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# The Virtual Casing Principle For 3D Toroidal Systems

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## BRIEF COMMUNICATION

# The virtual-casing principle for 3D toroidal systems*

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**Abstract**
The capability to calculate the magnetic field due to the plasma currents in a toroidally confined magnetic fusion equilibrium is of manifest relevance to equilibrium reconstruction and stellarator divertor design. Two methodologies arise for calculating such quantities. The first being a volume integral over the plasma current density for a given equilibrium. Such an integral is computationally expensive. The second is a surface integral over a surface current on the equilibrium boundary. This method is computationally desirable as the calculation does not grow as the radial resolution of the volume integral. This surface integral method has come to be known as the 'virtual-casing principle'. In this paper, a full derivation of this method is presented along with a discussion regarding its optimal application.

(Some figures may appear in colour only in the online journal)

## 1. Introduction

The calculation of three-dimensional magnetic fields due to the plasma currents, in regions external to three-dimensional plasma equilibria, is essential to equilibrium reconstruction and divertor design. The magnetic induction is a linear superposition of the vacuum field ($\vec{B}_{\text{Vacuum}}$) produced by the current flowing in conductors external to the plasma and the plasma field ($\vec{B}_{\text{Plasma}}$) generated by the currents flowing in the plasma:

$$\vec{B}_{\text{Total}} = \vec{B}_{\text{vacuum}} + \vec{B}_{\text{Plasma}}. \qquad (1)$$

Various analytic and numerical methods exist for calculating the vacuum component of the field (figure 1). As the vacuum component is a well-treated problem [1], we turn our attention towards the plasma component of the field. The most direct method for calculation of the magnetic field due to the plasma is to envoke the Biot–Savart integral:

$$\vec{B}_{\text{Plasma}}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \mathrm{d}^3x'. \qquad (2)$$

Here primes denote integration over the plasma volume, $\vec{j}$ is the plasma current density and $\mu_0$ is the permeability of free space. This volume integral is numerically expensive to evaluate and grows as a radial resolution of a given plasma equilibrium model. In this paper we are only interested in solving for the plasma field in regions external to the plasma domain thus, a 'virtual-casing' principle may be applied to convert the volume integral to a surface integral [2]. This significantly reduces the computational load required to obtain the plasma field. For an equilibria with 99 surfaces a surface integral would provide a nearly two orders of magnitude reduction in the number of operations over a volume integral with the same poloidal and toroidal resolution.

In past works, the derivation of this method involved the assumption that an infinitely conducting sheet exists between the plasma and external conductors. This sheet shields out the total magnetic induction, producing a current sheet. For 2D systems, the plasma response (external to the plasma volume) is purely in the poloidal direction producing a current sheet

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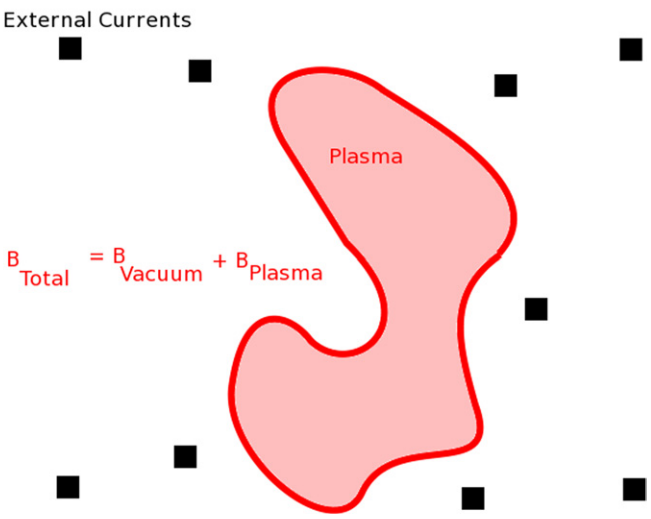


**Figure 1.** Depiction of plasma cross section and field coils.

in the toroidal direction. This allows the poloidal currents on this sheet to be explicitly set to zero. In 3D, this assumption is not valid. However, the fundamental method is correct and a generally valid result can be derived without invoking the existence of unphysical current sheets. It is worth noting that the 'virtual-casing' principle is just the magnetostatic limit of Love's equivalence theorem [3].

