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Linear Dispersion Relation For The Mirror Instability in Context Of The Gyrokinetic Theory

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Linear dispersion relation for the mirror instability in context of the gyrokinetic theory.

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The linear dispersion relation for the mirror instability is discussed in context of the gyrokinetic theory. The obejctive is to provide a coherent view of the various kinetic approaches used to derive the dispersion relation. The method based on gyrocenter phase space transformations is adopted in order to display the origin and ordering of various terms.

The linear dispersion relation for the mirror instability,[1] has been previously derived using the fully kinetic theory, for example in Refs. 2 and 3; and the gyrokinetic theory in Ref. 4. The purpose of this brief communication is to elaborate on the relationship between the results of these approaches. Specifically the source of various terms appearing in the gyrokinetic dispersion relation will be identified, and their derivation will be described in detail. Gyrokinetic models are important for efficient simulation of low frequency plasma phenomena, and understanding the relationship of these models to fully kinetic ones enables clear physical interpretation of results.

The derivation of the mirror mode dispersion relation will first be outlined in the framework of the fully kinetic theory, and then in the framework of gyrokinetic theory. The method based on gyrocenter phase space transformations[5] is adopted in order to display the origin and ordering of various terms.

For simplicity, a collisionless plasma with single bi-Maxwellian ion species and cold electron species is assumed, and only co-planar magnetic perturbations are allowed, i.e. $\mathbf{k}_\perp \cdot \delta\mathbf{B} = k_\perp \delta B_\perp$, where $\mathbf{k}_\perp$ is the wave vector perpendicular to the background magnetic field B_0 which is uniform and pointing in the z-direction; δ designates the perturbation.

I. "FULLY KINETIC" DERIVATION

The dominant component of the dielectric tensor is ϵ_{yy}, and the dispersion relation is

$$\epsilon_{yy} - \frac{k_\perp^2 c^2}{\omega^2} - \frac{k_\parallel^2 c^2}{\omega^2} = 0, \tag{1}$$

where the fully kinetic ϵ_{yy} for a bi-Maxwellian plasma, i.e. with $F_0 = n_0\sqrt{\dfrac{m^3}{(2\pi)^3 T_\perp^2 T_\parallel}}e^{-(mv_\parallel^2/2T_\parallel + \mu B_0/T_\perp)}$, may be written as[6]

$$\begin{aligned}\epsilon_{yy} =& 1 + \frac{\omega_p^2}{\omega^2}\sum_{n=-\infty}^{\infty} e^{-z}\left(\frac{n^2}{z}I_n(z) + 2zI_n(z) - 2zI_n'(z)\right)\\ &\times\left(\frac{T_\perp}{T_\parallel}\left[1+\zeta_n Z\left(\zeta_n\right)\right] - 1 + \frac{n\Omega_i}{\sqrt{2}k_\parallel v_{th\parallel}}Z\left(\zeta_n\right)\right),\end{aligned} \tag{2}$$

where $\zeta_n = \left(\omega - n\Omega_i\right)/\sqrt{2}k_\parallel v_{th\parallel}$, ω is the frequency of the wave, $k_\parallel$ is the wave vector parallel to the background mangetic field, $\Omega_i \equiv eB_0/mc$ is the ion cyclotron frequency, $v_{th\parallel,\perp} \equiv \sqrt{T_{\parallel,\perp}/m}$ is the parallel and perpendicular thermal velocity of ions, $T_{\parallel,\perp}$ is the parallel and perpendicular temperature, $v_\parallel$ is the parallel velocity coordinate, $\mu \equiv mv_\perp^2/2B_0$ is the ion magnetic moment coordinate, $\omega_p^2 = 4\pi n_0 e^2/m$, and n_0 is the plasma density which is constant. I_n are the Modified Bessel Functions of the first kind of order n, n being an integer, and their argument is $z \equiv k_\perp^2\rho_i^2$, where $\rho_i \equiv v_{th\perp}/\Omega_i$. Z is the plasma dispersion function. More detail can be found in Ref. 6.

