Princeton Plasma Physics Laboratory

PPPL- 4856

PPPL- 4856

On Variational Methods in the Physics of Plasma Waves

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March, 2013

Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

On variational methods in the physics of plasma waves

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(Dated: March 5, 2013)

A first-principle variational approach to adiabatic collisionless plasma waves is described. The focus is made on one-dimensional electrostatic oscillations, including phase-mixed electron plasma waves (EPW) with trapped particles, such as Bernstein-Greene-Kruskal modes. The well known Whitham's theory is extended by an explicit calculation of the EPW Lagrangian, which is related to the oscillation-center energies of individual particles in a periodic field, and those are found by a quadrature. Some paradigmatic physics of EPW is discussed for illustration purposes.

PACS numbers: 52.35.Mw, 52.35.Sb, 52.35.Fp, 45.20.Jj

I. INTRODUCTION

Studying waves has always been the bread and butter of plasma physics and, as such, requires periodic rethinking of its basic methodologies, especially because the ease of the standard Vlasov-Maxwell (VM) formulation [1] applies only to a limited class of problems. Traditionally, waves are treated as particular solutions of field equations, so studying them is often understood as formal exploration of asymptotics. But there is a limit to how much progress can be made through sacrificing physics to calculations, and sticking to *ad hoc* techniques is hardly promising in the long run. An alternative strategy would be to develop a theoretical framework first that would operate with conceptual blocks rather than specific equations. The variational approach comes in particularly handy here, as it permits one to select the level of detail *before* formulating any equations explicitly. Once variables of interest have been chosen, they can replace the traditional field variables in the plasma Lagrangian, so asymptotic dynamics is yielded independently from, rather than as a corollary of, exact dynamics. Not everything can be described this way; but certain theorems can be obtained that answer many physics questions rigorously, concisely, and often more generally than one could hope for within the standard reductionist approach [2].

The master wave equations yielded by the variational approach are generally known from Whitham's theory and its developments [3–8], but explicit Lagrangians have been available, at best, *ad hoc* [9–15]. This is particularly a concern for nonperturbative nonlinearities, where the variational formulation could, in fact, make most impact. An example here are periodic Bernstein-Greene-Kruskal (BGK) modes [16] or, more generally, waves with trapped particles (WTP), whose dynamics is often counterintuitive and remains controversial [17–19]. A consistent, first-principle formulation is thus needed to standardize the Lagrangian description of plasma waves, not necessarily in full detail but, at least, conceptually.

This paper aims to present an elementary tutorial on how to apply the variational formulation to collisionless-plasma waves in a standardized manner. The presentation is based on the results reported in Refs. [2, 17–21] and some earlier publications [22] and shows how a *practical*, albeit approximate, wave Lagrangian is constructed explicitly and concisely. As it turns out, the problem can be reduced to finding the so-called oscillation-center (OC) energies of individual particles in a strictly periodic field, and that is readily done by a quadrature both for passing and trapped trajectories. Then a variety of wave effects can be deduced straightforwardly, much along the lines of Whitham's theory, which we also revisit briefly for (relative) completeness. Note that we are only after the wave backbone dynamics, so, for clarity, dissipation is neglected (but see, e.g., Refs. [2, 23]). Discussed below will be only the very basic aspects of the theory and selected paradigmatic examples, namely, electrostatic one-dimensional (1D) oscillations. The specific focus is made on electron plasma waves (EPW) and phase-mixed electron WTP in particular. Note that surveying the (prohibitively extensive) relevant literature, especially of applied nature, is deliberately withheld here, as dictated by the format of this work. For historical background and applications, one is instead referred to the preceding papers [2, 17–21] and Ref. [24], where related theories are also reviewed more formally.

The paper is organized as follows. In Sec. II, we revisit some prerequisite concepts such as the Lagrangian OC formalism for single particle motion, yet only to the extent that is necessary for our purposes. In Sec. III, we apply this formalism toward explicitly calculating the Lagrangian density of a nondissipative plasma wave, allowing for nonlinearities and interaction with trapped particles in particular. In Sec. IV, we place our model into the context of Whitham's theory and revisit some general inferences from that, namely, regarding the wave transport, nonlinear group velocity, and modulational stability. In Sec. V, we illustrate how to apply the Lagrangian formulation to linear EPW. In Sec. VI, we extend the theory to WTP and show how it elucidates their dynamics and nonlinear dispersion. Some supplementary calculations are included as appendixes. Extension of the (largely known previously, but reworked) results that are presented in this paper can also be found in the aforementioned references [2, 17–21] and publications cited therein.

II. SINGLE PARTICLE MOTION

A. Adiabatic approximation

Basic equations. — In this paper, we assume dealing with a 1D electrostatic wave with the field $E = -\partial_x \varphi(t,x)$ oscillating at some frequency ω and wave number k, yet to be defined precisely (Sec. III B). For given $\varphi(t,x)$, the dynamics of each particle is governed by the least action principle,

$$\delta S = 0, \tag{1}$$

where $S \doteq \int L\, dt$ [25], L is a Lagrangian that can be expressed as a function of the physical coordinate x and $v \doteq \dot{x}$ [26, Sec. 1.5],

$$L(t,x,v) = mv^2/2 - e\varphi(t,x), \tag{2}$$

and m and e are the particle mass and charge. Equation (1) then takes the form $\delta_x S = 0$ and yields an Euler equation [27, Sec. 2] in the form

$$m\ddot{x} = eE(t,x). \tag{3}$$

OC variables. — We will assume that the field amplitude, ω, and k vary in space and time "slowly" (a term we are about to explain); then we can separate the average, or OC, motion with the coordinate and velocity (X,V) from the quiver motion, $(\tilde{x}, \tilde{v})$,

$$x = X + \tilde{x}, \quad v = V + \tilde{v}. \tag{4}$$

Such separation relies on having a small parameter,

$$\wp \doteq (\Omega\tau)^{-1} \ll 1, \tag{5}$$

where Ω is the frequency of the particle oscillations is the reference frame traveling with velocity V, called the OC rest frame, and τ is the characteristic time scale at which the wave parameters vary *in that frame*. Specifically, the quiver motion can be expressed as an asymptotic series,

$$\tilde{x}(t,X,V) = \mathcal{O}(1) + \mathcal{O}(\wp) + \mathcal{O}(\wp^2) + \dots, \tag{6}$$

and same for $\tilde{v}$. Keep in mind, however, that such a series generally diverges, so only few first terms are meaningful. Neglecting the difference between the true solution and its truncated asymptotic series constitutes the so-called adiabatic approximation. (The term "adiabatic" refers to the approximate conservation of a corresponding invariant, to be discussed shortly. For essentially nonadiabatic effects, see, e.g., Ref. [28] and references therein.) Below, only the leading term in Eq. (6) will be retained.