## 2. Method

The 'virtual-casing' principle makes use of the magnetostatic jump conditions present at the plasma–vacuum interface to describe volume integrals over currents in terms of surface integrals. The derivation of these jump conditions begins with the equations of electrodynamics [4]:

$$\frac{\partial \rho_c}{\partial t} = \nabla \cdot \vec{j}, \tag{3}$$

$$\nabla \cdot \vec{B} = 0, \tag{4}$$

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}, \tag{5}$$

$$\vec{B} = \mu \vec{H}, \tag{6}$$

where $\rho_c$ is the charge density, $\vec{B}$ is the magnetic induction, $\vec{H}$ is the magnetic field, $\vec{D}$ is the electric displacement, $\vec{j}$ is the current density and $\mu$ is the magnetic permeability of a given material. It is important to note that all quantities have a spatial dependence $f = f(\vec{x})$. In the magnetostatic limit (vanishing time derivatives) equations (3) and (5) become

$$\nabla \cdot \vec{j} = 0, \tag{7}$$

$$\nabla \times \vec{H} = \vec{j}. \tag{8}$$

We now imagine some toroidally closed surface, $S$, which encloses all the plasma currents and excludes any external conductors. For our purposes, we will use the plasma equilibrium surface. This geometry allows us to follow a similar exposition as found in many texts on classical electrodynamics. We consider an infinitesimal Stokesian loop, $C$, to be present across the surface, $S$, where two legs are directly parallel to the boundary surface ($S$) and the other two are perpendicular to the surface. The resulting normal vector ($\vec{t}$) to the loop is then tangential to the boundary surface, $S$ (figure 2). Equation (8) may be converted to a integral form

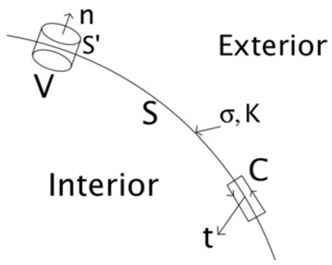


**Figure 2.** Depiction of the Stokesian loop (C) and the pillbox (V) spanning the plasma–vacuum interface (S). The jump conditions on the magnetic field allow the construction of a dipole moment density ($\sigma$) and surface current density (K), from which the plasma response outside the surface may be calculated.

$$\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = \int_{S'} \vec{j} \cdot \vec{t}\,\mathrm{d}A', \tag{9}$$

where $\vec{dl}$ is a line element on the contour $C$ and $S'$ is the surface spanning the loop and $\mathrm{d}A'$ is a surface element of $S'$.

Taking the contour $C$ to lay infinitesimally close to the surface $S$ the left-hand side of our modified Ampère–Maxwell equation (equation (9)) may be written as

$$\int_C \vec{H} \cdot \mathrm{d}\vec{l} = (\vec{t} \times \hat{n}) \cdot \left(\vec{H}_{\text{exterior}} - \vec{H}_{\text{interior}}\right) \Delta l, \tag{10}$$

where $\Delta l$ is the length of the curve parallel to the surface and $\hat{n}$ is the unit surface normal. The right-hand side of our modified Ampère–Maxwell equation becomes

$$\int_C \vec{j} \cdot \vec{t}\mathrm{d}A = \vec{K} \cdot \vec{t}\Delta l, \tag{11}$$

where $\vec{K}$ is an idealized surface current and the integral is over the surface area enclosed by the Stokesian loop (C). Here, the integral over a volume has been transformed into a surface integral. The implication is that $\vec{K}$ is a two-dimensional object. Taking these three equations ((9), (10) and (11)), the surface current can now be written in terms of a jump condition on the tangential component of the field

$$\vec{K} = \hat{n} \times \left(\vec{H}_{\text{exterior}} - \vec{H}_{\text{interior}}\right). \tag{12}$$

This transforms the current density of the plasma to a surface current on a boundary surface.