Considering only the $n = 0$ and $n = \pm 1$ contributions, Eq. (1) may be written as

$$\begin{aligned}1 &+ \left(\frac{k_\parallel}{k_\perp}\right)^2 - \frac{\omega^2}{k_\perp^2 c^2} - \frac{T_\perp}{T_\parallel}\left[\beta_\perp - \beta_\parallel + \beta_\perp\zeta_0 Z(\zeta_0)\right]\\ &\times\left[I_0(z) - I_0'(z)\right]e^{-z}\\ &- \beta_\perp\sum_{n=\pm1} e^{-z}\left(\frac{n^2}{2z^2}I_n(z) + I_n(z) - I_n'(z)\right)\\ &\times\left(\frac{T_\perp}{T_\parallel}\left[1+\zeta_n Z(\zeta_n)\right] - 1 + \frac{n\Omega_i}{\sqrt{2}k_\parallel v_{th\parallel}}Z(\zeta_n)\right) = 0,\end{aligned} \tag{3}$$

where $\beta_{\parallel,\perp} \equiv 8\pi n_0 T_{\parallel,\perp}/B_0^2$. With $\omega/\Omega_i, \sqrt{2}k_\parallel v_{th\parallel}/\Omega_i \ll 1$, the asymptotic expansion of the Z function may be used to write

$$\begin{aligned}\left(\frac{T_\perp}{T_\parallel}\left[1+\zeta_n Z(\zeta_n)\right] - 1 + \frac{n\Omega_i}{\sqrt{2}k_\parallel v_{th\parallel}}Z(\zeta_n)\right) \approx \frac{\omega}{n\Omega_i} + \frac{\omega^2}{n^2\Omega_i^2}\\ + \frac{1}{2}\left(1 - \frac{T_\perp}{T_\parallel}\right)\left(\frac{\sqrt{2}k_\parallel v_{th\parallel}}{n\Omega_i}\right)^2 \ldots\end{aligned} \tag{4}$$

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Substituting into Eq. (3) and performing the sum, the first order term cancels due to $n = \pm 1$ symmetry. The second order term in ω/Ω_i results in the fast compressional mode, and thus under the assumption of low frequency only the second order term in $k_\parallel v_{th\parallel}/\Omega_i$ survives. Explicitly, using Bessel Function indentities,[7] the following relation is obtained

$$1 + \left(\frac{k_\parallel}{k_\perp}\right)^2 - \frac{\omega^2}{k_\perp^2 c^2} - \frac{T_\perp}{T_\parallel}\left[\beta_\perp - \beta_\parallel + \beta_\perp \zeta_0 Z(\zeta_0)\right] \times \left[I_0(z) - I_0'(z)\right] e^{-z} - \frac{\alpha_b \beta_\perp}{z}\frac{\omega^2}{\Omega_i^2} - \alpha_b \beta_\perp \left(\frac{T_\parallel}{T_\perp} - 1\right)\left(\frac{k_\parallel}{k_\perp}\right)^2 = 0, \tag{5}$$

where $\alpha_b \equiv \left[-2zI_0(z) + I_1(z)/z + 2(1+z) I_1(z)\right] e^{-z}$ (the notation has been adopted from Ref. 4). Neglecting terms $\omega^2/k_\perp^2 c^2 \ll 1$ and $\omega^2/\Omega_i^2 \ll 1$, while using $I_0' = I_1$, the final form of the dispersion relation is

$$1 + \left(\frac{k_\parallel}{k_\perp}\right)^2 \left[1 + \alpha_b\left(\beta_\perp - \beta_\parallel\right)\right] - \frac{T_\perp}{T_\parallel}\left[\beta_\perp - \beta_\parallel + \beta_\perp \zeta_0 Z\left(\zeta_0\right)\right]\left[I_0(z) - I_1(z)\right] e^{-z} = 0. \tag{6}$$

