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The physics of the second-order gyrokinetic MHD Hamiltonian: μ conservation, Galilean invariance, and ponderomotive potential

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Some physical interpretations are given of the well-known second-order gyrokinetic Hamiltonian in the MHD limit. Its relations to the conservation of the true (Galilean-invariant) magnetic moment and fluid nonlinearities are described. Subtleties about its derivation as a cold-ion limit are explained; it is important to take that limit in the frame moving with the $\boldsymbol{E}\times\boldsymbol{B}$ velocity. The discussion also provides some geometric understanding of certain well-known Lie generating functions, and it makes contact with general discussions of ponderomotive potentials and the thermodynamics of dielectric media.

It is well known that gyrokinetics (see the review by Krommes[1] and references therein, as well as the pedagogical material in Ref. 2) can be described as a Lagrangian field theory.[3,4] In full generality, including gradients of the background magnetic field $\boldsymbol{B}$ and finite-Larmor-radius (FLR) effects, the gyrocenter Hamiltonian $\overline{H}$ is quite complicated, even at second order[5] in the gyrokinetic expansion parameter ϵ. However, in the absence of gradients of $\boldsymbol{B}$ and FLR effects, the second-order Hamiltonian simplifies dramatically to the well-known form[6–8]

$$\overline{H}_2 = -\frac{1}{2}Mu_E^2, \tag{1}$$

where $M \equiv m_i$ is the ion mass and $\boldsymbol{u_E} \doteq c\widehat{\boldsymbol{b}}\times\boldsymbol{\nabla}\phi/B$ is the $\boldsymbol{E}\times\boldsymbol{B}$ velocity. (I consider only electrostatics, with ϕ being the electrostatic potential such that $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$; the constant magnetic field is written as $\boldsymbol{B} = B\widehat{\boldsymbol{b}}$.) Authors such as Scott & Smirnov[9] have used this 'MHD Hamiltonian' to good advantage in order to illustrate various aspects of gyrokinetics such as conservation properties. Clearly $\overline{H}_2$ is the negative of the kinetic energy associated with the $\boldsymbol{E}\times\boldsymbol{B}$ motion of the gyrocenters; the reason for the minus sign may not be immediately apparent. In this note I give several relatively elementary derivations of $\overline{H}_2$ and discuss some subtleties that may not be generally appreciated.

In particular, note that the MHD Hamiltonian (1) does not involve the ion temperature T_i; it remains nonzero in the cold-ion limit $T_i \to 0$. Frequently the program for obtaining a gyrokinetic Hamiltonian is described as a systematic, order-by-order elimination of gyrophase ζ, which inevitably conjures up the idea of an average over a rapidly rotating gyroradius vector $\boldsymbol{\rho}$, where $\boldsymbol{\rho} \doteq \widehat{\boldsymbol{b}}\times\boldsymbol{v}_\perp/\omega_c = \rho\widehat{\boldsymbol{a}}$ is the gyroradius vector and $\omega_c \doteq qB/Mc$ is the gyrofrequency. (The symbol $\doteq$ is used for definitions.) It may seem that 'cold ions' implies $\boldsymbol{\rho}\to\boldsymbol{0}$ ($v_\perp \to 0$). If that were true, one could inquire whether in the limit there is anything left to rotate and whether the gyrokinetic expansion is valid.

