Princeton Plasma Physics Laboratory

PPPL- 5052

PPPL- 5052

Fusion Utility in the Knudsen Layer

Seth Davidovits, and Nathaniel J. Fisch

AUGUST 2014

Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

Princeton Plasma Physics Laboratory
Report Disclaimers

1 August 2014

Fusion Utility in the Knudsen Layer

Seth Davidovits[1] and Nathaniel J. Fisch[1]
Princeton University, Princeton Plasma Physics Laboratory, Princeton, New Jersey 08544, USA

In inertial confinement fusion, the loss of fast ions from the edge of the fusing hot-spot region reduces the reactivity below its Maxwellian value. The loss of fast ions may be pronounced because of the long mean free paths of fast ions, compared to those of thermal ions. We introduce a fusion utility function to demonstrate essential features of this Knudsen layer effect, in both magnetized and unmagnetized cases. The fusion utility concept is also used to evaluate restoring the reactivity in the Knudsen layer by manipulating fast ions in phase space using waves.

I. INTRODUCTION

Knudsen layer reduction of fusion reactivity occurs when some fusion fuel ions stream out of the fusing region of the capsule before having a chance to fuse. The fast fuel ions, which typically have the best chances of fusing, also have very long mean free paths compared to thermal ions. Thus, even when the bulk of the plasma is collisionally confined over the time of the fusion burn, the highly effective fast particles may be lost before fusing, substantially decreasing the fusion reactivity.

The possibility for Knudsen layer reduction of the fusion reactivity was initially explored by Henderson[1] and Petschek and Henderson[2]. Molvig et al. formulated an asymptotic, steady state theory of the effect, and applied it in radiation-hydrodynamic simulations of OMEGA implosions[3]. They found that including the Knudsen layer model significantly improved agreement in calculated D-T fusion neutron yield between the simulations and experiments. The treatment of the boundary in this model was subsequently improved by Albright et al.[4]. This past work found the effect to be pronounced only in capsules with small fuel ρr. Tang et al. considered larger capsules that may still have Knudsen losses due to hydrodynamic mix, and the recovery of some portion of the losses due to lost fast ions fusing in the surrounding cold fuel[5]. Tang et al.[6,7] and McDevitt et al.[8] studied a hierarchy of reduced Fokker-Planck operators to capture the essentials of the Knudsen layer effect and to compute the tail distributions at hot-cold plasma interfaces.

In the case where the implosion is magnetized, the picture of fast ion trajectories changes, since ions are limited in traveling in the direction perpendicular to the magnetic field. This fundamentally changes the length and velocity scalings. Schmit et al. considered the effect of magnetization on the Knudsen layer reduction of fusion reactivity, giving heuristic conditions for the reactivity to be largely restored, and showing the applicability of these conditions by numerically generating the steady state fast ion distribution function in cylindrical and spherical magnetized geometries[9].

The present work describes the Knudsen layer phenomenon in a way that is complementary to previous work. By identifying a fusion utility function for fast ions, we address the question, "How much fusion energy is an ion starting at position $\boldsymbol{x}_0$ with velocity $\boldsymbol{v}_0$ expected to produce over its lifetime?" After deriving the fusion utility function we show how it can be used to consider both the unmagnetized and magnetized Knudsen layer problems, while allowing for spatial density dependence.

In the unmagnetized case we find that it is the *total* density between a fast ion and the boundary that determines its lifetime fusion utility, not the absolute distance to the boundary. The fusion utility increases as the amount of density to the boundary increases until the fast ion is far from the boundary. For the magnetized case, the fusion utility is independent of the density.

The fusion utility function is a particularly powerful construct for evaluating incremental effects. For example, waves can be used to locally change the fast ion distribution function, both by increasing ion energy and by pushing ions away from the Knudsen layer. The utility function gives the change in lifetime fusion energy production that occurs on moving an ion from $\boldsymbol{x}_0, \boldsymbol{v}_0$ to $\boldsymbol{x}_1, \boldsymbol{v}_1$, thereby giving the effect on fusion energy production of such waves.

The utility function approach, in general, has been useful in considering incremental or differential effects of external perturbations in plasma. It has been particularly useful in resonant rf interactions with plasma, particularly in the case of wave-driven electrical current[10]. The rf waves diffuse particles along well-constrained diffusion paths, so that, essentially, the rf removes particles from one phase space location, and inserts those particles in an adjacent phase space location, with the phase space residing in the 6D space of velocity and position. By associating with each point in phase space a utility, the differential utility, as well as the energy cost, can be calculated under any rearrangement of the phase space by wave excitation. Thus, the current-carrying utility of a superthermal electron at an initial position in the 6D phase space may be used to calculate the current drive efficiency[11]. Similarly, a runaway probability can be associated with each initial position of an electron in the 6D phase space[12]. In both cases, the differential effect relates the rf power dissipation to either the generation of current or the production of runaways.

Here the fusion utility function gives the total extra fusion energy at the cost of moving the ion in phase space. The utility function may thus be used to answer whether it is useful to expend rf power to move particles away from the boundary, say if the rf power is simply applied from an external antenna. If, however, the power were supplied from tapping the alpha particle energy, say through an instability driven by the alpha particles themselves, then there would be a number of added benefits. Among the added benefits, for example, would be to avoid direct electron heating, thereby obtaining a hot ion mode, where the ion temperature exceeds the electron temperature[13]. Another benefit is that, if the rf wave is generated by the alpha particles, then the alpha particles may be transported toward the boundary. The present analysis considers only the direct utility of extra fusion energy, rather than these added benefits which depend on whether the rf power is internally generated or externally supplied. It is also beyond the scope of this paper to propose specific waves that might be destabilized by the alpha particles specifically near the Knudsen layer boundary.

The paper is organized as follows. In Sec. II, we describe the basic idea of the utility function and the scheme for mitigating Knudsen layer fusion reactivity losses. Next, Sec. III describes the fusion utility function more formally. Section IV shows example calculations of an unmagnetized utility function and a magnetized utility function. In Sec. V we find the theoretical fusion energy production gains from phase space manipulation in the Knudsen layer. Finally, Sec. VI discusses caveats for the reactivity restoration scheme, and possible improvements and generalizations of our work.

II. UTILITY FUNCTION

Consider tracking a fast ion moving through a plasma as it pitch angle scatters and slows down due to drag. The quantity of interest is the total expected fusion energy generated by the ion over its lifetime in the plasma. This lifetime is defined by following the fast ion until it slows down to thermal speed (at which point its chance for fusion is negligible) or until it leaves the plasma by exiting at a boundary. The boundary might be unreacting liner surrounding the ICF implosion hotspot.