It can now be seen that the $n = 0$ component of the sum in Eq. (3), while being of the lowest order in $k_\parallel v_{th\parallel}/\Omega_i$, is sufficient to recover the mirror mode dispersion relation as given in Eq. (35) of Ref. 2, for a homogeneous plasma – this is Eq. (6) without the α_b term. Meanwhile the $n = \pm 1$ components are of higher order, and are needed to recover the term proportional to α_b, which is present in the dispersion relation for a bi-Maxwellian plasma in Ref. 3 and 4. As $k_\perp \rho_i$ increases α_b decreases, consequently the term was dropped when finite-Larmor-radius effects were considered in Ref. 2. As $k_\perp \rho_i \to 0$, α_b approaches its maximum value of $1/2$, and the dispersion relation takes on the form of Eq. (19) of Ref. 2. In conclusion, from the fully kinetic theory it has been demonstrated that the α_b term enters the dispersion relation with the second order in $k_\parallel v_{th\parallel}/\Omega_i$. The gyrokinetic theory will now be used to recover the dispersion relation (6).

II. GYROKINETIC DERIVATION

Following Ref. 8, the lowest order transformation equations from particle coordinates $\mathbf{z} \equiv (\mathbf{x}, \mathbf{v})$ to guiding-center coordinates $\mathbf{Z}_0 \equiv (\mathbf{X}_0, v_{\parallel 0}, \mu_0, \xi_0)$ are,

$$\begin{aligned}
\mathbf{X}_0 &= \mathbf{x} - \boldsymbol{\rho}_0, \\
v_{\parallel 0} &= \hat{\mathbf{b}} \cdot \mathbf{v}, \\
\mu_0 &= m v_\perp^2 / 2B_0, \\
\xi_0 &= \tan^{-1}\left(\hat{\mathbf{e}}_1 \cdot \mathbf{v} / \hat{\mathbf{e}}_2 \cdot \mathbf{v}\right),
\end{aligned} \tag{7}$$

and from guiding-center to gyrocenter coordinates $\mathbf{Z} \equiv (\mathbf{X}, v_\parallel, \mu, \xi)$,

$$\mathbf{X} = \mathbf{X}_0 + \delta \mathbf{A}_\perp \times \frac{\hat{\mathbf{b}}}{B_0} - \frac{1}{m}\left(\frac{\hat{\mathbf{b}}}{\Omega_i} \times \partial_{\mathbf{X}_0} S_1 + \hat{\mathbf{b}} \partial_{v_{\parallel 0}} S_1\right),$$

$$v_\parallel = v_{\parallel 0} + \frac{e}{mc}\delta \tilde{A}_\parallel + \frac{\hat{\mathbf{b}}}{m} \cdot \partial_{\mathbf{X}_0} S_1, \tag{8}$$

$$\mu = \mu_0 + \frac{e^2}{mc^2}\delta \mathbf{A}_\perp \cdot \partial_{\xi_0} \boldsymbol{\rho}_0 + \frac{e}{mc}\partial_{\xi_0} S_1, \tag{9}$$

$$\xi = \xi_0 - \frac{e^2}{mc^2}\delta \mathbf{A}_\perp \cdot \partial_{\mu_0} \boldsymbol{\rho}_0 - \frac{e}{mc}\partial_{\mu_0} S_1,$$

to lowest order in perturbation amplitude, where $\mathbf{X}$ is the gyrocenter position vector, $v_\parallel$ the gyrocenter parallel velocity, μ the magnetic moment, which is an adiabatic invariant, and ξ is the gyro-phase angle. The definitions of various symbols in are, $v_\perp \equiv \left|\hat{\mathbf{b}} \times (\mathbf{v} \times \hat{\mathbf{b}})\right|$, $\boldsymbol{\rho}_0 \equiv \hat{\mathbf{b}} \times \mathbf{v}_\perp / \Omega_i = \rho_\perp(\hat{\mathbf{e}}_1 \cos\xi_0 - \hat{\mathbf{e}}_2 \sin\xi_0)$, $\rho_\perp^2 = v_\perp^2/\Omega_i^2 = 2\mu_0 B_0 / m\Omega_i^2$, and the unit vector $\hat{\mathbf{b}} = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$ points along the equilibrium magnetic field at particle position. The 0 subscript designates the slowly evolving background component, and δ the perturbed component. $\delta\phi$ and $\delta\mathbf{A}$ are the perturbed scalar potential and vector potential, respectively. Tilde designates the gyro-angle dependent part of the field, thus $\delta\tilde{A}_\parallel \equiv \delta A_\parallel - \langle \delta A_\parallel \rangle$, where $\langle \ ... \ \rangle$ is the gyro-phase average. $S_1(\mathbf{X}_0, v_{\parallel 0}, \mu_0, \xi_0, t)$ is the phase-space gauge-function,[9,10] and the subscript designates that it contains first order in perturbation amplitude. It is the generating function for the gyrocenter transformation and is the solution to the partial differential equation,[11,12]