One of the difficulties with such discussion is an imprecision about exactly to which coordinates one is referring and what quantities are held fixed as the limit $T_i \to 0$ is taken. In order to explain this further, let me note that the conceptual advance in the development of gyrokinetics was to shift the focus from rapidly gyrating particles to slowly drifting gyrocenters. (An important early paper was by Catto[10]; for more references, see Ref. 1.) Let us define the 'lowest-order' gyrocenter $\overline{\boldsymbol{X}}$ by $\overline{\boldsymbol{X}} \approx \boldsymbol{x} - \boldsymbol{\rho}$, where $\boldsymbol{x}$ is the particle position; gyrophase is defined in terms of $\boldsymbol{\rho}$ (or, equivalently, in terms of $\boldsymbol{v}_\perp = v_\perp\widehat{\boldsymbol{c}}$; note that $\widehat{\boldsymbol{a}}$, $\widehat{\boldsymbol{b}}$, and $\widehat{\boldsymbol{c}}$ form an orthonormal basis). This definition is natural, and the approximation becomes exact for purely circular motion. However, in the presence of an electrostatic potential ϕ, $\overline{\boldsymbol{X}}$ of course moves with (at least) the $\boldsymbol{E}\times\boldsymbol{B}$ velocity, so the particle motion is not precisely circular. Relative to the instantaneous center of gyration, defined in some systematic way, one can introduce a 'true' gyrophase $\overline{\zeta} \neq \zeta$ and a true magnetic moment $\overline{\mu} \neq \mu \doteq \frac{1}{2}Mv_\perp^2/\omega_c$.[11] One must be careful about the meaning of the 'zero-gyroradius limit'; at fixed ω_c, does one mean $\rho(\mu,\zeta)\to 0$ or $\overline{\rho}(\overline{\mu},\overline{\zeta})\to 0$? The answer is the latter, as I will demonstrate.

The definitions of the new, barred variables are not unique in the absence of a further constraint (which amounts to giving a precise meaning to 'instantaneous center of gyration'). That constraint is the adiabatic conservation of the true magnetic moment $\overline{\mu}$. To guess the form of $\overline{\mu}$, one can invoke the idea of Galilean invariance. The lowest-order quantity μ is not Galilean-invariant; it changes its value under a shift in velocity. Galilean invariance is restored if $\boldsymbol{v}_\perp$ is referred to a reference velocity, which is naturally chosen to be $\boldsymbol{u_E}$. Thus, one guesses

$$\overline{\mu} \approx \frac{1}{2}\frac{M|\boldsymbol{v}_\perp - \boldsymbol{u_E}(\boldsymbol{x})|^2}{\omega_c(\boldsymbol{x})}. \tag{2}$$

Indeed, it is easy to demonstrate from the equation of motion

$$\frac{d\boldsymbol{v}}{dt} = \frac{q}{M}(\boldsymbol{E} + c^{-1}\boldsymbol{v}\times\boldsymbol{B}) \tag{3}$$

that $\overline{\mu}$ is exactly conserved when $\boldsymbol{u_E}$ and $\boldsymbol{B}$ are constant. Putting that another way, if one makes the transformation $\boldsymbol{v}_\perp = \boldsymbol{u_E} + \delta\boldsymbol{v}_\perp$, one finds for constant $\boldsymbol{u_E}$ and $\boldsymbol{B}$

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that the equation of motion reduces to

$$\frac{d\delta \boldsymbol{v}_\perp}{dt} - \omega_c \delta \boldsymbol{v}_\perp \times \widehat{\boldsymbol{b}} = \boldsymbol{0}, \tag{4}$$

the solution of which is purely circular motion. That is, the motion is purely circular in a frame moving with a constant $\boldsymbol{u_E}$.

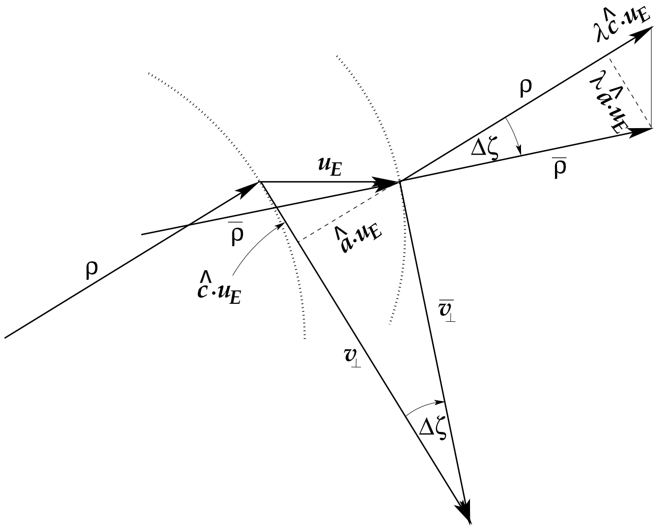

FIG. 1. Illustration of the relation between the original gyroradius $\boldsymbol{\rho}$ and the transformed gyroradius $\overline{\boldsymbol{\rho}}$, and the geometry that determines the first-order corrections to magnetic moment μ and gyrophase ζ. Here it is assumed that a constant magnetic field $\boldsymbol{B}$ is out of the page and that a constant electric field $\boldsymbol{E}$ points up, so that the $\boldsymbol{E} \times \boldsymbol{B}$ drift $\boldsymbol{u_E}$ is to the right; $\lambda \doteq \rho/v_\perp = \omega_c^{-1}$. Left-hand dotted path: circular motion in the unshifted frame; right-hand dotted path: motion around the instantaneous center of gyration in the moving frame.