Time-averaged Lagrangian. — On time scales large compared to Ω^{-1}, only the time-average part of L contributes to the action integral. Hence, one obtains a new variational principle,

$$\delta \mathcal{S} = 0, \tag{7}$$

for the reduced action $\mathcal{S} \doteq \int \mathcal{L}\, dx\, dt$, where $\mathcal{L} \doteq \langle L \rangle$ plays the role of the OC-motion Lagrangian. Specific dynamic equations that hence emerge are derived as follows.

B. Weak interaction

Ponderomotive potential. — Suppose first that the wave is approximately sinusoidal, and $k\tilde{x} \ll 1$, so the particle oscillations are linear. Then, $\Omega = |\omega - kV|$, and

$$\tilde{x} = -eE(t,X)/[m(\omega - kV)^2], \tag{8}$$

$$\tilde{v} = -i(\omega - kV)\tilde{x}. \tag{9}$$

Substituting the Taylor expansion $\varphi(t,x) \approx \varphi(t,X) - \tilde{x}E(t,X)$ in Eq. (2), we hence obtain

$$\mathcal{L} = mV^2/2 + m\langle\tilde{v}\rangle^2/2 + e\langle\tilde{x}E(t,X)\rangle. \tag{10}$$

A combination of Eqs. (8)-(10) then yields

$$\mathcal{L}(t,X,V) = mV^2/2 - \Phi(t,X,V), \tag{11}$$

where Φ is the so-called ponderomotive potential,

$$\Phi = \frac{e^2|E|^2}{4m(\omega - kV)^2}. \tag{12}$$

This leads to the Lagrange-Euler equation in the form $d_t(\partial_V \mathcal{L}) = \partial_X \mathcal{L}$, or, more specifically,

$$(m - \partial^2_{VV}\Phi)\,\dot{V} = -\partial_X \Phi + \partial^2_{tV}\Phi + V\,\partial^2_{XV}\Phi. \tag{13}$$

[In view of Eq. (13), the notorious concept of the ponderomotive force is generally ambiguous, whereas the ponderomotive potential is well defined as a part of $\mathcal{L}$.]

A number of comments are due at this point.

(i) The coefficient $m - \partial^2_{VV}\Phi \doteq m_\parallel$ in Eq. (13) acts as an effective longitudinal mass, which is generally a function of (t,X,V). Interestingly, the difference between $m_\parallel$ and m here is not a relativistic effect, and it does not vanish even when the wave is homogeneous and stationary. In the latter case, it can affect the particle response to quasistatic fields additional to the wave (say, a constant electric field or gravity). Also note that the ponderomotive modification of the longitudinal mass is known in other systems too. In some cases, $m_\parallel$ can even become negative, causing collective instabilities [29–31].

(ii) The ponderomotive potential is often derived by expanding Eq. (3) around X, substituting Eq. (8), and then time-averaging the equation. It is, however, difficult to do correctly, unless $\partial_V \Phi$ is neglected (Appendix A). The difficulty is seen from the complexity of Eq. (13) and is only aggravated beyond the 1D model.

(iii) Although it happens that $\Phi = m\langle\tilde{v}^2\rangle/2$ *here*, it is generally incorrect, contrary to a popular presumption, to consider the ponderomotive potential as the average energy of wave-induced oscillations. The representation that actually extends to all linear waves is [32, 33]

$$\Phi = -\alpha|E|^2/4, \tag{14}$$

where $\alpha E \doteq e\tilde{x}$ is the dipole moment associated with the particle oscillations around the OC trajectory $X(t)$, and α acts as the particle linear polarizability [34]. In other words, Φ is understood as the average potential energy of dipole interaction (cf. Ref. [35, Sec. 4.8]; an additional factor 1/2 stems from averaging over harmonic oscillations). Note also that Eq. (14) underlies the so-called K-χ theorem and can be connected with the *wave frequency shift* due to the interaction with an individual particle. See Ref. [32] and references therein for details and, e.g., Ref. [36] for a related quantum model.

OC Hamiltonian. — We can also cast the above equations in a canonical form, by introducing the OC momentum $P \doteq \partial_V \mathcal{L}$. Explicitly, the latter is given by

$$P = mV + \Delta P, \quad \Delta P \doteq -\partial_V \Phi, \tag{15}$$

where $\Delta P = \mathcal{O}(E^2)$ is known as the ponderomotive momentum. The OC Hamiltonian is then introduced in a usual manner, namely, $\mathcal{H} \doteq PV - \mathcal{L}$. Hence,

$$\mathcal{H}(t, X, P) = P^2/(2m) + \Phi(t, X, P), \tag{16}$$

where we neglected the term $\mathcal{O}(E^4)$ due to ΔP in Φ,

$$\Phi(t, X, P) \approx \frac{e^2|E|^2}{4m(\omega - kP/m)^2}. \tag{17}$$

Since $\dot{\mathcal{H}} = \partial_t \mathcal{H} = \partial_t \Phi$, the OC energy is conserved in a stationary wave. This can be attributed to the fact that, in the extended phase space, $\mathcal{H}/\omega$ serves as the conserved action, or the adiabatic invariant [27, Chap. 7], associated with oscillations at the (fixed) frequency ω [37, 38].