$$\left(\partial_t + v_\parallel \mathbf{b} \cdot \partial_{\mathbf{X_0}}\right) S_1 + \Omega_i \partial_{\xi_0} S_1 = e\left(\delta\phi - \langle\delta\phi\rangle\right) - \frac{e}{c}\left(\delta\mathbf{A} \cdot \mathbf{v} - \langle\delta\mathbf{A} \cdot \mathbf{v}\rangle\right) \tag{10}$$

In the present case $\delta\phi = 0 = \delta A_\parallel$. Moving to Fourier space with $\partial_t \to -i\omega$, $\partial_{\mathbf{X}_0} \to i\mathbf{k}$, and using $\omega \ll k_\parallel v_\parallel + \Omega_i$, together with

$$\left(\delta\mathbf{A}_\perp \cdot \mathbf{v}_\perp - \langle\delta\mathbf{A}_\perp \cdot \mathbf{v}_\perp\rangle\right) = \frac{c}{e}\mu_0 \delta B_\parallel \left(\frac{2i}{k_\perp \rho_\perp} e^{ik_\perp \rho_\perp \cos\xi_0} \cos\xi_0 - \frac{2}{k_\perp \rho_\perp} J_1(-k_\perp \rho_\perp)\right), \tag{11}$$

the equation for S_1 may be written as

$$\partial_{\xi_0} S_1 + \frac{ik_\parallel v_{\parallel 0}}{\Omega_i} S_1 = -\frac{mc}{e}\mu_0 \frac{\delta B_\parallel}{B_0}\frac{2}{k_\perp \rho_\perp}\left(ie^{ik_\perp \rho_\perp \cos\xi_0}\cos\xi_0 - J_1(-k_\perp \rho_\perp)\right). \tag{12}$$

Since $e^{\lambda\cos\theta}\cos\theta = \partial_\lambda e^{\lambda\cos\theta} = \sum_{n=-\infty}^{\infty} I_n'(\lambda) e^{in\theta}$, using $S_1 = A\sum_{n=-\infty}^{\infty} I_n'(ik_\perp \rho_\perp) e^{in\theta} + B$ as the trial function yields the solution to Eq. (12) in the form

$$S_1 = -\frac{mc}{e}\mu_0\frac{\delta B_\parallel}{B_0}\left\{\frac{\Omega_i}{k_\parallel v_{\parallel 0}}\frac{2i}{k_\perp\rho_\perp}J_1(-k_\perp\rho_\perp)+ \sum_{n=-\infty}^{\infty}\frac{\left[i^{-(n-1)}J_{n-1}(-k_\perp\rho_\perp)+i^{-(n+1)}J_{n+1}(-k_\perp\rho_\perp)\right]e^{in\xi_0}}{k_\perp\rho_\perp\left(n+k_\parallel v_{\parallel 0}/\Omega_i\right)}\right\} \tag{13}$$

Expanding the second term with $k_\parallel v_\parallel/\Omega_i \ll 1$ yields

$$S_1 = -\frac{mc}{e}\mu_0\frac{\delta B_\parallel}{B_0}\left\{\sum_{n\neq 0}\left[1-\frac{k_\parallel v_{\parallel 0}}{n\Omega_i}+\left(\frac{k_\parallel v_{\parallel 0}}{n\Omega_i}\right)^2...\right] \times\frac{\left[i^{-(n-1)}J_{n-1}(-k_\perp\rho_\perp)+i^{-(n+1)}J_{n+1}(-k_\perp\rho_\perp)\right]e^{in\xi_0}}{nk_\perp\rho_\perp}\right\}. \tag{14}$$