To discuss the cold-ion limit, one usually invokes the idea of an equilibrium Maxwellian distribution. An equilibrium solution of the gyrokinetic equation should be a function of the constants of motion. Indeed, a perpendicular ion Maxwellian shifted by $\boldsymbol{u_E}$ is proportional to

$$\exp\left(-\frac{1}{2}|\boldsymbol{v}_\perp - \boldsymbol{u_E}|^2/v_{ti}^2\right) = e^{-\overline{\mu}\omega_c/T_i}. \tag{5}$$

Thus the limit $T_i \to 0$ constrains $\overline{\mu}$ to vanish [the distribution function is proportional to $\delta(\overline{\mu})$]. With gyroradius defined by $\rho \doteq v_\perp/\omega_c = (2\mu/M\omega_c)^{1/2}$ and similarly $\overline{\rho} \doteq (2\overline{\mu}/M\omega_c)^{1/2}$, one sees that $\overline{\rho} \to 0$ in the cold-ion limit. ρ, however, does not vanish, since $\overline{\mu} = 0$ constrains $\boldsymbol{v}_\perp$ to be equal to $\boldsymbol{u_E}$, or $\rho = u_E/\omega_c$. This length is not associated with circular motion.

Thus it is crucial to understand the distinction between the barred and unbarred variables, and the definitions of each. At first order, it is not hard to use geometric reasoning to obtain the barred quantities; see Fig. 1. That figure illustrates that the transformed gyroradius is based on the perpendicular velocity relative to the drift motion ($\boldsymbol{E} \times \boldsymbol{B}$ here); thus

$$\overline{\boldsymbol{\rho}} = \omega_c^{-1}\widehat{\boldsymbol{b}} \times \overline{\boldsymbol{v}}_\perp, \quad \overline{\boldsymbol{v}}_\perp \doteq \boldsymbol{v}_\perp - \boldsymbol{u_E}. \tag{6}$$

This relation can be used to determine the first-order corrections $\Delta\mu$ and $\Delta\zeta$ (later to be identified with the first-order 'Lie generating functions' w_1^μ and w_1^ζ) as follows. From the figure, one sees that approximately

$$\Delta\rho/\rho = -\widehat{\boldsymbol{c}} \cdot \boldsymbol{u_E}/v_\perp; \tag{7}$$

from $\Delta\rho/\rho = \frac{1}{2}\Delta\mu/\mu$ and the definition $\mu \doteq \frac{1}{2}Mv_\perp^2/\omega_c$, one obtains

$$\Delta\mu = -M\rho\widehat{\boldsymbol{c}} \cdot \boldsymbol{u_E}. \tag{8}$$

Similarly, one sees that $\rho\Delta\zeta = \lambda\widehat{\boldsymbol{a}}\cdot\boldsymbol{u_E}$, where $\lambda \doteq \rho/v_\perp = \omega_c^{-1}$ is the scale factor relating gyroradius to velocity; thus

$$\Delta\zeta = \widehat{\boldsymbol{a}} \cdot \boldsymbol{u_E}/v_\perp. \tag{9}$$

More formally, one has

$$\begin{aligned} \overline{\boldsymbol{\rho}} &= \overline{\rho}(\overline{\mu})\widehat{\boldsymbol{a}}(\overline{\zeta}) && \text{(10a)} \\ &\approx \overline{\rho}(\mu + \Delta\mu)\widehat{\boldsymbol{a}}(\zeta + \Delta\zeta) && \text{(10b)} \\ &\approx \boldsymbol{\rho} + \Delta\mu\frac{\partial\rho}{\partial\mu}\widehat{\boldsymbol{a}}(\zeta) + \Delta\zeta\rho(\mu)\widehat{\boldsymbol{c}}(\zeta), && \text{(10c)} \end{aligned}$$