Returning to our surface, $S$, let us now imagine an infinitesimal pillbox of volume ($V$), where the ends of the pillbox have normal components parallel to the surface normal (figure 2). If the surface ($S$) is a flux surface the solenoidal constraint on the magnetic induction (equation (4)) is identically satisfied on this surface, we may write

$$\int_{S'} \vec{B} \cdot \hat{n}\mathrm{d}A = \left(\vec{B}_{\text{exterior}} - \vec{B}_{\text{interior}}\right) \cdot \hat{n}\Delta A = 0, \tag{13}$$

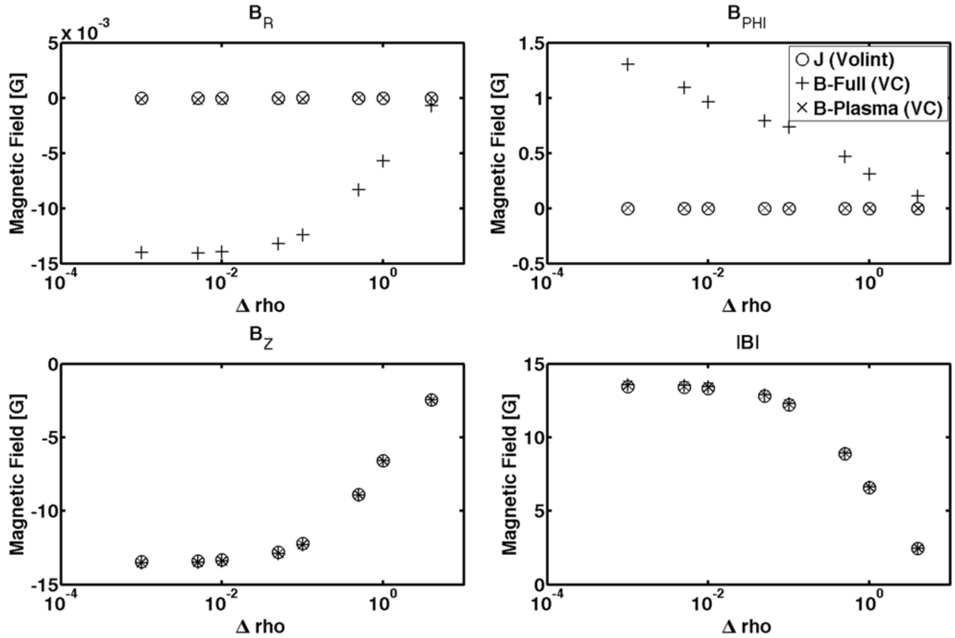


**Figure 3.** Calculation of plasma field outside a circular cross section VMEC equilibria ($R = 100$ m, $a = 1$ m). The horizontal axis indicates distance from the plasma surface (normalized to the plasma radius 1 m). Circles indicate field as calculated by a volume integral over the plasma volume. Crosses indicate fields as calculated by a virtual casing method with the full (vacuum and plasma) field. Xs indicate field as calculated by virtual-casing with just the plasma field utilized. Errors in the toroidal field persist for distances up to four minor radii from the equilibrium surface.

where $\Delta A$ is the surface area of our pillbox and $S'$ is the surface of the pillbox. This gives the jump condition on the normal component of the field, namely

$$\left(\vec{B}_{\text{exterior}} - \vec{B}_{\text{interior}}\right) \cdot \hat{n} = 0. \tag{14}$$

In general our surface ($S$) need not be a flux surface. We now introduce a dipole moment density ($\sigma_{\text{dipole}}$) with the property

$$\int_S \vec{B} \cdot \hat{n} \mathrm{d}A = \int_S \sigma_{\text{dipole}} \mathrm{d}A = 0. \tag{15}$$

In the limit that the volume of the pillbox vanishes we may then write

$$\left(\vec{B}_{\text{exterior}} - \vec{B}_{\text{interior}}\right) \cdot \hat{n} = \sigma_{\text{dipole}}. \tag{16}$$

This dipole moment density allows our surface to become arbitrary with respect to the magnetic topology [5].