The gauge function given in Eq. (14) completely determines the transformation between particle coordinates and gyrocenter coordinates and will now be used to express the ion current in terms of the gyrocenter distribution function. The relationship between the particle distribution function f and the gyrocenter distribution function F is given by the scalar invariance property as $f(\mathbf{x},\mathbf{v},t)\equiv F\left[\mathbf{X}(\mathbf{x},\mathbf{v},t),v_\parallel(\mathbf{x},\mathbf{v},t),\mu(\mathbf{x},\mathbf{v},t),\xi(\mathbf{x},\mathbf{v},t),t\right]$. However, to lowest order in ω/Ω_i, F is in fact independent of the gyrophase angle ξ. When expanded about the background fields, to first order in perturbation amplitude δ, the scalar invariance relation becomes

$$\delta f(\mathbf{z},t)\approx\delta F\left(\mathbf{Z}_0,t\right)+\delta\mathbf{Z}\cdot\partial_{\mathbf{Z}_0}F_0\left(\mathbf{Z}_0\right), \tag{15}$$

where $f_0(\mathbf{z}) = F_0(\mathbf{Z}_0)$ was used, and $\delta\mathbf{Z}$ stands for the perturbed part of the transformation equations between gyrocenter coordinate and the guiding-center coordinates, i.e. $\delta\mathbf{Z} = \mathbf{Z}-\mathbf{Z}_0$. The perpendicular ion current to be used in the perpendicular Ampere's law is therefore,

$$\begin{aligned}\mathbf{J} &\equiv e\int\mathbf{v}_\perp\delta f(\mathbf{x},\mathbf{v},t)d\mathbf{v}\\ &= e\int\mathbf{v}_\perp e^{-\boldsymbol{\rho}_0\cdot\nabla_0}\delta F(\mathbf{x},v_{\parallel 0},\mu_0,t)\frac{B_0}{m}dv_{\parallel 0}d\mu_0 d\xi_0\\ &+ e\int\mathbf{v}_\perp e^{-\boldsymbol{\rho}_0\cdot\nabla_0}\left(\delta v_\parallel\partial_{v_{\parallel 0}}F_0+\delta\mu\partial_{\mu_0}F_0\right)\frac{B_0}{m}dv_{\parallel 0}d\mu_0 d\xi_0,\end{aligned} \tag{16}$$

where it was assumed that F_0 is uniform in space, and the exponential notation was used to designate the transformation from guiding center position to particle position (pull-back).

From Eq. (8) and Eq. (14) it is seen that to lowest order in $k_\parallel v_{\parallel 0}/\Omega_i$, $\delta v_{\parallel 0}$ is independent of $v_\parallel$. Consequently, if F_0 is symmetric in $v_{\parallel 0}$, $\delta v_\parallel^{(0)}$ will not contribute to the perpendicular current given in Eq. (16), where the ordering is indicated by the superscript. For $\delta\mu^{(0)}$, $n\neq 0$ components of the second term in Eq. (9) cancel with the $\partial_{\xi_0}S_1^{(0)}$ term, and the $n=0$ component then yields

$$\delta\mu^{(0)} = \mu_0\frac{\delta B}{B_0}\frac{2}{k_\perp\rho_\perp}J_1(-k_\perp\rho_\perp), \tag{17}$$

which can also be seen by substituting Eq. (12) into Eq. (9), with $\partial_{\xi_0}\boldsymbol{\rho}_0 = \mathbf{v}_\perp$, and keeping only the lowest order terms. Thus, to the lowest order, $\delta\mu^{(0)}\partial_{\mu_0}F_0$ in Eq. (16) is the only contribution to the perpendicular current originating from the gyrocenter coordinate transformation.