where $\partial_\zeta\widehat{\boldsymbol{a}} = \widehat{\boldsymbol{c}}$ was used. From $\mu \doteq \frac{1}{2}Mv_\perp^2/\omega_c$, one has $\partial\rho/\partial\mu = (Mv_\perp)^{-1}$. One can therefore determine the corrections by resolving the $\boldsymbol{u_E}$ term in Eq. (6) onto the $\widehat{\boldsymbol{a}}$ and $\widehat{\boldsymbol{c}}$ directions:

$$\begin{aligned} &(-\omega_c^{-1}\widehat{\boldsymbol{b}} \times \boldsymbol{u_E}) \cdot (\widehat{\boldsymbol{a}}\,\widehat{\boldsymbol{a}} + \widehat{\boldsymbol{c}}\,\widehat{\boldsymbol{c}}) && \text{(11a)} \\ &\quad = -(\omega_c^{-1}\widehat{\boldsymbol{c}} \cdot \boldsymbol{u_E})\widehat{\boldsymbol{a}} + (\omega_c^{-1}\widehat{\boldsymbol{a}} \cdot \boldsymbol{u_E})\widehat{\boldsymbol{c}}. && \text{(11b)} \end{aligned}$$

Upon comparing with the form (10c), one once again obtains Eqs. (8) and (9).

To proceed to higher order, one requires a systematic procedure. Thus one changes variables from the particle phase-space variables z to gyrocenter variables $\overline{z} \doteq \{\overline{\boldsymbol{X}}, \overline{U}, \overline{\mu}, \overline{\zeta}\}$, where $\overline{\boldsymbol{X}}$ is the gyrocenter position, $\overline{U}$ is the effective parallel velocity of the gyrocenter, $\overline{\mu}$ is the (adiabatically conserved) magnetic moment, and $\overline{\zeta}$ is the transformed gyrophase. One standard way of deriving the transformation $\overline{z} = \mathrm{T}z$ is *via* a Lie transformation $\mathrm{T} = \exp(\mathcal{L}_w)$, where $\mathcal{L}_w \doteq w^\nu\partial_\nu$). For asymptotic, order-by-order expansion, one can use either the original Deprit series[8,12] for w or the compound transformation advocated by Dragt & Finn[8,13]:

$$\mathrm{T} = \ldots e^{\mathcal{L}_2}e^{\mathcal{L}_1} = 1 + \mathcal{L}_1 + (\mathcal{L}_2 + \frac{1}{2}\mathcal{L}_1^2) + \cdots, \tag{12}$$

where $\mathcal{L}_n \doteq \epsilon^n\mathcal{L}_{w_n}$. The first-order generating functions are recorded in the review article by Brizard & Hahm.[14] Specialized to the case of constant magnetic field, they are

$$\boldsymbol{w}^{\boldsymbol{x}}_{1\perp} = -\boldsymbol{\rho}, \tag{13a}$$
$$w^U_1 = 0, \tag{13b}$$
$$w^\mu_1 = \omega_c^{-1}\boldsymbol{\rho}\cdot\boldsymbol{\nabla}\widehat{\phi}, \tag{13c}$$
$$w^\zeta_1 = (\omega_c v_\perp)^{-1}\widehat{\boldsymbol{c}}\cdot\boldsymbol{\nabla}\widehat{\phi}, \tag{13d}$$

where $\widehat{\phi} \doteq q\phi/M$ (having dimensions of velocity squared). Upon noting that $\boldsymbol{u_E} = \omega_c^{-1}\widehat{\boldsymbol{b}}\times\boldsymbol{\nabla}\widehat{\phi}$ and that $\Delta\mu = \mathcal{L}_1\mu = w^\mu_1$ and similarly for $\Delta\zeta$, one can readily verify that Eqs. (8) and (13c) are in agreement, as are Eqs. (9) and (13d). It is also not difficult to find that

$$\boldsymbol{w}^{\boldsymbol{x}}_{2\perp} = \frac{1}{2}\omega_c^{-2}\boldsymbol{\nabla}_\perp\widehat{\phi}, \tag{14a}$$
$$w^U_2 = 0, \tag{14b}$$
$$w^\mu_2 = 0; \tag{14c}$$

w^ζ_2 will not be needed here.