We write the expected fusion energy generated by the fast ion over its lifetime as $\mathbb{E}\left(\boldsymbol{x}_0, \boldsymbol{v}_0\right)$, where $\boldsymbol{x}_0$ and $\boldsymbol{v}_0$ are the fast ion's initial position and velocity. In the limit where the fast ion starts very far (in mean free paths) from the boundary, the chance it leaves the plasma before slowing down to thermal speed is negligible, and the expected lifetime fusion energy will tend to depend only on the initial velocity, $\mathbb{E}\left(\boldsymbol{x}_0, \boldsymbol{v}_0\right) \rightarrow \mathbb{E}_0(\boldsymbol{v}_0)$. As $\boldsymbol{x}_0$ gets closer and closer to the boundary, the fast ion is more and more likely to leave the plasma before slowing down completely, decreasing the expected fusion yield. This region of decreased yield coincides with the Knudsen layer.

It is possible in some circumstances to use plasma waves to change the velocity and position of particles, for example, in alpha channeling in tokamaks[14]. If a fast ion that starts near to the boundary is pushed in position away from the boundary by Δx while being heated in energy by $\Delta\epsilon$, the expected lifetime fusion energy yield (the utility) $\mathbb{E}$ will increase,

$$\mathbb{E} \rightarrow \mathbb{E} + \frac{\partial \mathbb{E}}{\partial \boldsymbol{x}} \cdot \Delta \boldsymbol{x} + \frac{\partial \mathbb{E}}{\partial \epsilon} \Delta\epsilon. \tag{1}$$

The gain, g, from such pushing will be the incremental fusion energy produced divided by the energy required to do such pushing, represented by the change in the fast ion's energy in the push, $\Delta\epsilon$

$$g = \left(\frac{\partial \mathbb{E}}{\partial \boldsymbol{x}} \cdot \Delta \boldsymbol{x} + \frac{\partial \mathbb{E}}{\partial \epsilon} \Delta\epsilon\right) / \Delta\epsilon. \tag{2}$$

In certain cases[14], the spatial push Δx can be proportional to the energy push $\Delta\epsilon$ - a larger push in energy yields a larger spatial push. Moreover, the direction of this push can be arranged through the wave polarizations. In regions where the fusion yield is lost most rapidly $|\partial \mathbb{E}/\partial \boldsymbol{x}|$ is large, so that a small spatial push can give significant gains. Thus, the regions of rapid yield loss are also those where yield can be regained with lowest energy cost. Indeed, we will show that in some circumstances, the gain may be high enough to consider such a mitigation strategy for counteracting Knudsen layer losses of fusion yield.

III. APPROACH

To write a fusion utility function as described in Sec. II, consider a function $g(\boldsymbol{x}, \boldsymbol{v}, t; \boldsymbol{x}', \boldsymbol{v}', t')$ that gives the probability an ion initialized at phase space point $\boldsymbol{x}', \boldsymbol{v}'$ at time t' is found later at time t at phase space point $\boldsymbol{x}, \boldsymbol{v}$. The instantaneous expected fusion production at time t for this ion is

$$E(t; \boldsymbol{x}', \boldsymbol{v}', t') = \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{v}\, g(\boldsymbol{x}, \boldsymbol{v}, t; \boldsymbol{x}', \boldsymbol{v}', t')\, W(\boldsymbol{x}, \boldsymbol{v}) \tag{3}$$

where $W\left(\boldsymbol{x}, \boldsymbol{v}\right)$ is the fusion energy production rate for a fast ion located at point $\boldsymbol{x}, \boldsymbol{v}$. Here W is taken as,

$$W(\boldsymbol{x}, \boldsymbol{v}) = E_f \left\langle \sigma v \right\rangle \tag{4}$$

the Maxwellian averaged fusion reactivity multiplied by the energy from fusion, E_f, with σ the velocity dependent fusion cross-section. The function W may depend on position through its dependence on the density. One could generalize this to account for multiple species reacting, but for simplicity we consider the fast ion reacting with only one species here. The integrals in Eq. (3) are carried out over the domain of g in the unprimed variables - anywhere the ion may exist at time t. An integration of g in the unprimed variables may give a total probability less than 1 if the ion can be lost, say through a boundary.

The fusion utility is the integral of Eq. (3) over all time,

$$\mathbb{E}_{t'}(\boldsymbol{x}', \boldsymbol{v}') = \int_{t'}^{\infty} \mathrm{d}t\, E(t; \boldsymbol{x}', \boldsymbol{v}', t') \,. \tag{5}$$

This gives a more precise definition of the utility function appearing in Eqs. (1),(2). Finding the function g and then integrating it to find $\mathbb{E}$ is the Langevin approach to finding the fusion utility.

There is another approach, based on an adjoint formalism. This approach gives a more direct way of solving for the utility, and shows that the utility is the function that connects forcings on the distribution function to changes in fusion energy production. The adjoint approach is outlined here; it is discussed more completely elsewhere[15].

Consider the general kinetic equation for a single plasma species, with a collision operator that includes all relevant collisions (e.g. electron-ion, electron-electron, for a two species plasma),

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{F}\cdot\frac{\partial f}{\partial \boldsymbol{v}} - \mathrm{C}\,[f] = -\frac{\partial}{\partial \boldsymbol{v}}\cdot\boldsymbol{\Gamma}_v - \frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{\Gamma}_x. \tag{6}$$

Here $\boldsymbol{\Gamma}_{x,v}$ are wave induced fluxes in space and velocity respectively, which will be useful when we consider wave manipulation of ions.

Expand the distribution function f as

$$f = f_M\,(1+\chi) \tag{7}$$

with χ assumed to be a small correction, induced in our case by an external perturbation such as waves or boundary effects. Plugging Eq. (7) into Eq. (6), assuming that the background non-drifting Maxwellian quantities (density, temperature) have no time dependence, and linearizing the collision operator, yields

$$\frac{\partial}{\partial t}(f_M\chi) + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}(f_M\chi) + \boldsymbol{F}\cdot\frac{\partial}{\partial \boldsymbol{v}}(f_M\chi) - \hat{\mathrm{C}}\,(\chi) = - \boldsymbol{v}\cdot\frac{\partial f_M}{\partial \boldsymbol{x}} - \boldsymbol{F}\cdot\frac{\partial f_M}{\partial \boldsymbol{v}} - \frac{\partial}{\partial \boldsymbol{v}}\cdot\boldsymbol{\Gamma}_v - \frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{\Gamma}_x. \tag{8}$$

Equation 8 has the form

$$\hat{\mathrm{L}}\chi = s \tag{9}$$

where $\hat{\mathrm{L}}$ is a linear operator and s is a source term.