C. General interaction

When the particle average velocity V approaches the phase velocity, the approximation of linear oscillations [Eqs. (8) and (9)] breaks down. Then, one needs a nonlinear theory, which can be constructed as follows.

Homogeneous stationary wave. — First, consider a strictly homogeneous stationary wave with a potential of the form $\varphi(x - ut)$. The function $\varphi(x)$ need not be sinusoidal, but is assumed to have a spatial period $\lambda \doteq 2\pi/k$ and a single minimum per period. The temporal period of the potential in the laboratory frame, $2\pi/\omega$, then equals λ/u, so $u = \omega/k$; therefore, u can be understood as the phase velocity. In the wave rest frame, the potential is stationary, so particles conserve their energy,

$$\varepsilon = mw^2/2 + e\varphi, \tag{18}$$

where $w \doteq v - u$. Hence, depending on whether ε is larger or smaller than the separatrix value, ε_* (equal to the maximum $|e\varphi|$), a particle either has nonzero average velocity, $\langle w \rangle$, or is confined to a local potential well. The particle is then called passing or trapped, respectively.

Let us now allow the wave to evolve, slowly, and discuss how this affects particles of both types, for simplicity ignoring transitions through the separatrix [39].

Passing particles. — Consider a passing particle first. As its unperturbed phase space is effectively a cylinder [due to the periodicity of $\varphi(x)$], it is convenient to describe it in terms of (canonical) angle-action coordinates (θ, J), where [27, Sec. 52]

$$J \doteq \frac{1}{2\pi} \int_0^\lambda p(x)\, dx, \tag{19}$$

and $p = mw$ is the particle momentum in the wave rest frame. Those translate into the the OC canonical coordinate X and momentum P in the laboratory frame,

$$P = mu + kJ \operatorname{sgn}(w), \tag{20}$$

governed by the Hamiltonian [17]

$$\mathcal{H} = \varepsilon + Pu - mu^2/2. \tag{21}$$

The corresponding canonical equations are $\dot{X} = \partial_P \mathcal{H}$ and $\dot{P} = -\partial_X \mathcal{H}$. In particular, P is conserved (is an adiabatic invariant) when $\mathcal{H}$ is independent of X, i.e., when the wave is homogeneous. The applicability condition for Eq. (21) has the form (5), where Ω is now understood as the canonical frequency,

$$\Omega = \varepsilon'(J), \tag{22}$$

a definition still consistent with $\Omega = |\omega - kV|$. The results of Sec. II B are reproduced from here, as a limit, when J is much larger than the separatrix action, J_*.

Trapped particles. — Undergoing bound oscillations, trapped particles also can be assigned angle-action variables (θ, J). Although the corresponding J cannot be an *analytic* continuation of the passing-particle action (as the singularity at the separatrix is essential), we can at least make $J(\varepsilon)$ continuous and invertible. This is done by taking the trapped-particle action J to be

$$J \doteq \frac{1}{4\pi} \oint p(x)\, dx, \tag{23}$$

where the integration is performed over the bounce period. As the coefficient here is half of its standard value [27, Sec. 52], the associated canonical frequency (22) will be twice the true bounce frequency. Figure 1 in Ref. [18] makes these definitions transparent.

If a wave slowly evolves [e.g., $J_*(t)$ changes], a once-trapped particle remains trapped and conserves its J, as long as $J_*(t) > J$ [40]. The corresponding bounce oscillations are described by the Hamiltonian [17]

$$\mathcal{H} = \varepsilon - mu^2/2, \tag{24}$$

for which the validity condition is, again, Eq. (5). This yields $\dot{J} = 0$ and $\dot{\theta} = \Omega(J)$, with Ω given by Eq. (22). (The term $mu^2/2$ has no effect on these equations, but it depends on ω and k and thus affects the wave dynamics, as will be discussed below.) We will call this $\mathcal{H}$ an OC Hamiltonian, too, because the time dependence associated with the wave rapid oscillations is mapped out from Eq. (24). Keep in mind, however, that the OC physical coordinate X is now tied to the wave phase, and $V = u$, so X cannot serve as an independent coordinate.

Area function. — Both for passing and trapped particles, the function $J(\varepsilon)$ is the phase space area encircled by the trajectory of a particle with given ε in the wave rest frame, up to a coefficient. (Similarly, $P\lambda$ is the phase space area encircled by a passing particle in the laboratory frame.) This "area function", as it is called below, entirely determines *all* the properties of the particle adiabatic trajectory, including both OC Hamiltonians $\mathcal{H}$ [Eqs. (21) and (24)] and, in particular, Ω [Eq. (22)] and P [Eq. (20)]. Considering Eqs. (19) and (23), one can hence say that those are found by a quadrature [as w is known from Eq. (18)] for any given $\varphi(x)$.

III. WAVE LAGRANGIAN DENSITY

Application of the OC formalism often renders collisionless wave physics transparent even without the VM system *per se*. We discuss basics of this approach below, by casting the wave dynamics in the appropriate format and considering sample applications. In a related context, supplemental paradigmatic calculations are also revisited in Appendix B, merely for (relative) completeness of our presentation.

A. General model

Basic equations. — Let us approach the plasma collective dynamics in the same manner as we did for the single particle dynamics. We again start out with the least action principle, Eq. (1), except now the action is

$$S = \int \mathscr{L}\, dx\, dt. \tag{25}$$

Here $\mathscr{L}$ is the plasma Lagrangian density [26, Sec. 11.5],

$$\mathscr{L} = \hat{\mathscr{L}} + \sum_i \delta(x - x_i)\, L_i(t, x, v_i), \tag{26}$$

and $\hat{\mathscr{L}} \doteq (\partial_x \varphi)^2/(8\pi)$ is the field Lagrangian density. The sum in Eq. (26) is taken over individual particles, x_i and v_i are the corresponding coordinates and velocities, and L_i are of the form (2). We study 1D dynamics; unit transverse area is assumed to simplify notation.