Upon using Eq. (12) to calculate $\overline{\mu} = \mathrm{T}\mu$, one finds that

$$\overline{\mu} = \mu + w^\mu_1 + w^\mu_2 + \frac{1}{2}\mathcal{L}_1 w^\mu_1 + \cdots. \tag{15}$$

Simple algebra using Eqs. (13a) and (14a) leads to $\mathcal{L}_1 w^\mu_1 = (w^\mu_1\partial_\mu + w^\zeta_1\partial_\zeta)(\omega_c^{-1}\boldsymbol{\rho}\cdot\boldsymbol{\nabla}\widehat{\phi}) = \omega_c^{-2}|\boldsymbol{\nabla}_\perp\widehat{\phi}|^2$ and finally to

$$\overline{\mu} = \frac{1}{2}\frac{M|\boldsymbol{v}_\perp - \boldsymbol{u_E}|^2}{\omega_c} + O(\epsilon^3), \tag{16}$$

which agrees with the heuristic result (2). I will not review the Lie-theoretic derivation of $\overline{H}_2$, but it follows immediately once $\boldsymbol{w}^{\boldsymbol{x}}_2$ is known.

Because the variable transformation was chosen to preserve the symplectic structure,[14] the Poisson brackets retain their guiding-center forms.[14] In particular,

$$M\frac{d\overline{\boldsymbol{X}}}{dt} = \{\overline{\boldsymbol{X}}, \overline{H}\} = \frac{c}{B}\widehat{\boldsymbol{b}}\times\overline{\boldsymbol{\nabla}}\,\overline{H}(\overline{z}) + \cdots. \tag{17}$$

The contribution from $\overline{H}_1 = q\phi$ gives the usual $\boldsymbol{E}\times\boldsymbol{B}$ drift $\boldsymbol{u_E}$. From $\overline{H}_2$, one obtains a ponderomotive correction

$$\Delta\overline{\boldsymbol{u}} = \frac{c}{B}\widehat{\boldsymbol{b}}\times\overline{\boldsymbol{\nabla}}\left(-\frac{1}{2}Mu_E^2\right) \tag{18}$$

stemming from the ponderomotive potential $q^{-1}[-\frac{1}{2}Mu_E^2(\overline{\boldsymbol{x}})]$. To interpret that potential, one may turn to the equation of motion

$$M\frac{d\boldsymbol{v}}{dt} = q\boldsymbol{E}(\boldsymbol{x}) + M\omega_c\boldsymbol{v}\times\widehat{\boldsymbol{b}}, \tag{19}$$

which is written in terms of particle variables. Now

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}(\overline{\boldsymbol{X}} + \boldsymbol{\rho}) = \boldsymbol{E}(\overline{\boldsymbol{X}}) + \boldsymbol{\rho}(z)\cdot\overline{\boldsymbol{\nabla}}\boldsymbol{E} + \cdots. \tag{20}$$

Here $\boldsymbol{\rho}(z) = \rho(\mu)\widehat{\boldsymbol{a}}(\zeta)$, again written in terms of particle variables. We encounter here the same paradox alluded to earlier, which is that if one were to equate the cold-ion limit with the limit $\rho(z)\to 0$, the gyroradius correction (which will ultimately lead to the ponderomotive effect) would vanish. We know, however, that is incorrect because $\boldsymbol{v}_\perp$ contains a $\boldsymbol{u_E}$ part. What one must do is write $\boldsymbol{\rho}(z)$ in terms of the barred variables (in terms of which the conservation of $\overline{\mu}$ is manifest), then take the limit. One has