In the present application, our interest is in the change in fusion energy production in the plasma when it is perturbed from Maxwellian, not the full solution of Eq. (8). That is to say, the relevant quantity is a *moment* of the distribution function χ, rather than χ itself. In this case, it is natural to use an adjoint formulation, which allows us to write an equation for a general moment M of the distribution,

$$\mathrm{M}(t) = \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{v}\, W(\boldsymbol{x},\boldsymbol{v})\, f_M \chi(\boldsymbol{x},\boldsymbol{v},t)\,. \tag{10}$$

For purposes of calculating the fusion energy production W is given by Eq. (4).

The Green's function solution to an equation of the form of Eq. (9) is

$$\chi(\boldsymbol{x},\boldsymbol{v},t) = \int \mathrm{d}t' \int \mathrm{d}\boldsymbol{x}' \int \mathrm{d}\boldsymbol{v}'\, g(\boldsymbol{x},\boldsymbol{v},t;\boldsymbol{x}',\boldsymbol{v}',t')\, s(\boldsymbol{x}',\boldsymbol{v}',t') - \int \mathrm{d}t' \int \mathrm{d}\boldsymbol{\sigma}'\cdot\mathbb{J}[\chi(\boldsymbol{x}',\boldsymbol{v}',t')\,, g(\boldsymbol{x},\boldsymbol{v},t;\boldsymbol{x}',\boldsymbol{v}',t')] \tag{11}$$

with the Green's function g solving

$$\hat{\mathrm{L}}g = \delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(\boldsymbol{v}-\boldsymbol{v}')\,\delta(t-t')\,. \tag{12}$$

Integrals in $\boldsymbol{x}', \boldsymbol{v}'$ are carried out over the interior of a general, possibly bounded domain, while the integral in $\boldsymbol{\sigma}'$ is over the x', v' domain boundary. Time integrals are carried out over an appropriate time domain (e.g. $[0,\infty]$). The operator $\mathbb{J}$ is defined through the relation

$$\int \mathrm{d}t \Big(\Big\{\psi, \hat{\mathrm{L}}\chi\Big\} - \Big\{\chi, \hat{\mathrm{L}}^{\dagger}\psi\Big\} \Big) = \int \mathrm{d}t \int \mathrm{d}\boldsymbol{\sigma}\cdot\mathbb{J}[\chi,\psi]\,. \tag{13}$$

This relation also serves to define the adjoint operator $\hat{\mathrm{L}}^{\dagger}$. The inner product is defined as

$$\Big\{\psi, \hat{\mathrm{L}}\chi\Big\} = \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{v}\, \psi \hat{\mathrm{L}}\chi. \tag{14}$$

Given an operator $\hat{\mathrm{L}}$, one uses Eq. (13) to find $\hat{\mathrm{L}}^{\dagger}$ and $\mathbb{J}$. Substituting χ from Eq. (11) into M, Eq. (10), it is possible to write an equation for M and the moment of χ over the domain boundary. Carrying out this procedure for the current $\hat{\mathrm{L}}$, and specializing for homogeneous boundary conditions on ψ gives an equation for M,

$$\int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{v}\, W(\boldsymbol{x},\boldsymbol{v})\, f_M \chi(\boldsymbol{x},\boldsymbol{v},t) = \int \mathrm{d}t' \int \mathrm{d}\boldsymbol{x}' \int \mathrm{d}\boldsymbol{v}' \left(\frac{\partial \psi}{\partial \boldsymbol{x}'}\cdot\boldsymbol{S}_x + \frac{\partial \psi}{\partial \boldsymbol{v}'}\cdot\boldsymbol{S}_v \right) \tag{15}$$

with the fluxes

$$\boldsymbol{S}_x = \boldsymbol{v}' f_M + \boldsymbol{\Gamma}_x, \tag{16a}$$

$$\boldsymbol{S}_v = \boldsymbol{F} f_M + \boldsymbol{\Gamma}_v. \tag{16b}$$

With W given by Eq. (4), Eq. (15) gives the volume averaged fusion energy production of the perturbed distribution χ.

The function ψ is defined as

$$\psi(t;\boldsymbol{x}',\boldsymbol{v}',t') = \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{v}\, g(\boldsymbol{x},\boldsymbol{v},t;\boldsymbol{x}',\boldsymbol{v}',t')\, W(\boldsymbol{x},\boldsymbol{v})\,. \tag{17}$$

It obeys the adjoint equation

$$\hat{\mathrm{L}}^{\dagger}\psi = -f_M\frac{\partial \psi}{\partial t'} - f_M\boldsymbol{v}'\cdot\frac{\partial \psi}{\partial \boldsymbol{x}'} - f_M\boldsymbol{F}\cdot\frac{\partial \psi}{\partial \boldsymbol{v}'} - \hat{\mathrm{C}}[\psi] = 0 \tag{18}$$

with an initial condition given by W, and homogeneous boundary conditions for the present work. The function ψ in Eq. (17) is the same as the fusion energy production rate, Eq. (3).

In order to get the utility, $\mathbb{E}$, ψ must be integrated in t,

$$\mathbb{E}_{t'}(\boldsymbol{x}', \boldsymbol{v}') = \int_{t'}^{\infty} \mathrm{d}t \psi(t; \boldsymbol{x}', \boldsymbol{v}', t') \,. \tag{19}$$

Thus, in the case that the ion obeys a linear equation, the utility can be found by integrating in time the solution to the adjoint equation with initial condition W. Furthermore, the function g becomes the Green's function for the kinetic equation.

In the applications considered here ψ will only depend on the time difference $t - t'$, and an integration in t corresponds to integrating the adjoint equation in t' from $-\infty$ to an initial condition at t.

Since ψ is the fusion production rate, Eq. (15) shows that the instantaneous fusion energy production of the perturbation χ can be written in terms of fluxes and this fusion production rate. The same relationship will hold after time integration; the total fusion energy production of the perturbation can be written in terms of fluxes and the utility.

IV. EXAMPLE UTILITY CALCULATIONS

We make a number of simplifying assumptions in our example calculations for both the unmagnetized and magnetized cases, but the adjoint formulation is also applicable to more complicated scenarios. The unmagnetized case contains no magnetic fields, while the magnetized case has a constant z directed field. We assume that the plasma region of interest has no electric fields, and allow spatial dependence in only one direction, the z direction in the unmagnetized case, and the x direction in the magnetized case. This dependence is on the half line $z, x \in [0, \infty]$, with an absorbing boundary at $z, x = 0$, so that we can isolate the effects of the boundary.