The particle motion, Eq. (3), follows as usual from $\delta_{x_i} S = 0$, since $\varphi(t, x)$ is independent of x_i. A partial differential equation (PDE) for the electrostatic field is obtained from $\delta_\varphi S = 0$. Specifically, in conjunction with Eq. (2), the latter leads to Poisson's equation,

$$\partial^2_{xx}\varphi(t, x) = -4\pi\rho(t, x), \tag{27}$$

where $\rho(t, x) = \sum_i e_i \delta(x - x_i(t))$ is the charge density.

For simplicity, let us assume single species for now. Let us also replace the discrete sum in Eq. (26) with an average over a continuous distribution, $\mathsf{f}_0(\mathsf{z}_0)$, where z_0 are some tags assigned to individual particles, say, their canonical coordinates at $t = 0$. This leads to the so-called Low's Lagrangian density [41],

$$\mathscr{L} = \hat{\mathscr{L}} + \int \delta(x - x_t(\mathsf{z}_0))\, L(t, x, v_t(\mathsf{z}_0))\, \mathsf{f}_0(\mathsf{z}_0)\, d\mathsf{z}_0.$$

The field equation then is again Eq. (27), yet with

$$\rho = e \int \delta(x - x_t(\mathsf{z}_0))\, \mathsf{f}_0(\mathsf{z}_0)\, d\mathsf{z}_0, \tag{28}$$

and the particle motion is derived via $\delta_{x_t} S = 0$, where $x_t(\mathsf{z}_0)$ is the trajectory starting from z_0, and $v_t \doteq \dot{x}_t$.

Reduced model. — Consider now developing a reduced theory by extracting the slow dynamics from the above equations. As in Sec. II, this can be done by locally averaging the Lagrangian density $\mathscr{L}$ over time, namely, over all characteristic periods in the system that are relevant. This includes the wave period, but also the largest characteristic canonical period of particle oscillations, $2\pi/\Omega_c$. To be able to use the adiabatic model for particles, we must then assume $\Omega_c \tau \gg 1$ [Eq. (5)]. In cold plasma limit, $\Omega_c \approx \omega$ for most particles, so we need to require only $\omega\tau \gg 1$ (and averaging over the field period is enough). Otherwise, however, additional applicability conditions must be satisfied.

Suppose that there are resonant particles in the system, i.e., ones that are trapped or passing close to the separatrix. (More formally, this means that J does not exceed a few J_*.) They generally have $\Omega \sim \Omega_\star$, where

$$\Omega_\star \doteq (|e|\mathscr{E}k/m)^{1/2} \tag{29}$$

is the characteristic frequency at the bottom of a (locally parabolic) wave trough [1, Sec. 8.6], and $\mathscr{E}$ is the electric field amplitude; $k > 0$ is assumed for clarity. Then, a necessary condition under which such particles can be accommodated within our model is

$$\wp_\star \doteq (\Omega_\star \tau)^{-1} \ll 1. \tag{30}$$

(Note, in particular, that sideband instabilities [42] are thereby ignored.) Yet Eq. (30) is not sufficient, as the adiabaticity condition (5) is always violated at the separatrix. We thus will assume one of the following: (i) There are no particles in the immediate vicinity of the separatrix. This is a valid assumption for certain scenarios of WTP formation that produce deeply-trapped distributions (see, e.g., Refs. [20, 43]). (ii) The separatrix wiggles little, so only few particles violate the condition (5), so nonadiabatic effects are negligible. This is valid, e.g., for calculating the dispersion of homogeneous stationary waves (Sec. VI B). With few exceptions, such as Ref. [44], the assumption (ii) is also commonly (albeit often tacitly) accepted in literature.

Dissipation. — As collisions are ignored, dissipative effects within the model can stem only from wave interaction with resonant particles. But the latter are allowed only under the assumption (30), in which case their distribution is automatically phase-mixed, so there is no collisionless dissipation either [45–47]. That said, however, any dissipation that is local and linear can be accommodated later as a perturbation [2]. Strong dissipation, in contrast, would negate the wave concept altogether, so it is irrelevant in the context of this paper.

B. Field Lagrangian density

Let us start out with the field Lagrangian density, $\hat{\mathscr{L}}$ and, as in Sec. II, adopt the approximation for φ that is of the zeroth order in $\wp$. This leads to the eikonal approximation,

$$\varphi = \bar{\varphi} + \frac{1}{2}\sum_h \left(\varphi_h\, e^{ih\xi} + \varphi_h^*\, e^{-ih\xi}\right), \tag{31}$$

where the number of Fourier harmonics that need to be retained depends on the desired accuracy of the model. The phase $\xi(t,x)$ is considered a rapid variable, but its gradients must be slow functions, which then serve to define the local frequency and wave number:

$$\omega \doteq -\partial_t \xi(t,x), \quad k \doteq \partial_x \xi(t,x). \tag{32}$$

Also assumed slow are the harmonic amplitudes $\varphi_h(t,x)$ and the quasistatic potential $\bar{\varphi}(t,x)$, if any. [Strictly speaking, $e\bar{\varphi}$ must be added to Eqs. (16), (21), and (24).] Then, averaging $\hat{\mathscr{L}}$ over time gives

$$\begin{aligned}\langle \hat{\mathscr{L}}\rangle^{(t)} = \langle \hat{\mathscr{L}}\rangle^{(x)} = \langle \hat{\mathscr{L}}\rangle^{(t,x)} \\ = \frac{(\partial_x\bar{\varphi})^2}{8\pi} + \sum_h \frac{k^2h^2|\varphi_h|^2}{16\pi} \risingdotseq \hat{\mathfrak{L}},\end{aligned} \tag{33}$$

where $\langle\ldots\rangle^{(t)}$ denotes time-averaging, $\langle\ldots\rangle^{(x)}$ denotes space-averaging, and $\langle\ldots\rangle^{(t,x)}$ denotes both.

The particle contribution into the plasma action can be considered similarly but depends on whether particles are passing or trapped. We consider passing particles in Sec. III C and trapped particles in Sec. III D.