The next order correction to the perpendicular current is derived by taking the first order terms in Eq. (14) when obtaining $\delta v_\parallel$, and second order terms when obtaining $\delta\mu$ from Eqs. (8) and (9) respectively. Retaining only $n=\pm 1$ components of the sum in Eq. (14), the result is

$$\delta v_\parallel^{(1)\pm 1} = v_{\parallel 0}\frac{k_\parallel^2}{\Omega_i^2}\frac{\mu_0\delta B_\parallel}{m}\frac{4i}{k_\perp\rho_\perp}J_1'(-k_\perp\rho_\perp)\cos\xi_0 \tag{18}$$

$$\delta\mu^{(2)\pm 1} = -\mu_0\frac{k_\parallel^2 v_{\parallel 0}^2}{\Omega_i^2}\frac{\delta B_\parallel}{B_0}\frac{4i}{k_\perp\rho_\perp}J_1'(-k_\perp\rho_\perp)\cos\xi_0. \tag{19}$$

Next, since $\delta\mu^{(2)\pm 1} = -\left(mv_{\parallel 0}\right)\delta v_\parallel^{(1)\pm 1}/B$, and $\partial_{\mu_0}|_E = \partial_{\mu_0}|_{v_\parallel} - B_0/mv_{\parallel 0}\partial_{v_{\parallel 0}}|_\mu$, where $E = mv_{\parallel 0}^2/2+\mu_0 B_0$, the integral corresponding to the second order correction to ion current in Eq. (16) can be written as

$$e\int\mathbf{v}_\perp e^{-i\boldsymbol{\rho}_0\cdot\mathbf{k}_\perp}\delta\mu^{(2)\pm 1}\,\partial_{\mu_0}F_0|_E\,\frac{B_0}{m}dv_{\parallel 0}d\mu_0 d\xi_0 = -8\pi c\hat{\mathbf{b}}\times i\mathbf{k}_\perp\frac{k_\parallel^2}{k_\perp^2}\frac{\delta B_\parallel}{B_0}\int\mu_0 v_{\parallel 0}^2 J_1'^2(-k_\perp\rho_\perp)\,\partial_{\mu_0}F_0|_E\,dv_{\parallel 0}d\mu_0, \tag{20}$$

where $\mathbf{v}_\perp = (\hat{\mathbf{b}}\times\mathbf{k}_\perp\cos\xi_0 - \mathbf{k}_\perp\sin\xi_0)\Omega_i\rho_\perp/k_\perp$, and $\int_0^{2\pi}\cos^2\xi_0 e^{-ik_\perp\rho_\perp\cos\xi_0}d\xi_0 = 2\pi J_1'(-k_\perp\rho_\perp)$ was used. Substituting (17) and (20) into (16) the perpendicular ion current to 2nd order in $k_\parallel v_{\parallel 0}/\Omega_i$ can be written as

$$\mathbf{J} = i\mathbf{k}\times\hat{\mathbf{b}}c\int\Bigg[-2\pi\frac{\Omega_i^2\rho_\perp}{k_\perp}J_1(-k_\perp\rho_\perp)\delta F - 4\pi\frac{\delta B_\parallel}{B_0}\frac{\Omega_i^2}{k_\perp^2}\mu_0 J_1^2(-k_\perp\rho_\perp)\,\partial_{\mu_0}F_0|_{v_{\parallel 0}} + 8\pi\frac{\delta B_\parallel}{B_0}\frac{k_\parallel^2}{k_\perp^2}\mu_0 v_{\parallel 0}^2 J_1'^2(-k_\perp\rho_\perp)\,\partial_{\mu_0}F_0|_E\Bigg]dv_{\parallel 0}d\mu_0. \tag{21}$$

This is the perturbed ion current, due to magnetic compressional perturbation $\delta B_\parallel$, in terms of the gyrocenter distribution function δF with up to second order correction in $k_\parallel v_{\parallel 0}/\Omega_i$.