Additionally, we use the high velocity limit of the collision operator. In this limit, the fast ion only undergoes velocity drag and pitch angle scattering, with frequencies ν_E and ν_μ, respectively,

$$\hat{C}[\phi] = \frac{1}{2}\nu_\mu \frac{V_T^3}{v'^3}\frac{\partial}{\partial \mu'}\left(1-\mu'^2\right)\frac{\partial \phi}{\partial \mu'} + \nu_E \frac{V_T^3}{v'^2}\frac{\partial \phi}{\partial v'}. \tag{20}$$

The lack of dependence on the thermal velocity is made clear by writing

$$V_T^3 \nu_E = C_E n(z'), \tag{21a}$$

$$V_T^3 \nu_\mu = C_\mu n(z'), \tag{21b}$$

where C_E and C_μ are constants independent of the thermal velocity and density (and any coordinates), and $n(z)$ is the density, which we allow to vary in the z direction. While the collisional dynamics are somewhat more complicated for the magnetized case, they can still be expressed in terms of these frequencies. In factoring out the density we assume that collision partner species all have the same functional form of dependence in z, although they need not appear in equal amounts. In other words, consider

$$V_T^3 \nu_\mu = \frac{e^4 Z_a^2 \ln(\Lambda)}{4\pi m_a^2 \epsilon_0^2}\sum_b Z_b^2 n_b(z') \tag{22}$$

where we have ignored dependence inside the Coulomb logarithm, $Z_{a,b}$ are the charge numbers of the fast ion and collision partner species, m_a is the fast ion mass, and ϵ_0 is the permittivity of free space. Each $n_b(z')$ is assumed to have the same functional form, $n_b(z') = n_{b0} n(z')$, so that the functional dependence can be factored out of the sum,

$$\sum_b Z_b^2 n_b(z') = n(z') \sum_b Z_b^2 n_{b0}. \tag{23}$$

The coefficients n_{b0} are dimensionless. Thus C_μ is defined by Eqs. (21b), (22), and (23). The same process gives C_E,

$$C_E = \frac{e^4 Z_a^2 \ln(\Lambda)}{4\pi m_a^2 \epsilon_0^2}\sum_b Z_b^2 \frac{m_a}{m_b} n_{b0}. \tag{24}$$

As previously mentioned, in the unmagnetized case, we allow z dependence, for the magnetized case, the dependence is in the x direction, and so z should be replaced with x in the preceding expressions.

A. Unmagnetized utility

When the pitch angle scattering frequency is much larger than the rate at which the fast ion slows down, the particle motion is diffusive on scales longer than the mean free path. This is shown formally through an expansion and averaging in μ, see, for example Melrose[16], or Albright[4]. Such an approximation would be most valid for, say, protons in proton-boron fusion where the high Z and mass of boron make pitch angle scattering occur significantly faster than slowing down for protons. Its application to D-T fusion has also been discussed[3,4]. The presence of impurities in the plasma can also make the approximation more valid. To derive the fusion utility in this limit, the kinetic equation

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{C_\mu n(z)}{2v^3}\frac{\partial}{\partial \mu}\left(1-\mu^2\right)\frac{\partial f}{\partial \mu} + \frac{C_E n(z)}{v^2}\frac{\partial f}{\partial v} \tag{25}$$

is rewritten in terms of the variable Z,

$$Z(z) = \int_0^z n(\hat{z})\,\mathrm{d}\hat{z}. \tag{26}$$

Performing the expansion and averaging in μ gives the diffusive kinetic equation

$$\frac{1}{C_E n(Z)}\frac{\partial f}{\partial t} = \frac{v^5}{3 C_E C_\mu}\frac{\partial^2 f}{\partial Z^2} + \frac{1}{v^2}\frac{\partial f}{\partial v}. \tag{27}$$

The adjoint equation, as defined by Eq. 13, rewritten in primed variables, is

$$-\frac{1}{C_E n(Z')}\frac{\partial \psi}{\partial t'} = \frac{v'^5}{3C_E C_\mu}\frac{\partial^2 \psi}{\partial Z'^2} - \frac{1}{v'^2}\frac{\partial \psi}{\partial v'}. \tag{28}$$

To get the utility $\mathbb{E}$, Eq. (19), integrate this equation in t' from $-\infty$ to an initial condition of W at t, per the discussion at the end of Sec. III,

$$\frac{C_E}{v'^2}\frac{\partial \mathbb{E}}{\partial v'} = \frac{1}{3}\frac{v'^5}{C_\mu}\frac{\partial^2 \mathbb{E}}{\partial Z'^2} + \frac{W(Z', v')}{n(Z')}. \tag{29}$$

The dependence of W on Z is only through the density. To see this, consider, for simplicity,

$$W(Z, v) = E_f n_{f0} n(Z)\, v\sigma(v) \tag{30}$$

which approximates the collision partners in the Maxwellian average to all be stationary with respect to the fast ion. The density of collision scattering centers is $n_f = n_{f0}n(Z)$, typically representing a single species from the sum in Eq. (22). Then the solution of Eq. (29) on the domain $z \in [0, \infty]$ is

$$\mathbb{E}(\bar{v}', \bar{z}') = \mathbb{E}_0 \int_0^{\bar{v}'} \mathrm{d}\bar{v}\bar{\sigma}(\bar{v})\,\bar{v}^3 \mathrm{Erf}\left(\frac{\sqrt{6}}{\sqrt{\bar{v}'^8 - \bar{v}^8}}\int_0^{\bar{z}'} \bar{n}(\bar{z})\,\mathrm{d}\bar{z}\right) \tag{31}$$

in variables where the velocity is normalized to the velocity at the peak of the fusion cross-section, the density is normalized to a mean density, the cross section to the peak cross section, and distance is normalized to a hybrid mean free path that naturally appears,

$$\bar{v} = \frac{v}{v_G}, \tag{32a}$$

$$\bar{n} = \frac{n}{n_0}\ ,\ \bar{\sigma} = \frac{\sigma}{\sigma_0}, \tag{32b}$$

$$\bar{z} = \frac{z}{\lambda_*} = \frac{z}{v_G/\sqrt{\nu_\mu^G \nu_E^G}}. \tag{32c}$$

The constants are grouped into $\mathbb{E}_0 = E_f v_G^4 \sigma_0 n_{f0}/C_E$, which has units of energy. The superscript G on the collision frequencies indicates the collision frequency at velocity v_G and mean density n_0. Note that n_{f0} will also appear in the sum in C_E Eq. (24), so that $\mathbb{E}_0$ does not scale linearly with it. The contours of the fusion utility $\mathbb{E}$, as given in Eq. (31), normalized to E_f are shown in Fig. (1). Normalizing to E_f effectively turns the utility into the lifetime probability of the particle fusing, assuming no removal of the particle after a fusion event. The limit $Z \to \infty$ (equivalently $z \to \infty$ for non-vanishing density profiles) gives the utility in the absence of a boundary. Horizontal contours in Fig. 1 indicate the region where fast ions are fully utilized; moving the ion spatially in a region of horizontal contours has no effect on the utility. In other words, the ion does not feel the boundary. Figure 1 shows that as z approaches the boundary, there