C. Contribution of passing particles

Let us assume, until Sec. III D, that the wave interacts with passing particles only and switch to their OC coordinates in Eq. (26). Time-averaging gives

$$\begin{aligned}&\int \langle \delta(x - X_t - \tilde{x}_t)\, L(t,x,v_t)\rangle^{(t)}\, dx \\ &= \int \langle \delta(x - X_t)\, L(t, x+\tilde{x}_t, V_t + \tilde{v}_t)\rangle^{(t)}\, dx \\ &= \int \delta(x - X_t)\, \langle L(t, x+\tilde{x}_t, V_t + \tilde{v}_t)\rangle^{(t)}\, dx \\ &= \int \delta(x - X_t)\, \mathcal{L}(t,x,V_t)\, dx.\end{aligned} \tag{34}$$

After substituting also $\mathcal{L} = PV - \mathcal{H}$, we then arrive at an equivalent time-averaged $\mathscr{L}$ of the form

$$\begin{aligned}\langle \mathscr{L}\rangle^{(t)} = \hat{\mathfrak{L}} - \mathscr{H} \\ + \int \delta(x - X_t(\mathsf{z}_0))\, P_t(\mathsf{z}_0)\dot{X}_t(\mathsf{z}_0)\, \mathsf{f}_0(\mathsf{z}_0)\, d\mathsf{z}_0,\end{aligned} \tag{35}$$

where, together with z_t as a shortened notation for (X_t, P_t), we introduced

$$\mathscr{H} \doteq \int \delta(x - X_t(\mathsf{z}_0))\, \mathcal{H}(t, \mathsf{z}_t(\mathsf{z}_0))\, \mathsf{f}_0(\mathsf{z}_0)\, d\mathsf{z}_0. \tag{36}$$

Requiring $\delta_{X_t} S = 0$ and $\delta_{P_t} S = 0$ leads to canonical equations, as usual [27, Sec. 40]:

$$\dot{X}_t = \partial_{P_t}\mathcal{H}(t,\mathsf{z}_t), \quad \dot{P}_t = -\partial_{X_t}\mathcal{H}(t,\mathsf{z}_t). \tag{37}$$

Their solution determines a mapping between the initial phase space variables z_0 and the instantaneous phase space coordinates z at time t; i.e., $\mathsf{z}_t : \mathsf{z}_0 \mapsto \mathsf{z}$. We can then map the distribution f_0 into the instantaneous canonical distribution f. Phase space conservation yields

$$\mathsf{f}(t,\mathsf{z}) = \mathsf{f}_0(\mathsf{z}_t^{-1}(\mathsf{z})), \quad \mathsf{f}(t, \mathsf{z}_t(\mathsf{z}_0)) = \mathsf{f}_0(\mathsf{z}_0), \tag{38}$$

meaning that the distribution is conserved along z_t (Appendix B 1). In particular, differentiating this with respect to time leads to the OC Vlasov equation,

$$\partial_t \mathsf{f} + (\partial_P \mathcal{H})\,(\partial_X \mathsf{f}) - (\partial_X \mathcal{H})\,(\partial_P \mathsf{f}) = 0. \tag{39}$$

The field equation (or equations; see below) comes from $\delta \int \langle \mathscr{L}\rangle^{(t)}\, dx\, dt = 0$. The variation is taken at fixed z_t, so the third term in Eq. (35) does not contribute. Thus, one can use $\delta \int \mathscr{R}\, dx\, dt = 0$ instead, where $\mathscr{R} \doteq \langle \hat{\mathscr{L}}\rangle^{(t)} - \mathscr{H}$ is understood as the Routhian density [48]. But notice that the integral of $\mathscr{R}$ also permits averaging over the spatial volume, just like averaging over time that was done earlier. Hence, one obtains a new variational principle,

$$\delta \mathfrak{S} = 0, \tag{40}$$

for the reduced action

$$\mathfrak{S} \doteq \int \mathfrak{L}\, dx\, dt, \tag{41}$$

where $\mathfrak{L} \doteq \langle \mathscr{R}\rangle^{(x)}$. The function $\mathfrak{L}$ serves as the wave Lagrangian density, and is given by

$$\mathfrak{L} = \hat{\mathfrak{L}} - \langle \mathscr{H}\rangle. \tag{42}$$

To calculate $\langle \mathscr{H}\rangle \doteq \langle \mathscr{H}\rangle^{(x)}$, let us rewrite Eq. (36) as

$$\mathscr{H} = \int \delta(x - X)\, \mathcal{H}(t,\mathsf{z})\, \mathsf{f}(t,\mathsf{z})\, d\mathsf{z}. \tag{43}$$

Since $d\mathsf{z} = dX\, dP$, the integration over X is straightforward, and one gets

$$\langle \mathscr{H}\rangle = \int_{-\infty}^{\infty} \mathcal{H}(t,x,P)\, \mathsf{F}(t,x,P)\, dP. \tag{44}$$

Here $\mathsf{F}(t,x,P) \doteq \langle \mathsf{f}(t,x,P)\rangle^{(x)}$ is the local average of f over the wave spatial period, so one may recognize $\langle \mathscr{H} \rangle$ as the locally-averaged OC energy density. As a side note, the OC space-averaged density, n, can be written as

$$n(t,x) = \int_{-\infty}^{\infty} \mathsf{F}(t,x,P)\, dP. \tag{45}$$

As another side note, the Vlasov equation for $\mathsf{F}(t,X,P)$ is obtained from space-averaging of Eq. (39). Since $\mathcal{H}$ is a slow function of space, this yields

$$\partial_t \mathsf{F} + (\partial_P \mathcal{H})\,(\partial_X \mathsf{F}) - (\partial_X \mathcal{H})\,(\partial_P \mathsf{F}) = 0. \tag{46}$$

Combining Eqs. (42), (33), and (44), we now obtain an explicit representation of the wave Lagrangian density:

$$\mathfrak{L} = \frac{(\partial_x \bar{\varphi})^2}{8\pi} + \sum_h \frac{k^2 h^2 |\varphi_h|^2}{16\pi} - \sum_s \int_{-\infty}^{\infty} \mathcal{H}_s(t,x,P)\,\mathsf{F}_s(t,x,P)\, dP, \tag{47}$$

where contributions of multiple species s were reintroduced. Generalization to electromagnetic waves is also straightforward [17].

