The factor Q is 1 for ions and -1 for electrons. Because we have normalized time in terms of the gyro frequency, both $\vec{A}$ and $\vec{v}$ change sign for electrons. We also have

$$\frac{v^2}{c^2} = \frac{(H - \Phi)^2 - m^2c^4}{(H - \Phi)^2}. \tag{3}$$

Equilibrium field quantities are given by $\vec{A} = (\psi\nabla\theta - \psi_p\nabla\zeta)$ and $\vec{B} = g\nabla\zeta + I\nabla\theta + \delta\nabla\psi$ with ψ the toroidal flux, ψ_p the poloidal flux, θ a poloidal angle coordinate, ζ a toroidal angle coordinate, and g, I and δ are equilibrium functions. The vector potential through $\vec{B} = \nabla \times \vec{A}$ provides a contravariant representation of the field. The function δ is given by the nonorthogonality of the coordinate system and is normally small[7]. We will consider an axisymmetric equilibrium, so field quantities I, δ are functions of ψ_p and θ and $\gamma = \gamma(\psi_p, \theta, \rho_\|)$.

Introduce a field perturbation of the form $\delta\vec{B} = \nabla \times \alpha\vec{B}$ with α a function of ψ_p, θ, ζ and time[8], and substitute $\vec{B}$ and $\vec{A}$ in the Lagrangian $L(\rho_\|, \psi_p, \theta, \zeta, \dot{\rho}_\|, \dot{\psi}_p, \dot{\theta}, \dot{\zeta})$, giving

$$L = [\psi + (Q\rho_\|\gamma + \alpha)I]\dot{\theta} + [(Q\rho_\|\gamma + \alpha)g - \psi_p]\dot{\zeta} + Q\delta q\gamma\rho_\|\dot{\psi}_p + \mu\dot{\xi} - H. \tag{4}$$

Lagrange's equations are,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}, \tag{5}$$

immediately giving the constancy of the magnetic moment, and taking the equilibrium axisymmetric, $\partial_\zeta I = \partial_\zeta g = 0$ and using $g = g(\psi_p)$ we find

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}_\|} = 0 = QI\gamma\dot{\theta} + Qg\gamma\dot{\zeta} + QI\rho_\|\gamma'_{\rho_\|}\dot{\theta} + Qg\rho_\|\gamma'_{\rho_\|}\dot{\zeta} + Q\delta q\gamma\dot{\psi}_p + Q\delta q\gamma'_{\rho_\|}\rho_\|\dot{\psi}_p - \partial_{\rho_\|}H, \tag{6}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}_p} = Q\delta'_\theta q\gamma\rho_\|\dot{\theta} + Q\delta q\rho_\|\gamma'_\theta\dot{\theta} + Q\delta q\rho_\|\gamma'_{\rho_\|}\dot{\rho}_\| = q\dot{\theta} + Q\rho_\| I\gamma'_{\psi_p}\dot{\theta} \\ +(Q\rho_\|\gamma + \alpha)I'\dot{\theta} + I\alpha'_{\psi_p}\dot{\theta} \\ +Q\rho_\| g\gamma'_{\psi_p}\dot{\zeta} + Q\rho_\|\gamma g'\dot{\zeta} + (g\alpha'_{\psi_p} + \alpha g' - 1)\dot{\zeta} - \partial_{\psi_p}H,\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = q\dot{\psi}_p + Q\dot{\rho}_\|\gamma I + (Q\rho_\|\gamma + \alpha)I'\dot{\psi}_p + Q\rho_\| I(\gamma'_\theta\dot{\theta} + \gamma'_{\psi_p}\dot{\psi}_p + \gamma'_{\rho_\|}\dot{\rho}_\|) \\ +I(\alpha'_\theta\dot{\theta} + \alpha'_\zeta\dot{\zeta} + \alpha'_{\psi_p}\dot{\psi}_p + Q\partial_t\alpha) = Q\rho_\| I\gamma'_\theta\dot{\theta} + I\alpha'_\theta\dot{\theta} + Q\rho_\| g\gamma'_\theta\dot{\zeta} + g\alpha'_\theta\dot{\zeta} \\ +Q\delta'_\theta q\gamma\rho_\|\dot{\psi}_p + Q\delta q\gamma'_\theta\rho_\|\dot{\psi}_p - \partial_\theta H,\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{\zeta}} = -\dot{\psi}_p + Q\dot{\rho}_\|\gamma g + (Q\rho_\|\gamma + \alpha)g'\dot{\psi}_p + Q\rho_\| g(\gamma'_\theta\dot{\theta} + \gamma'_{\psi_p}\dot{\psi}_p + \gamma'_{\rho_\|}\dot{\rho}_\|) \\ +g(\alpha'_\theta\dot{\theta} + \alpha'_\zeta\dot{\zeta} + \alpha'_{\psi_p}\dot{\psi}_p + Q\partial_t\alpha) = I\alpha'_\zeta\dot{\theta} + g\alpha'_\zeta\dot{\zeta} - \partial_\zeta H,\end{aligned} \tag{9}$$