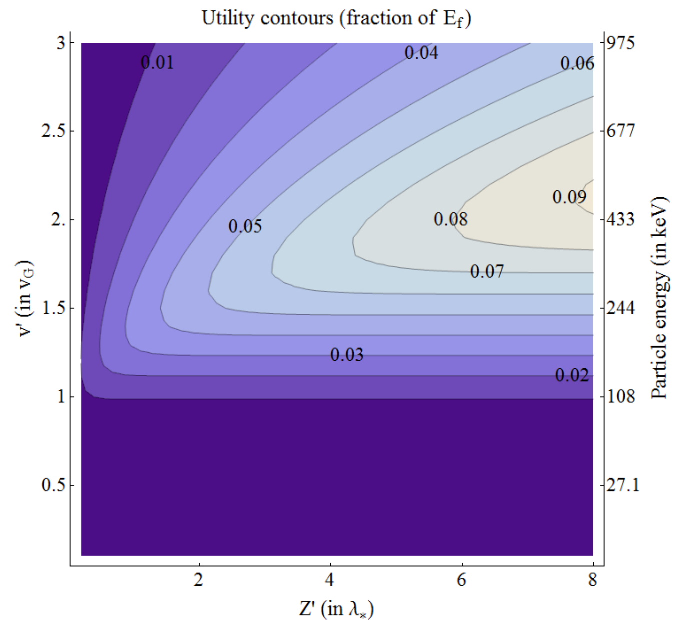

FIG. 1. Normalized utility function $\mathbb{E}/E_f$ in the unmagnetized case for a 50/50 D-T plasma. The normalized utility gives the lifetime probability of fusing for a particle starting with velocity v' at position Z'. The expected lifetime energy production is the contour value times the 17.6 MeV fusion energy release. The absorbing boundary is located at $Z' = 0$. The right axis gives the particle energy corresponding to v', assuming the particle is a deuteron. The utility values shown are for a deuteron. For a triton, the plot is identical except that both the normalized utility values and right axis energy values are higher by a factor of 3/2.

is increasing underutilization of fast ions that is characteristic of Knudsen layer effects. The utility in Fig. 1 is plotted using the D-T cross section[17] for $\sigma(v)$ and assuming a 50/50 D-T plasma, but would have a similar structure for any cross section in these normalized units, assuming it is qualitatively similar in being peaked, and that the background plasma temperature is well below the peak so that the scattering centers can be approximated as stationary. Using a fusion reaction other than D-T would scale the utility by affecting various factors in $\mathbb{E}_0$, especially E_f and σ_0. Subsequent figures also use the D-T cross section and a 50/50 D-T plasma. The fact the plasma is 50/50 D-T, instead of some other ratio, affects the distance scale through the collision frequencies in the hybrid mean free path, Eq. (32c), and the utility scaling, $\mathbb{E}_0$, through n_{f0} and C_E, but not the overall structure of the utility contours. The density dependence of Eq. (31) makes clear an earlier assertion – that it is the *total* density between the fast ion's starting position and the edge that matters in determining its utility, not the absolute distance. Furthermore, as a result of the error function, there are initially large gains in utility for adding density between a particle and the edge, which then quickly become diminishing.

Figure 1 shows, for example, that a particle at $v' = 1$

(deuteron energy of 108 keV) has a normalized utility of 0.01, so that it is expected to produce 176 keV of energy in its lifetime as a fast particle. On the other hand, a particle that is $\sim$ 300 keV hotter, and it is located at least 8 λ_* from the boundary, has a normalized utility of 0.09, so that it produces nine times the energy, 1.58 MeV, while having approximately only four times the energy. Far from a boundary, the ratio of utility to particle energy increases rapidly up to a peak around $v' = 2$ (deuteron energy of 433 keV) and then falls off gradually. Note that, infinitely far from a boundary, the utility itself will be a strictly increasing function of velocity, which can be seen from Eq. (31), since the cross section is positive and the error function evaluates to one.

B. Magnetized utility

To treat the magnetized case we use a guiding center Fokker-Planck collision operator, specializing for simplicity to a uniform magnetic field in the z direction[18,19]. In order for the fast ions to be treated by this collision operator, their cyclotron frequency ω_c must be greater than the collision frequency for fast particles, $\omega_c \gg \nu_\mu^G$. The length scale of the density variation allowed must also be larger than the gyroradius scale.

$$\frac{\partial f}{\partial t} = -v\mu\frac{\partial f}{\partial z} + \frac{C_\mu n(x)}{2v^3}\frac{\partial}{\partial \mu}\left(1-\mu^2\right)\frac{\partial f}{\partial \mu} + \frac{\mu^2}{v}\frac{C_\mu}{4\omega_c^2}\frac{\partial^2}{\partial x^2}\left[n(x)\, f\right] + \frac{C_E n(x)}{v^2}\frac{\partial f}{\partial v}. \quad (33)$$

The first and second terms after the equal sign are precisely those that lead to diffusive transport in the unmagnetized case, and will lead to similar transport along the magnetic field in this case. To isolate the cross field effects, we ignore the z dynamics. We also average over a Maxwellian in pitch angle. This assumption of uniformity in pitch angle breaks down when near a boundary. After dropping terms and averaging out μ, the kinetic equation is

$$\frac{\partial f}{\partial t} = \frac{C_\mu}{6v\omega_c^2}\frac{\partial^2}{\partial x^2}\left(n(x)\, f\right) + \frac{C_E n(x)}{v^2}\frac{\partial f}{\partial v} \quad (34)$$

for which the adjoint must be found and then integrated in time, as in Sec. IV A. The adjoint equation is

$$-\frac{\partial \psi}{\partial t'} = \frac{C_\mu n(x')}{6v'\omega_c^2}\frac{\partial^2 \psi}{\partial x'^2} - \frac{C_E n(x')}{v'^2}\frac{\partial \psi}{\partial v'}. \quad (35)$$

Integrating in time gives the lifetime utility equation

$$\frac{C_E}{v'^2}\frac{\partial \mathbb{E}}{\partial v'} = \frac{C_\mu}{6v'\omega_c^2}\frac{\partial^2 \mathbb{E}}{\partial x'^2} + \frac{W(x', v')}{n(x')}. \quad (36)$$

Solving this equation gives the lifetime utility function

$$\mathbb{E}(\bar{v}', \bar{z}') = \mathbb{E}_0 \int_0^{\bar{v}'} \mathrm{d}\bar{v}\bar{\sigma}(\bar{v})\, \bar{v}^3 \mathrm{Erf}\left(\frac{\sqrt{3}\bar{x}'}{\sqrt{\bar{v}'^2 - \bar{v}^2}}\right). \quad (37)$$