$$\boldsymbol{\rho}(z) = \rho(\mu)\widehat{\boldsymbol{a}}(\zeta) \tag{21a}$$
$$\approx \rho(\overline{\mu} - \Delta\mu)\widehat{\boldsymbol{a}}(\overline{\zeta} - \Delta\zeta) \tag{21b}$$
$$\approx \rho(\overline{\mu})\widehat{\boldsymbol{a}}(\overline{\zeta}) - \Delta\mu\,\partial_{\overline{\mu}}\rho(\overline{\mu})\widehat{\boldsymbol{a}}(\overline{\zeta}) - \Delta\zeta\,\rho(\overline{\mu})\partial_{\overline{\zeta}}\widehat{\boldsymbol{a}}(\overline{\zeta}) + \cdots \tag{21c}$$
$$= \overline{\boldsymbol{\rho}} - w^\mu_1(\omega_c\overline{v}_\perp)^{-1}\widehat{\boldsymbol{a}}(\overline{\zeta}) - w^\zeta_1\rho(\overline{\mu})\widehat{\boldsymbol{c}}(\overline{\zeta}) + O(\epsilon^3) \tag{21d}$$
$$= \overline{\boldsymbol{\rho}} - \omega_c^{-2}\overline{\boldsymbol{\nabla}}_\perp\widehat{\phi}(\overline{\boldsymbol{X}}) + O(\epsilon^3). \tag{21e}$$

(The operations are the same as those involved in the calculation of $\mathcal{L}_1\boldsymbol{w}^{\boldsymbol{x}}_1$.) In the cold-ion limit, $\overline{\boldsymbol{\rho}}\to\boldsymbol{0}$ and the $\boldsymbol{\rho}$ correction in Eq. (20) is at second order proportional to $-\overline{\boldsymbol{\nabla}}_\perp\phi\cdot\overline{\boldsymbol{\nabla}}(-\overline{\boldsymbol{\nabla}}\phi) = \frac{1}{2}\overline{\boldsymbol{\nabla}}|\overline{\boldsymbol{\nabla}}_\perp\phi|^2$. Upon comparing with Eq. (18), one sees that this gives rise exactly to the force arising from the ponderomotive potential $q^{-1}\overline{H}_2$: $q\Delta\boldsymbol{E} = q[\boldsymbol{\rho}(z)\cdot\overline{\boldsymbol{\nabla}}(-\overline{\boldsymbol{\nabla}}\phi)] = \cdots = -\overline{\boldsymbol{\nabla}}\,\overline{H}_2$.

So far I have interpreted the gyrokinetic Hamiltonian $\overline{H}_2$ by using detailed considerations about the motion of a particle gyrating in a strong magnetic field, since I wished to emphasize that the cold-ion limit must be taken in the moving frame. However, there is a both simpler and more general route to the answer. Dodin & Fisch have shown,[15] by using considerations relating to action conservation, that in a system of weakly interacting waves the ponderomotive potential energy Φ can be written to lowest (dipole–field interaction) order as $\Phi_2 = -\frac{1}{4}\boldsymbol{E}^*\cdot\widehat{\mathsf{A}}\cdot\boldsymbol{E}$, where $\widehat{\mathsf{A}} \doteq \frac{1}{4\pi}\delta\widehat{\mathsf{X}}/\delta n$, $\widehat{\mathsf{X}}$ is the susceptibility tensor, n is the density, a species label is omitted, and only one mode is considered. The applicability of this wave-based formula to the physics of a gyrating particle requires further discussion, but can be heuristically justified by drawing an analogy between (i) time averaging over the fast quiver of a particle in a wave field, and (ii) averaging over the rapidly changing gyration phase of a particle in a magnetic field; cf. the general discussion in Sec. 2.1 of Ref. 15. In electrostatic gyrokinetics, the cold-ion permittivity, which describes the crucial process of ion polarization, is[1,2]

$$\chi_\perp = \frac{\omega_{pi}^2}{\omega_{ci}^2} = \frac{4\pi n M c^2}{B^2}. \tag{22}$$

After allowing for the fact that a field in real space involves both positive and negative wave numbers, which introduces a factor of 2, one finds that the previous formulas reduce to

$$\Phi_2 = -\frac{1}{2}\frac{Mc^2}{B^2}|\boldsymbol{\nabla}_\perp\phi|^2 = -\frac{1}{2}Mu_E^2 = \overline{H}_2, \tag{23}$$

in agreement with the previous results. That an interaction potential can be negative is, of course, well known: an elementary discussion in the context of gravitational energy is given in Ref. 16; the general formula for the interaction of a dipole with an external field, which contains an explicit minus sign, is derived in Ref. 17; and the fact that a ponderomotive potential in the presence of a magnetic field can be negative has been known since at least 1958.[18]