The ion current given in Eq. (21) will now be used to derive the dispersion relation for the mirror mode assuming a bi-Maxwellian ion distribution, and co-planar magnetic perturbation such that $\mathbf{k}\cdot\delta\mathbf{B}_\perp = k_\perp\delta B_\perp$, yielding

$\delta B_\perp = -\delta B_\parallel k_\parallel / k_\perp$. The Ampere's law, in the Darwin's approximation, can then be written as

$$i\mathbf{k}\times\delta B_\parallel\hat{\mathbf{b}}\left(1+\frac{k_\parallel^2}{k_\perp^2}\right)= -\frac{4\pi}{c}i\mathbf{k}\times\hat{\mathbf{b}}c\int\Bigg[-2\pi\frac{\Omega_i^2\rho_\perp}{k_\perp}J_1(-k_\perp\rho_\perp)\delta F + 4\pi\frac{\delta B_\parallel}{T_\perp}\frac{\Omega_i^2}{k_\perp^2}\mu_0 J_1^2(-k_\perp\rho_\perp)F_0 + 8\pi\frac{\delta B_\parallel}{T_\perp}\frac{k_\parallel^2}{k_\perp^2}\left(\frac{T_\perp}{T_\parallel}-1\right)\mu_0 v_{\parallel 0}^2 J_1'^2(-k_\perp\rho_\perp)F_0\Bigg]dv_{\parallel 0}d\mu_0, \tag{22}$$

Removing the $i\mathbf{k}\times\hat{\mathbf{b}}$ operator from both sides, performing the integrals over F_0, and rearranging, Eq. (22) becomes

$$\frac{\delta B_\parallel B_0}{4\pi}\left\{1+\frac{k_\parallel^2}{k_\perp^2}\left[1+\left(\beta_\perp-\beta_\parallel\right)\alpha_b\right]+\beta_\perp\left[I_0(z)-I_1(z)\right]e^{-z}\right\} = -2\pi\int\mu_0 B_0\delta F\frac{2}{k_\perp\rho_\perp}J_1(k_\perp\rho_\perp)\frac{B_0}{m}dv_{\parallel 0}d\mu_0. \tag{23}$$

Eq. (23) has a similar form to the low frequency force-balance of Ref. 13, where the off-diagonal "gyroviscous" components of the pressure tensor were included.

The solution of δF is now required to complete the derivation of the disperion relation. The gyrokinetic ion response to coplanar magnetic perturbations is then governed by

$$\partial_t F + \dot{\mathbf{X}}\cdot\partial_{\mathbf{X}}F + \dot{v}_\parallel\partial_{v_\parallel}F = 0, \tag{24}$$

where, the lowest order equations of motion are,[8,9]

$$\dot{\mathbf{X}} = v_\parallel\left(\hat{\mathbf{b}}+\frac{\langle\delta\mathbf{B}_\perp(\mathbf{x})\rangle}{B_0}\right)+\frac{c}{eB_0}\hat{\mathbf{b}}\times\nabla\mu\left\langle\left\langle\delta B_\parallel(\mathbf{x})\right\rangle\right\rangle \tag{25}$$

$$\dot{v}_\parallel = -\frac{\mu}{m}\left(\hat{\mathbf{b}}+\frac{\langle\delta\mathbf{B}_\perp(\mathbf{x})\rangle}{B_0}\right)\cdot\nabla\left\langle\left\langle\delta B_\parallel(\mathbf{x})\right\rangle\right\rangle. \tag{26}$$

$F(\mathbf{X}, v_\parallel, \mu)$ is the gyrocenter distribution function, whose formal dependance on the gyroangle ξ has been removed due to the frequency ordering, and $\mathbf{X}$ is the gyrocenter position, $v_\parallel$ is the parallel gyrocenter velocity, and μ is the gyrocenter magnetic moment. The magnetic moment is a conserved quantity, $\dot{\mu}=0$. The background magnetic field is $B_0\hat{\mathbf{b}}$. The magnetic perturbation parallel to the background magnetic field is designated by $\delta B_\parallel$, and perpendicular to it by $\delta\mathbf{B}_\perp$. The symbols $\langle\langle\ ...\ \rangle\rangle$ stand for the operation $\frac{1}{\pi\rho_\perp^2}\int_0^{2\pi}\int_0^{\rho_\perp}\ ...\ r dr d\xi_0$, i.e. averaging over the surface enclosed by the gyro-orbit of radius $\rho_\perp = v_\perp/\Omega_i = \sqrt{2\mu B_0/m}/\Omega_i$. The first term of equation (25) corresponds to the velocity along the total magnetic field (unperturbed + perturbed), and the second term corresponds to the perturbed magnetic gradient drift. Equation (26) describes the mirror force along the total magnetic field, due to the parallel magnetic perturbation.