In this case the distance coordinate is normalized to a modified fast particle gyro-radius,

$$\bar{x} = x/\rho_* \quad (38)$$

where

$$\rho_* = \sqrt{R}\left(\frac{v_G}{\omega_c}\right) \quad (39)$$

and R is the ratio of collision frequencies, $R = C_\mu/C_E$. Figure 2 shows this utility function on similar (normalized) axes as Fig. 1 for the unmagnetized case. As might be expected, given the effects included, we can see the weaker penetration of the Knudsen layer effect, as well as the altered scaling of utility with increased velocity. In the unmagnetized case, fast ion utility decreases beyond a certain velocity, due to much higher edge loss probability outcompeting gains in fusion production. In the magnetized case, this is no longer true, and the utility increases with increasing velocity, albeit at a much slower rate near the edge than would occur with no boundary. Note that the z scales in the magnetized and unmagnetized cases are very different. The observation of this decreased Knudsen penetration with magnetization is consistent with the work of Schmit[9]. With the present approximations, the magnetized utility is independent of density. The much higher maximum normalized utility values in the magnetized case compared to the unmagnetzied one result from full fast ion utilization in the highest velocity phase space region shown. In other words, at 3 ρ_* and $v' \sim 2.75$, the normalized utility of 0.15 in the magnetized case is the same value as would be achieved in the absence of a boundary. The unmagnetized case will reach this same normalized utility, at the same velocity (as it must), at a distance much greater than the maximum distance shown in Fig. 1.

V. INCREMENTAL UTILITY CALCULATION AND REACTIVITY LOSS MITIGATION

Figures 1 and 2 show that the energy produced by a fast ion in the edge region can be increased substantially by moving the ion away from the boundary. For example, Fig. 1 shows that moving an ion with velocity $v' = 1.5$ by one λ_* away from the boundary, from $Z' \sim 1$ to $Z' \sim 2$, increases the expected fusion probability by more than 50% (from ∼0.03 to ∼0.05), and therefore also the expected energy production. All the while, the particle energy remains the same. While the numbers are different for the magnetized case, the effect is clear. In the magnetized case, the magnetic field can link wave pushes in energy to pushes in space, making it possible to move hot ions away from the boundary and increase their utility. Here we calculate the fusion gains possible from such pushing.

Using the magnetized fusion utility, Eq. (37), and simplifying the moment Eq. (15) under the same set of assumptions, we can write the expected change in fusion

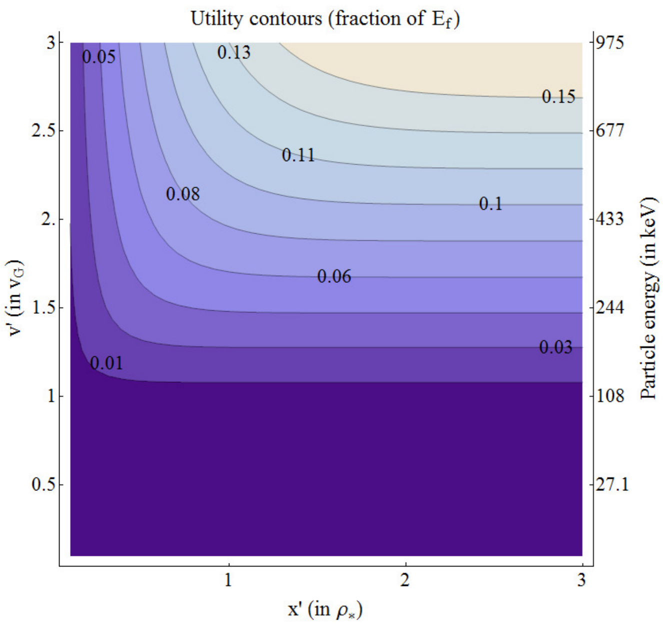

FIG. 2. Same as Fig. 1, the normalized utility function $\mathbb{E}/E_f$, but in the magnetized case. The horizontal axis scale is now ρ_* instead of λ_*.

power production as a result of wave induced fluxes in space and velocity,

$$\mathcal{E} = \int \mathrm{d}\boldsymbol{v} \int \mathrm{d}\boldsymbol{x}\, W(\boldsymbol{v},\boldsymbol{x})\, f_M \chi(\boldsymbol{v},\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{v}' \int \mathrm{d}x' \frac{\partial \mathbb{E}}{\partial x'}\Gamma_x + \frac{\partial \mathbb{E}}{\partial \boldsymbol{v}'}\cdot \Gamma_v. \quad (40)$$

To isolate the impact of the waves in phase space, consider localized fluxes

$$\boldsymbol{\Gamma}_{\boldsymbol{x},\boldsymbol{v}} = \boldsymbol{\Gamma}_{x0,v0}\delta(x'-x_0)\,\delta(\boldsymbol{v}'-\boldsymbol{v}_0)\,. \quad (41)$$

If pushing the fast ion to a new point in phase space produces more net fusion energy over its lifetime than energy required to push, there will be a net gain in energy. This gain is defined by

$$g\left(x_0,\boldsymbol{v}_0\right) = \frac{\mathcal{E}}{\epsilon_\Delta} \quad (42)$$

where ϵ_Δ is the energy absorbed by the ion during the push,

$$\epsilon_\Delta = \int \mathrm{d}x' \int \mathrm{d}v' \boldsymbol{\Gamma}_{v0}\cdot \frac{\mathrm{d}\left(mv'^2/2\right)}{\mathrm{d}\boldsymbol{v}'}. \quad (43)$$

The gain for the magnetized case is then

$$g(z_0,v_0) = \left(\frac{\partial \mathbb{E}}{\partial x_0}\Gamma_{x0} + \frac{\partial \mathbb{E}}{\partial v_0}\Gamma_{v0}\right)/(mv_0) = \frac{\mathbb{E}_0}{mv_G^2}\left(\bar{\sigma}(\bar{v}_0)\,\bar{v}_0^3 + 2\sqrt{\frac{3}{\pi}}\int_0^{v_0}\mathrm{d}\bar{v}\frac{\bar{\sigma}(\bar{v})\,\bar{v}^3}{\sqrt{\bar{v}_0^2-\bar{v}^2}}\exp\left(-\frac{3\bar{z}_0^2}{\bar{v}_0^2-\bar{v}^2}\right)\left[\frac{\mathrm{d}\bar{x}_0}{\mathrm{d}\bar{v}_0} - \frac{\bar{x}_0\bar{v}_0}{\bar{v}_0^2-\bar{v}^2}\right]\right). \quad (44)$$