D. Contribution of trapped particles

Consider now applying the adiabatic model to WTP. In this case, the model requires Eq. (30), so the waves must be phase-mixed, i.e., similar to periodic BGK modes (Appendix B 3). The difference from BGK modes, however, is that we allow waves to evolve, in both time and space. We proceed as in Sec. III C but, this time, omit the passing-particle contribution for brevity [except in the final answer, Eq. (54)]. The initial coordinates are hence (θ_0, J_0), but, as the distribution f_0 is phase-mixed, one can consider it as a function of J_0 alone. On the other hand, this function may also vary from one trapping island, or a bucket b, to another, so one gets

$$\langle \mathscr{L} \rangle^{(t)} = \hat{\mathfrak{L}} - \mathscr{H} + \sum_b \int \delta(x - X_t(\xi_b)) \times J_t(J_0)\dot{\theta}_t(J_0)\,\mathsf{f}_{b0}(J_0)\, 2\pi\, dJ_0.$$

Here X_t is bth bucket's location at time t, expressed in terms of the corresponding wave phase, ξ_b. Let us yet think of X_t as a (step-like) function of the continuous phase ξ, so we obtain

$$\langle \mathscr{L} \rangle^{(t)} = \hat{\mathfrak{L}} - \mathscr{H} + \int \delta(x - X_t(\xi)) \times J_t(J_0)\dot{\theta}_t(J_0)\,\mathsf{f}_0(\xi, J_0)\, dJ_0\, d\xi,$$

and, similarly,

$$\mathscr{H} = \int \delta(x - X_t(\xi))\, \mathcal{H}(t, X_t(\xi), J_0)\, \mathsf{f}_0(\xi, J_0)\, dJ_0\, d\xi.$$

The canonical equations are then derived as usual,

$$\dot{\theta}_t = \Omega, \quad \dot{J}_t = 0, \tag{48}$$

and particle conservation hence takes the form

$$\mathsf{f}(\xi, J) = \mathsf{f}_0(\xi, J). \tag{49}$$

Along the lines of Sec. III C, let us now rewrite $\mathscr{H}$ as an integral over the instantaneous phase space,

$$\mathscr{H}(t,x) = \int \delta(x - X_t(\xi))\, \mathcal{H}(t,x,J)\, \mathsf{f}(\xi, J)\, dJ\, d\xi,$$

and perform averaging over some spatial interval Λ, which is large compared to λ yet small compared to the inhomogeneity scale. From $\langle \mathscr{H} \rangle = \Lambda^{-1} \int \mathscr{H}\, dx$, one gets

$$\langle \mathscr{H} \rangle = k \int_0^{J_*} \mathcal{H}(t,x,J)\, \mathsf{F}(\xi(t,x), J)\, dJ, \tag{50}$$

where $\mathsf{F} \doteq (k\Lambda)^{-1} \int_\Xi \mathsf{f}\, d\xi$, and Ξ is the phase interval corresponding to those Λ/λ buckets that are contained in Λ. [The integration limit J_* in Eq. (50) is symbolic, as the trapped distribution distribution "close enough" to the separatrix is assumed zero in any case (Sec. III A).]

Since the length of Ξ is equal to $k\Lambda$, one can understand F as the local average of f. Also keep in mind that, like f, the function F can depend on (t,x) only through ξ (taken, say, at the center of Ξ), as reflected in Eq. (50). For $\mathsf{F}(t,x,J) \doteq \mathsf{F}(\xi(t,x), J)$, this yields

$$\partial_t \mathsf{F} = -\omega\, \partial_\xi \mathsf{F}, \quad \partial_x \mathsf{F} = k\, \partial_\xi \mathsf{F}. \tag{51}$$

Thus, differentiating the local average of Eq. (49) with respect to time leads to the following Vlasov equation:

$$\partial_t \mathsf{F} + u\, \partial_x \mathsf{F} = 0. \tag{52}$$

As a side note, one can as well construct a WTP theory by assuming $\mathsf{F} = \mathsf{F}(t,x,J)$ *if* Eq. (52) is imposed as an additional variational constraint [49]. As another side note, F can be expressed in terms of the locally-averaged trapped-particle density, n, as follows:

$$\int_0^{J_*} \mathsf{F}(\xi, J)\, dJ = \frac{n}{k} \doteqdot \ell(\xi). \tag{53}$$

The right-hand side here, $\ell(\xi)$, is equal to the (local average of) the total number of trapped particles per bucket, divided by 2π; we will call it loading function. It is seen from Eq. (53) that having the trapped-particle distribution homogeneous renders the loading function a constant, a fact to be used in Sec. VI.

Combining Eq. (50) with Eqs. (42), (33), the contribution of passing particles [Eq. (44)], and of multiple species s too, we finally arrive at the following general $\mathfrak{L}$:

$$\mathfrak{L} = \frac{(\partial_x \bar{\varphi})^2}{8\pi} + \sum_h \frac{k^2 h^2 |\varphi_h|^2}{16\pi} - \sum_{\text{pass},\, s} \int_{-\infty}^{\infty} \mathcal{H}_s(t,x,P)\,\mathsf{F}_s(t,x,P)\, dP - \sum_{\text{trap},\, s} \int_0^{J_*} \mathcal{H}_s(t,x,J)\,\mathsf{F}_s(\xi,J)\, k\, dJ. \tag{54}$$

[Notice that, when F_s is homogeneous and stationary, the passing and trapped contributions here are remarkably similar, as $|dP| = k\,|dJ|$ due to Eq. (20).] The qualitative distinction of Eq. (54) from Eq. (47) is that now $\mathfrak{L}$ can depend on ξ explicitly, namely, through the trapped-particle distribution. But, of course, $\mathcal{H}_s$ for passing and trapped particles are also very different (Sec. II C).