The surface current density and dipole moment density may then be utilized to construct the plasma component of the magnetic field in the vacuum region:

$$\vec{B}_{\text{plasma}}\left(\vec{x}\right) = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \left(\vec{x} - \vec{x}'\right)}{|\left(\vec{x} - \vec{x}'\right)|^3} \mathrm{d}A' + \frac{1}{4\pi} \int \frac{\sigma_{\text{dipole}}\left(\vec{x} - \vec{x}'\right)}{|\left(\vec{x} - \vec{x}'\right)|^3} \mathrm{d}A'. \tag{17}$$

This is the 'virtual-casing' principle where primes denote integration over our surface, $S$, and $\vec{x}$ is the point at which we desire the plasma field.

## 3. Comments

The application of the 'virtual-casing' principle to calculate the plasma response is easily understood but care must be taken to use the vacuum or plasma field where appropriate. In the Shafranov and Zakharov paper [2], the method is derived for an axisymmetric tokamak. The clear result is that there can only be a poloidal plasma response in the exterior region. They then use this information to construct their current sheet. A toroidal plasma response is never even calculated because there is an implicit assumption that the integral over such a field is zero.

The effect of including the toroidal field can be evaluated for this case. The magnetic induction is a linear superposition of fields so the total field can be written as before:

$$\vec{B}_{\text{Total}} = \vec{B}_{\text{vacuum}} + \vec{B}_{\text{Plasma}}. \tag{18}$$

Assuming for simplicity that the toroidal surface is a flux surface (dipole moment density vanishes), the surface current density becomes a linear superposition as well:

$$\vec{K}_{\text{Total}} = \vec{K}_{\text{vacuum}} + \vec{K}_{\text{Plasma}}. \tag{19}$$

The $\vec{K}_{\text{vacuum}}$ portion of the surface current density arises from sources external to the surface. This requires the following integral relation to be fulfilled:

$$\int \vec{B}_{\text{vacuum}} \cdot \mathrm{d}\vec{l} = \mu_0 \int \vec{K}_{\text{vacuum}} \cdot \mathrm{d}\vec{A}' = \mu_0 I_{\text{enclosed}} = 0. \tag{20}$$

Recently, the magnetic diagnostics code DIAGNO [6] has been modified to incorporate a virtual-casing principle and a volume integral over current densities into its diagnostic calculation methods. Figure 3 depicts the magnetic field as calculated by eight field probes located on the outboard mid-plane ($\phi = 0, \theta = 0$) of the equilibria. The VMEC code [7] was utilized to calculate an aspect ratio 100 ($R = 100$ m, $a = 1$ m) circular cross section equilibria with zero pressure. A total toroidal current of 7000 A with a current profile of $\mathrm{d}I/\mathrm{d}s = I_0(1-s-s^4+s^5)$ was assumed, where $I_0$ is calculated to guarantee the total toroidal current to be correct and $s$ is the normalized toroidal flux. This configuration was chosen to minimize effects of torodicity and any uncertainties regarding the equilibrium itself. In this configuration the polodial field is generated by plasma currents alone and the toroidal field is generated by external conductors. The plot indicates the plasma field as calculated by a volume integral over the current densities (o), a virtual-casing principle utilizing the full field on the equilibrium boundary (+), and just the plasma field on the equilibrium boundary (x). It is clear that by utilizing the total field an erroneous toroidal plasma field is generated outside the equilibria in both the radial and toroidal directions. Although both methods seem to properly calculate the correct vertical plasma response ($B_Z$), this problem was separable with no vacuum field present in the vertical direction. A situation which in 3D is not necessarily true.

In general, both the vacuum and plasma fields will have toroidal, poloidal and radial components. Any surface current contribution, due to the vacuum field, must integrate to zero explicitly ($\int \vec{B}_{\mathrm{vac}} \cdot \mathrm{d}\vec{l} = 0$). The vacuum field does not contain any source terms inside the equilibrium boundary. As a result, the field due to this vacuum surface current must be zero in the region outside the equilibrium. This result can be identically satisfied by explicitly assuming this contribution is zero and thus only using the plasma field to calculate a surface current sheet. This significantly reduces the demands on the accuracy of any numerical integration scheme.

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