The first and third terms in Eq. (44) occur due to changes in utility with changing velocity, the second term (first in the square brackets) occurs due to changes in utility with changing position. In writing this expression, we have made the replacement

$$\Gamma_{x0} = \frac{\mathrm{d}x_0}{\mathrm{d}v_0}\Gamma_{v0} \quad (45)$$

without loss of generality. The flux Γ_{v0} is set to 1 so that g represents the single particle gain, which is useful to see for gaining intuition. The amount of gain in the edge region depends heavily on the factor $\mathrm{d}\bar{x}_0/\mathrm{d}\bar{v}_0$, which represents the amount of change in spatial position a wave can impart for a given velocity change. For an ion gyro-orbiting a z directed magnetic field, and a wave directed in the y direction, we can write a simple resonance condition as $\omega - k_y v_y = 0$. If the wave imparts a velocity kick $\mathrm{d}v_y$ and a corresponding energy change $mv_y\mathrm{d}v_y$, then the change in guiding center for the particle is

$$\frac{\mathrm{d}\bar{x}_{gc}}{\mathrm{d}\bar{v}_0} = -\frac{v_0 k_y}{\omega}. \quad (46)$$

Equation 46 shows that the amount of change in position for a given velocity change is in large part determined by the wave properties. Figure 3 shows the gain, Eq. (44), plotted for $\mathrm{d}\bar{x}_0/\mathrm{d}\bar{v}_0 = 2$. In this case, the gains may be quite high in the region where Knudsen effects are prominent. For ions pushed over a non-infinitesimal path through phase space, the gain would be averaged along the path. In Fig. 3 this path is constrained to be a line of slope 1/2, since the gain contours are calculated assuming $\mathrm{d}\bar{x}_0/\mathrm{d}\bar{v}_0 = 2$.

As before, regions of horizontal contours indicate where Knudsen effects cease to have an impact. Larger values of $\mathrm{d}\bar{x}_0/\mathrm{d}\bar{v}_0$ will result in even larger gains, but these gains won't extend into regions of horizontal contours. For example, Knudsen impacted gains for particles less than 1.5 v_G (or equivalently 244 keV deuteron energy in the D-T plasma considered here) have a maximum extent of approximately 1 ρ_* from the boundary. Far from the boundary, pushing in space has no effect, so that the gains are due purely to a baseline gain from pushing in velocity. This baseline is given by the first term in Eq. (44). In the present case, the baseline has a maximum gain of approximately 49.2/ln(Λ) for fast ions near the peak of the fusion cross section. Gains here are calculated using the full 17.6 MeV fusion energy for E_f, and must be scaled down accordingly if one is only interested in the 3.5 MeV alpha particle energy. Figure 3 uses ln(Λ) = 8. A different Coulomb logarithm value would, again, affect the scaling of the figure but not the structure. Having large gains requires being able to find a wave with the right properties (e.g. phase velocity, wavenumber) in the edge region of the ICF plasma.

The gains calculated here assume that the only increase in fusion energy production as a result of the injected energy is that generated by increases in the ion chance of fusion. However, the effective gains may be increased by the fact that the some portion of the injected energy will be transferred by collisions from the ion to other plasma particles, heating them. Energy transferred from the fast ion that helps generate other fast ions would most increase the gain in fusion reactions, but any energy going into ion heating is useful.

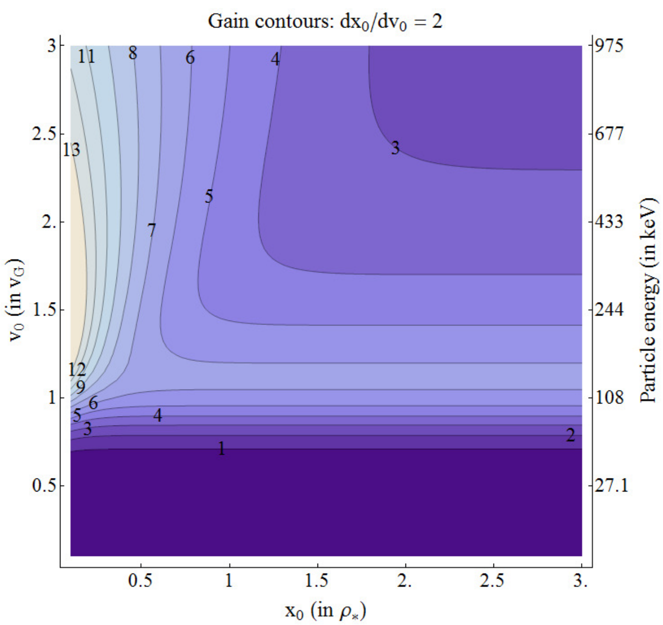

FIG. 3. Gain function g in near boundary region, giving the multiplier between input energy and increase of expected fusion energy output, when a fast ion at x_0, v_0 is pushed incrementally in space and velocity. Axis scales are the same as discussed in Fig. 1, but with a horizontal scale of ρ_* instead of λ_*. Gain values are the same for both deuterons and tritons.

VI. DISCUSSION

Ultimately, the usefulness of the scheme for restoring fusion reactivity lost to the deleterious effects of the Knudsen layer depends on two factors: the efficiency with which reactivity can be restored, and the total amount of reactivity that can be restored compared to the entire hot spot reactivity. With the right wave, the theoretical single particle efficiency may be high. Pushing many fast particles from the edge region towards the interior may result in a lower individual efficiency, since such pushing requires diffusion paths in phase space - once a particle has been pushed inwards to a new position in the phase space, it raises the phase space density there, eventually making it infeasible to push to the same location. Since the distribution function drops off quite rapidly as a function of velocity in the region of velocity space occupied by fast particles, pushing particles more in velocity for a given spatial push opens up more phase space, but lowers the efficiency. Tackling the global efficiency of the scheme in a dynamic situation is a challenging problem.

The second factor depends largely on the design of magnetized ICF experiments. The larger the fraction of the burning plasma volume that is subject to the depletion of fast ions due to edge loss, the more theoretically useful the mitigation scheme. The point design for magnetized liner experiments and recent magnetized OMEGA implosions are not expected to suffer substantial Knudsen related losses[9]. However, it is possible that future magnetized ICF experiments may be in a regime where there is some level of magnetized Knudsen edge loss. Unforeseen kinetic or dynamical effects may also cause more ion loss than currently expected. The mitigation strategy presented here should remain relevant for more inclusive physics models of edge ion loss, so long as the loss is kinetic in nature. While no mitigation may be needed, it is reassuring that the more necessary it is the more theoretically efficient it may become - when the utility decreases rapidly near the edge, large restorations can be had for small spatial pushes.

Note that neither utility function derived here is expected to be accurate immediately near the boundary. This is due to the breakdown of underlying assumptions in each model at the boundary, particularly the lack of dependence of the distribution function on μ, which is not sensible for an absorbing boundary but underlies the diffusive approximation. This is a well known problem. Albright et al. have demonstrated the implementation of an improved boundary condition for the unmagnetized diffusive model used here in the Knudsen layer context[4]. Improvements for the unmagnetized case, beyond the diffusive model used here, have been discussed by Tang et al.[6] and McDevitt et al.[8]. For simplicity of demonstrating the technique and ideas, we have used a zero boundary condition and diffusive approximations.