IV. GENERAL WAVE PROPERTIES

A. Euler-Lagrange equations

The wave Lagrangian densities (47) and (54) both can be represented in the form

$$\mathfrak{L} = \mathfrak{L}(\varphi_1, \varphi_1^*; \varphi_2, \varphi_2^*; \ldots; \xi, \underbrace{-\partial_t\xi}_{\omega}, \underbrace{\partial_x\xi}_{k}). \tag{55}$$

Keep in mind that the dynamics of F is not contained in Eq. (55) [but see Eqs. (46) and (52) instead]; the presence of F in Eq. (47) merely determines parametric dependence of $\mathfrak{L}$ on t and x (or ξ), which below we conceal for brevity. We will also conceal the dependence on $\bar{\varphi}$ and $\partial_x\bar{\varphi}$, as it is similar to that in Eq. (26) and need not be revisited. The independent functions thus retained are φ_h, φ_h^*, and ξ, and Eq. (40) requires that the functional derivative $\mathfrak{S}$ be zero with respect to *each* of them. This leads to the following wave equations.

Dispersion relation. — First, consider varying $\mathfrak{S}$ with respect to harmonic amplitudes, φ_h:

$$\delta\mathfrak{S} = \int (\partial_{\varphi_h}\mathfrak{L})\,\delta\varphi_h\,dx\,dt. \tag{56}$$

Requiring that this be zero for any $\delta\varphi_h$, and similarly for φ_h^*, leads the following equations:

$$\partial_{\varphi_h}\mathfrak{L} = 0, \quad \partial_{\varphi_h^*}\mathfrak{L} = 0. \tag{57}$$

The advantage of Eqs. (57) [say, compared to Poisson's equation, Eq. (27)] is that they are *non-differential* and thus allow one, in principle, to explicitly obtain $\omega = \omega(k; \varphi_1, \varphi_1^*, \ldots)$ without integrating any PDEs. By definition, the latter constitutes the nonlinear dispersion relation (NDR), so Eqs. (57) can be referred as such.

Action conservation. — Now consider varying $\mathfrak{S}$ with respect to the wave phase, ξ:

$$\begin{aligned}\delta\mathfrak{S} &= \int [\mathfrak{L}_\omega\,\delta(-\partial_t\xi) + \mathfrak{L}_k\,\delta(\partial_x\xi) + \mathfrak{L}_\xi\,\delta\xi]\,dx\,dt \\ &= \int (\partial_t\mathfrak{L}_\omega - \partial_x\mathfrak{L}_k + \mathfrak{L}_\xi)\,\delta\xi\,dx\,dt,\end{aligned} \tag{58}$$

where integration by parts was used. (It is hence adopted that, when used as indexes, the tags ω, k, and ξ denote partial derivatives.) Requiring $\delta_\xi\mathfrak{S} = 0$ for all $\delta\xi$ yields

$$\partial_t\mathfrak{L}_\omega - \partial_x\mathfrak{L}_k + \mathfrak{L}_\xi = 0. \tag{59}$$

If $\mathfrak{L}_\xi$ is zero, implying that a wave has no trapped particles or that their distribution is homogeneous, Eq. (59) leads to a conservative equation,

$$\partial_t\mathcal{I} + \partial_x\mathcal{J} = 0, \tag{60}$$

called the action conservation theorem (ACT). Equation (60) can be understood as a continuity equation, so

$$\mathcal{I} \doteq \mathfrak{L}_\omega, \quad \mathcal{J} \doteq -\mathfrak{L}_k \tag{61}$$

are called the density of the wave "action" and the action flux density, correspondingly. Integrating Eq. (60) over the volume yields conservation of the total action [50],

$$I \doteq \int \mathcal{I}\,dx = \text{const.} \tag{62}$$

The normalized action, $I/\hbar$, can be understood as the number of wave photons (plasmons, or, more generally, wave quanta), and $\mathcal{I}/\hbar$ serves as the photon density. For details see Ref. [2], which also puts the wave equations in their most natural form, invariant with respect to variable transformations in arbitrarily curved spacetime. (For additional discussions on the ACT in curved spacetime, see, e.g., Refs. [51–53].)

If $\mathfrak{L}_\xi$ is nonzero, implying inhomogeneous loading of trapped particles, then Eq. (59) yields

$$\partial_t\mathcal{I} + \partial_x\mathcal{J} = \sum_{\text{trap},\,s} \int_0^{J_*} \mathcal{H}_s(t,x,J)\,\partial_x\mathsf{F}_s(t,x,J)\,dJ, \tag{63}$$

where Eqs. (54) and (51) were substituted. Clearly, such a WTP does not conserve its total action. The cause of this is that the trapped-particle distribution propagates as a material wave [Eq. (52)] phase-locked with the field wave, so the two can exchange quanta through resonant interaction. The effect is similar to those described in Refs. [54, 55] and also of entropy waves on magnetohydrodynamic oscillations reported in Refs. [56, 57].

Consistency relation. — A yet another equation is obtained from Eqs. (32) and an identity $\partial^2_{xt}\xi = \partial^2_{tx}\xi$,

$$\partial_t k(t,x) = -\partial_x\omega(t,x). \tag{64}$$

It is called a consistency relation, and additional relations of this type also emerge when one deals with multiple spatial dimensions [2]. Equation (64) too can be cast in the form of a continuity equation; namely,

$$\partial_t k + \partial_x(ku) = 0, \tag{65}$$

where $\omega = ku$ was substituted. The phase velocity, u, is hence seen to act as some flow velocity, while k acts as the density of *something*. This something can be identified as wave crests [3], so Eq. (65) is understood as the crest conservation theorem. Although seemingly trivial, Eqs. (64) and (65) are essential for completing the set of wave equations [Eqs. (57) and (60)]. For instance, see Ref. [19] for its application in numerical simulations.