where we use the notation $f'_\beta = \partial_\beta f$.

Write these in matrix form

$$\begin{vmatrix} 0 & W & L & M \\ -W & 0 & F & C \\ -L & -F & 0 & K \\ -M & -C & -K & 0 \end{vmatrix} \begin{vmatrix} \dot{\rho}_\parallel \\ \dot{\psi}_p \\ \dot{\theta} \\ \dot{\zeta} \end{vmatrix} = \begin{vmatrix} \partial_{\rho_\parallel} H \\ \partial_{\psi_p} H \\ \partial_\theta H + I\partial_t \alpha \\ \partial_\zeta H + g\partial_t \alpha \end{vmatrix} \tag{10}$$

where $K = g\alpha'_\theta - I\alpha'_\zeta + Q\rho_\parallel g\gamma'_\theta$, $C = -1 + Q\rho_\parallel \gamma g' + \alpha g' + g\alpha'_{\psi_p} + Q\rho_\parallel g\gamma'_{\psi_p}$, $M = Qg\gamma + Qg\rho_\parallel \gamma'_{\rho_\parallel}$, $F = q + (Q\gamma\rho_\parallel + \alpha)I' + Q\rho_\parallel I\gamma'_{\psi_p} + I\alpha'_{\psi_p} + Q\delta'_\theta q\gamma\rho_\parallel + Q\delta q\gamma'_\theta \rho_\parallel$, $L = QI\gamma + QI\rho_\parallel \gamma'_{\rho_\parallel}$, $W = Q\delta q\gamma + Q\delta q\gamma'_{\rho_\parallel}\rho_\parallel$.

Inverting this equation gives

$$\begin{vmatrix} \dot{\rho}_\parallel \\ \dot{\psi}_p \\ \dot{\theta} \\ \dot{\zeta} \end{vmatrix} = \frac{1}{D} \begin{vmatrix} 0 & -K & C & -F \\ K & 0 & -M & L \\ -C & M & 0 & -W \\ F & -L & W & 0 \end{vmatrix} \begin{vmatrix} \partial_{\rho_\parallel} H \\ \partial_{\psi_p} H \\ \partial_\theta H + QI\partial_t \alpha \\ \partial_\zeta H + Qg\partial_t \alpha \end{vmatrix} \tag{11}$$

with denominator

$$D = WK + FM - CL. \tag{12}$$

Note that there are no derivatives of I with respect to θ.

Consider the resulting stepping equations with no field perturbation and an axisymmetric equilibrium, so $\partial_\zeta H = 0$ and $\alpha = 0$. The terms in δ appear only in F, W and D. The time derivatives $\dot{\rho}_\parallel$, $\dot{\psi}_p$, and $\dot{\theta}$ are given by terms independent of δ divided by D. Thus a renormalization of time using D leaves invariant the projection of the orbits in the poloidal plane, and thus this projection is independent of δ. Now consider mean toroidal precession

$$\Delta\zeta = \int \dot{\zeta} dt = \oint \frac{\dot{\zeta}}{\dot{\theta}} d\theta = \oint \frac{F\partial_{\rho_\parallel} H - L\partial_{\psi_p} H + W\partial_\theta H}{M\partial_{\psi_p} H - C\partial_{\rho_\parallel} H} d\theta \tag{13}$$

where the integral is taken following an orbit. The terms in δ are

$$\Delta\zeta_\delta = \oint \frac{Qq\rho_\parallel \partial_\theta(\gamma\delta)\partial_{\rho_\parallel} H + Qq\delta\partial_{\rho_\parallel}(\rho_\parallel \gamma)\partial_\theta H}{M\partial_{\psi_p} H - C\partial_{\rho_\parallel} H} d\theta \tag{14}$$