The linearization in Eq. (7) is not strictly valid near boundaries, where past work[3,4,9] has indicated the excursion from Maxwellian in the tail of the distribution function can be rather large. This affects the validity of equations involving χ, like the adjoint moment equation for the differential fusion energy production, Eq. (15). However, the utility, given by Eq. (19), is relatively insensitive to the background distribution so that it is still well defined and valid within the approximations used in its calculation. The utility is in essence a single particle calculation that helps determine what we can say without calculating the actual distribution function.

The high velocity approximation discussed in Sec. IV can be accurate for the calculation of the utility of very fast particles because the vast majority of the expected lifetime fusion energy created by a particle that starts fast will occur while it is fast. Even if we fail to accurately capture the particle dynamics when it starts getting closer to thermal speed (for example, velocity diffusion which starts to kick in), the contribution to the fusion utility is negligible there, so we will still get a reasonable estimate of the expected fusion energy production.

Note that this is true since we are considering a finite 'lifetime', i.e., there is some velocity, say the thermal velocity or some substantial fraction of it, below which we stop tracking the particle. This means we are not treating the circumstance when a fast ion has slowed down to nearly zero velocity and is then jostled back into being a fast ion. This circumstance does not matter when calculating the incremental energy production due to an initial energy or spatial push, since the ion loses memory of the push after it slows down. However, the finite lifetime may limit the applicability of the utility functions given here for other problems. This is not a fundamental limitation of the utility function formulation, but rather of the present approximations.

For the approximately 10 keV operating temperature targeted in typical ICF experiments, the Gamow peak in a 50-50 DT plasma is located at approximately 3 times the thermal energy (less than 2 V_T). The Gamow peak gives the particle energy value where the maximum fusion production occurs, when both the fusion cross section and the number of particles at each energy in a Maxwellian distribution are taken into account. The high velocity approximation means utility values for these particles will not be quantitatively accurate, although trends in the utility at these lower velocities can still be correct. As the temperature considered decreases, the broad Gamow peak will start to contain more and more high (normalized to V_T) velocity particles. Then the high velocity approximation will yield increasingly accurate utility results. (For reference, far from the boundary, Monte Carlo simulations indicate that the high velocity utility for a particle starting at 4 V_T is off by $\sim 10\%$ compared to a utility calculated with the next order velocity diffusion term included.) For an accurate utility function across the full width of the Gamow peak, the high velocity approximation should be relaxed.

The utility function and adjoint approach could be applied to calculate a total reactivity reduction, which past Knudsen layer work has focused on[3–6,9]. However, a full consideration of the relative benefits of different approaches for calculating reactivity reduction is beyond the scope of this present work.

The adjoint approach for the utility function can be systematically generalized to increasingly complex situations. More general moment equations than Eq. (15) can be written, allowing for more complicated boundary conditions. One could include other effects not considered here, such as electric fields or more complex collisional dynamics. The adjoint formulation can also be expanded to include a time evolving background[15].

While the examples given in this work could be made quantitatively more accurate, the approach should be useful as more inclusive and accurate pictures of ion kinetic physics in ICF implosions are developed. With a simple application, it has given us insight into the density dependence of magnetized and unmagnetized Knudsen dynamics, as well as a basic evaluation of a scheme for combating Knudsen losses.

ACKNOWLEDGEMENTS

This work was supported by DOE through contracts DE-AC02-09CH1-1466 and 67350-9960 (Prime # DOE DE-NA0001836). Seth Davidovits would like to acknowledge support by the DOE-CSGF program under grant DE-FG02-97ER25308.

[1] D. B. Henderson, Phys. Rev. Lett. **33**, 1142 (1974).
[2] A. Petschek and D. Henderson, Nuclear Fusion **19**, 1678 (1979).
[3] K. Molvig, N. M. Hoffman, B. J. Albright, E. M. Nelson, and R. B. Webster, Phys. Rev. Lett. **109**, 095001 (2012).
[4] B. J. Albright, K. Molvig, C.-K. Huang, A. N. Simakov, E. S. Dodd, N. M. Hoffman, G. Kagan, and P. F. Schmit, Physics of Plasmas **20**, 122705 (2013).
[5] X.-Z. Tang, C. J. McDevitt, Z. Guo, and H. L. Berk, Europhysics Letters **105**, 32001 (2014).
[6] X.-Z. Tang, H. L. Berk, Z. Guo, and C. J. McDevitt, Physics of Plasmas **21**, 032707 (2014).
[7] X.-Z. Tang, C. J. McDevitt, Z. Guo, and H. L. Berk, Physics of Plasmas **21**, 032706 (2014).
[8] C. J. McDevitt, X.-Z. Tang, Z. Guo, and H. L. Berk, Physics of Plasmas **21**, 032708 (2014).
[9] P. F. Schmit, K. Molvig, and C. W. Nakhleh, Physics of Plasmas **20**, 112705 (2013).
[10] N. J. Fisch, Rev. Mod. Phys. **59**, 175 (1987).
[11] N. J. Fisch and A. H. Boozer, Phys. Rev. Lett. **45**, 720 (1980).
[12] C. F. F. Karney and N. J. Fisch, Physics of Fluids **29**, 180 (1986).
[13] N. Fisch and M. Herrmann, Nuclear Fusion **34**, 1541 (1994).
[14] N. J. Fisch and J.-M. Rax, Phys. Rev. Lett. **69**, 612 (1992).
[15] N. J. Fisch, Physics of Fluids **29**, 172 (1986).
[16] J. G. Kirk, D. B. Melrose, and E. R. Priest, *Plasma Astrophysics*, edited by A. O. Benz and T.-L. Courvoisier, Saas-Fee Advanced Course, Vol. 24 (Springer-Verlag, Berlin Heidelberg, 1994).
[17] H.-S. Bosch and G. Hale, Nuclear Fusion **32**, 611 (1992).
[18] A. J. Brizard, Physics of Plasmas **11**, 4429 (2004).
[19] E. Hirvijoki, A. Brizard, A. Snicker, and T. Kurki-Suonio, Physics of Plasmas **20**, 092505 (2013).

Princeton Plasma Physics Laboratory
Office of Reports and Publications

Managed by
Princeton University

under contract with the
U.S. Department of Energy
(DE-AC02-09CH11466)

P.O. Box 451, Princeton, NJ 08543
Phone: 609-243-2245
Fax: 609-243-2751
E-mail: publications@pppl.gov
Website: http://www.pppl.gov