Although the ponderomotive potential is negative, the energy constant of the gyrokinetic–Poisson system is positive, consisting of the zeroth-order gyrocenter kinetic energy plus a polarization energy whose value in the present limit is[19] $\mathcal{E}_{\rm pol} = \int d\boldsymbol{x}\, n(\frac{1}{2}Mu_E^2)$. This coincides with the well-known expression for the polarization free-energy density of a linear dielectric medium, $\mathcal{E}_{\rm pol} = \frac{1}{8\pi}\int d\boldsymbol{x}\,\chi E^2$, when formula (22) is used. This result is frequently derived[20] by calculating the incremental energy $\boldsymbol{E}\cdot\Delta\boldsymbol{P}$ [$\boldsymbol{P} \approx (4\pi)^{-1}\chi\boldsymbol{E}$ is the polarization) of a small amount of additional free charge added to an existing dielectric configuration that is frozen so no mechanical work is done, then summing the increments to obtain the final result. This corresponds to the procedure followed by Dubin *et al.*,[7] who obtained their energy constant by transforming the particle Hamiltonian. The alternate possible expression $\boldsymbol{E}\cdot\boldsymbol{P} \approx M\boldsymbol{u}_{\boldsymbol{E}}^2$, which is written in terms of the fields of the final state, includes the contribution of the mechanical work that was used to achieve that state, so to obtain the proper free energy one must add the (negative) interaction potential. In this limit, that turns out to mean subtracting the guiding-center kinetic energy associated with the final configuration: $M\boldsymbol{u}_{\boldsymbol{E}}^2 - \frac{1}{2}M\boldsymbol{u}_{\boldsymbol{E}}^2 = \frac{1}{2}M\boldsymbol{u}_{\boldsymbol{E}}^2$. This is a special case of the general expression for gyrokinetic energy that was given by Brizard[3]: $\mathcal{E} = \int d\overline{z}\,\overline{F}(\overline{z},t)[\overline{H} - q\langle(\mathrm{T}_{\rm gy}^*)^{-1}\phi_{\rm gc}\rangle]$, where $\overline{F}$ is the gyrocenter distribution function and T^* is the pullback transformation[2,14,21] such that the particle distribution f is $f = \mathrm{T}^*\overline{F}$; see Ref. 3 for further definitions. The last term corresponds to $\boldsymbol{E}\cdot\boldsymbol{P}$; the term in $\overline{H}$ contains the negative $\overline{H}_2$ that must be added to give the correct conserved quantity for the entire system, which contains both free (gyrocenter) and bound (polarization) charge. The thermodynamics associated with the various equivalent procedures are spelled out clearly in §10 and §11 of Ref. 20. In particular, $\overline{H}_2$ represents the polarization free energy relative to the state in which polarization does not occur (the hypothetical state that would pertain "if the body were absent").

These calculations show, in the simplest possible context, how ponderomotive nonlinearities and the second-order gyrokinetic Hamiltonian are intimately related to each other, to the thermodynamics of a gyrokinetic plasma, which behaves as a polarizable medium, and to conservation of the magnetic moment $\overline{\mu}$; they also provide a physical interpretation of the somewhat abstract Lie generating functions. In general, ponderomotive effects to higher order are most easily obtained by systematic calculation of gyrokinetic $\overline{\mu}$-conserving variables, but the effects are physical and the ponderomotive force will show up as well in a calculation based on particle, not gyrocenter, variables, as we have seen. (A related calculation is by Scott,[22] who discussed the relationship between gyrofluid equations and the Braginskii equations.)

The last observation has implications for recent discussions of gyrokinetic momentum conservation laws. In numerous papers (many of which are referenced and discussed in Ref. 2), Parra & Catto have considered aspects of momentum conservation in the laboratory (particle) frame. Scott & Smirnov[9] and Brizard & Tronko[23] have instead considered the momentum conservation law for the gyrokinetic–Maxwell system. There are definite advantages of simplicity for the latter approach. However, since $f = \mathrm{T}^*\overline{F}$, the same physics results relating to momentum conservation will follow from either the particle-based or the gyrocenter-based moment equations provided that one deals properly with the ponderomotive nonlinearities. The present calculations provide the simplest example of this fact.

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