Use the identies

$$\partial_\theta H = \frac{\rho_\parallel \gamma B^2 \partial_\theta(\rho_\parallel \gamma) + (m\gamma^2 \rho_\parallel^2 B + \mu)\partial_\theta B}{\rho_\parallel \gamma^2 B^2} \partial_{\rho_\parallel} H, \tag{15}$$

and

$$\partial_{\psi_p} H = \frac{(m\gamma^2 \rho_\|^2 B + \mu)\partial_{\psi_p} B}{\rho_\| \gamma^2 B^2} \partial_{\rho_\|} H. \tag{16}$$

Following an orbit $\gamma\rho_\|$ and ψ_p are functions of θ, but to lowest order in ρ, ψ_p is constant, so to lowest order energy conservation following an orbit involves only the variation of $\gamma\rho_\|$ and B, giving $HdH = [m^2c^2B^2\rho_\|\gamma\partial_\theta(\rho_\|\gamma) + (m^2c^2\rho_\|^2\gamma^2 B + mc^2\mu)\partial_\theta B]d\theta = 0$, or

$$(m\gamma^2\rho_\|^2 B + \mu)\partial_\theta B = -m\gamma\rho_\|\partial_\theta(\rho_\|\gamma)B^2, \tag{17}$$

and keeping only first order in ρ in Eq 14 we find

$$\Delta\zeta_\delta = Q \oint \partial_\theta[q\rho_\|\gamma\delta]d\theta = 0. \tag{18}$$

Thus δ produces a nonsecular modification of the toroidal precession but no change in the poloidal projection of the orbit. Deleting δ results in a simple periodic modification of the toroidal position of the guiding center. We thus drop the term δ in the expression for the equilibrium. This leaves the Lagrangian in Hamiltonian form with canonical variables

$$P_\theta = \psi + (\rho_\|\gamma + \alpha)I, \qquad \theta, \tag{19}$$

$$P_\zeta = (\rho_\|\gamma + \alpha)g - \psi_p, \qquad \zeta. \tag{20}$$

Dropping δ produces an implicit change of the guiding center coordinate to produce canonical variables, leaving invariant both the poloidal projection of the orbit and the mean toroidal precession. There is no explicit redefinition of these coordinates in this process. Guiding center equations are only accurate to second order in ρ/R, including particle drifts but not higher order corrections. However it is important that this truncation in the expansion in orders of ρ/R maintain the Hamiltonian character of the equations, to guarantee the Liouville theorem and other properties.

We also have

$$\partial_\theta v^2 = 2\frac{\rho_\|^2 B + \mu/m\gamma^2}{1 + 2\mu B/mc^2}\partial_\theta B, \qquad \partial_\zeta v^2 = 2\frac{\rho_\|^2 B + \mu/m\gamma^2}{1 + 2\mu B/mc^2}\partial_\zeta B \tag{21}$$

$$\partial_{\psi_p} v^2 = 2\frac{\rho_\|^2 B + \mu/m\gamma^2}{1 + 2\mu B/mc^2}\partial_{\psi_p} B, \qquad \partial_{\rho_\|} v^2 = \frac{2\rho_\| B^2}{1 + 2\mu B/mc^2} \tag{22}$$

$$\partial_\beta \gamma^2 = \frac{\gamma^4}{c^2}\partial_\beta v^2, \quad \beta = \theta, \quad \zeta, \quad \rho_\|, \quad \psi_p. \tag{23}$$

$$H'_\beta = \frac{m}{2(1+v^2\gamma^2/c^2)^{1/2}}[v^2\partial_\beta\gamma^2 + \gamma^2\partial_\beta v^2] + \Phi'_\beta, \quad \beta = \theta, \quad \zeta, \quad \rho_\parallel, \quad \psi_p. \tag{24}$$

The resulting guiding center equations are a straightforward modification of the classical case derived previously[9] and used in the code Orbit. It is convenient to use energy $E = H - \Phi - mc^2$ so that it agrees with the kinetic energy in the nonrelativistic limit. For initial particle values, if given position, energy and pitch $\lambda = v_\parallel/v$, one uses Eq. 3 to find v, then $\rho_\parallel = \lambda v/B$, and then $\mu B = mv^2 - m\rho_\parallel^2 B^2\gamma^2$ to define μ. Given H and μ one uses Eq. 3 to find v and γ and then this last equation to find $\rho_\parallel$.

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