Linearizing the gyrokinetic equation (24), and using a bi-Maxwellian distribution for F_0, the linear response of the ion gyrocenters is

$$\delta F = -\frac{\mu\delta B_\parallel}{T_\parallel}\frac{2}{k_\perp\rho_\perp}J_1(k_\perp\rho_\perp)\left(1-\frac{\omega}{k_\parallel}\frac{1}{\omega/k_\parallel - v_\parallel}\right)F_0. \tag{27}$$

Substituting into Eq. (23) and integrating, the right hand side of Eq. (22) becomes,

$$-2\pi\int\mu_0 B_0\delta F\frac{2}{k_\perp\rho_\perp}J_1(k_\perp\rho_\perp)\frac{B_0}{m}dv_{\parallel 0}d\mu_0 = = \frac{\delta B_\parallel B_0}{4\pi}\frac{T_\perp}{T_\parallel}\beta_\perp\left[1+\zeta_0 Z(\zeta_0)\right]\left[I_0(z)-I_1(z)\right]e^{-z}. \tag{28}$$

Thus, after cancelling the magnetic perturbation from both sides of Eq. (22), the dispersion relation as given by Eq. (6) is recovered.

III. DISCUSSION

It has been demonstrated that keeping the lowest order in $k_\parallel v_{th\parallel}/\Omega_i$ in the gyrocenter transformation is sufficient to recover the essential features of the dispersion relation for the mirror instability. Such dispersion relation corresponds to the fully kinetic derivation given in Eq. (6), without the $(k_\parallel/k_\perp)^2\alpha_b\left(\beta_\perp-\beta_\parallel\right)$ term. To recover this term it is necessary to keep up to second order in $k_\parallel v_{th\parallel}/\Omega_i$ in the solution of the gauge function S_1, given in Eq. (14). This term has no affect on the mirror instability threshold, which is still $\beta_\perp(T_\perp/T_\parallel - 1) > 1$, and only slightly decreases the growth rate of the most unstable mode. It may however be more important for the compressional branch of the firehose instability where it is responsible for its destabilization.[14]

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[1] S. Chandrasekhar, A. N. Kaufman, and K. M. Watson, Royal Society of London Proceedings Series A **245**, 435 (1958).
[2] A. Hasegawa, Physics of Fluids **12**, 2642 (1969).
[3] O. A. Pokhotelov, R. Z. Sagdeev, M. A. Balikhin, and R. A. Treumann, Journal of Geophysical Research (Space Physics) **109**, 9213 (2004).

[4] H. Qu, Z. Lin, and L. Chen, Physics of Plasmas **14**, 042108 (2007).
[5] A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007).
[6] T. H. Stix, (1992).
[7] M. Abramowitz and I. A. Stegun, (1964).
[8] A. J. Brizard, Phys. Fluids B **4**, 1213 (1992).
[9] A. J. Brizard, J. Plasma Phys. **41**, 541 (1989).
[10] R. G. Littlejohn, J. Plasma Phys. **29**, 111 (1983).
[11] H. Qin, *Gyrokinetic theory and computational methods for electromagnetic perturbations in tokamaks*, Phd. thesis, Princeton University (1998).
[12] H. Qin and W. M. Tang, Physics of Plasmas **11**, 1052 (2004).
[13] C. Z. Cheng and J. R. Johnson, J. Geophys. Res. **104**, 413 (1999).
[14] A. Hasegawa, Rev. Geophys. **9**, 703 (1971).

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