B. Geometrical optics of simple waves

Simple-wave model. — To simplify the discussion, let us assume from now on that $\mathfrak{L}_\xi = 0$. Let us also assume that a wave is approximately monochromatic, which is often an accurate model even for essentially nonlinear waves [58–60]. Then,

$$\sum_h \frac{k^2 h^2 |\varphi_h|^2}{16\pi} = \frac{\mathscr{E}^2}{16\pi} \mathrel{=\!\!\cdot} \mathscr{A}, \tag{66}$$

where $\mathscr{E} \doteq |E|$ is the amplitude of the wave electric field. The new independent function (to replace φ_1 and φ_1^*) can hence be $\mathscr{E}$ itself, or $\mathscr{A}$, or, even more generally, any other invertible $a = a(\mathscr{E}, \omega, k)$. Thus, we hereupon adopt

$$\mathfrak{L} = \mathfrak{L}(a, \omega, k), \tag{67}$$

where parametric dependence on t and x is allowed too.

Equation (67) happens to constitute the essence of the so-called geometrical-optics (GO) approximation and can, in fact, be used as a *definition* of GO [2], alternative to the standard definition [61–68]. If understood this way, GO becomes a *field theory*, also broadly known as Whitham's theory. Below, we use it to infer some basic properties of waves governed by a Lagrangian density of the form (67), henceforth termed simple waves. Note that, in doing so, we closely follow Refs. [2, 3, 19].

Basic notation. — Let us start with summarizing Whitham's equations for simple waves:

$$\mathfrak{L}_a = 0, \tag{68}$$
$$\partial_t k + \partial_x \omega = 0, \tag{69}$$
$$\partial_t \mathfrak{L}_\omega - \partial_x \mathfrak{L}_k = 0. \tag{70}$$

As an index, a is used here to denote the corresponding partial derivative, just like ω and k. Same will be assumed for other amplitude-related quantities [Eq. (73)]. Let us also adopt the same convention for t and x; e.g., for any $\beta \doteq \beta(a, \omega, k, t, x)$, the symbol β_t denotes the partial derivative with respect to the fourth argument. Those partial derivatives must be distinguished from the "full" derivatives ∂_t and ∂_x, which treat *all* arguments of (any) β as functions of, correspondingly, t and x. For instance, for β that we have just introduced, one gets

$$\partial_t \beta = \beta_a \, \partial_t a + \beta_\omega \, \partial_t \omega + \beta_k \, \partial_t k + \beta_t, \tag{71}$$
$$\partial_x \beta = \beta_a \, \partial_x a + \beta_\omega \, \partial_x \omega + \beta_k \, \partial_x k + \beta_x, \tag{72}$$

while $\partial_t a(t, x) = a_t(t, x)$, etc.

In summary, the complete list of symbols that we henceforth use to denote spatial derivatives is as follows:

$$\beta_t, \quad \beta_x, \quad \beta_\omega, \quad \beta_k, \quad \beta_a, \quad \beta_A, \quad \beta_{\mathscr{E}}, \quad \beta_{\mathcal{I}}. \tag{73}$$

Here β is a tag for an arbitrary function. Double indexes will denote second-order derivatives, correspondingly.

Wave energy and momentum. — Equations (68)-(70) have the following corollaries:

$$\partial_t(\omega \mathfrak{L}_\omega - \mathfrak{L}) - \partial_x(\omega \mathfrak{L}_k) = -\mathfrak{L}_t, \tag{74}$$
$$\partial_t(k \mathfrak{L}_\omega) + \partial_x(\mathfrak{L} - k \mathfrak{L}_k) = \mathfrak{L}_x. \tag{75}$$

Since the right-hand side here is the canonical force density (in spacetime representation), the quantities

$$\mathcal{W} \doteq \omega \mathfrak{L}_\omega - \mathfrak{L}, \quad \mathcal{P} \doteq k \mathfrak{L}_\omega \tag{76}$$

must serve as the densities of the wave canonical energy and momentum. (For 3D waves, angular momentum can be introduced similarly too, including a classical interpretation of the photon spin [2].) Correspondingly,

$$\mathcal{Q} \doteq -\omega \mathfrak{L}_k, \quad \Pi \doteq \mathfrak{L} - k \mathfrak{L}_k \tag{77}$$

must be the energy flux density and the momentum flux density. The respective flow velocities are

$$v^{(\mathcal{W})} = -\frac{\omega \mathfrak{L}_k}{\omega \mathfrak{L}_\omega - \mathfrak{L}}, \quad v^{(\mathcal{P})} = -\frac{k \mathfrak{L}_k - \mathfrak{L}}{k \mathfrak{L}_\omega}, \tag{78}$$

and they are generally different from each other and from the action flow velocity, $v^{(\mathcal{I})} = -\mathfrak{L}_k / \mathfrak{L}_\omega$ [cf. Eq. (70)]. Keep in mind, however, that this difference is a purely nonlinear effect, so it exists only *within* the wave envelope. At its front and tail, a wave is always linear [69], and linear waves are described as follows.

Linear waves. — By definition, a linear wave has $\omega(k)$ independent of a. It is seen from Eq. (68) that $\mathfrak{L}_a$ must then be separable as $\mathfrak{L}_a = \mathfrak{D}(\omega, k) \, A_a$, where A is some function such that A_a is nonzero. Hence

$$\mathfrak{L} = \mathfrak{D}(\omega, k) A, \tag{79}$$

so Eq. (68) leads to

$$\mathfrak{D}(\omega, k) = 0, \tag{80}$$

and thus, *on the solution* of Eq. (80), one has

$$\mathfrak{L} = 0. \tag{81}$$

(For instance, this means $\partial_t \mathfrak{L} = 0$, whereas $\mathfrak{L}_t$ need not be zero.) The action flow velocity is then given by

$$v^{(\mathcal{I})} = -\mathfrak{L}_k / \mathfrak{L}_\omega = -\mathfrak{D}_k / \mathfrak{D}_\omega. \tag{82}$$

But the latter equals $\omega_k \mathrel{=\!\!\cdot} v_g$ [as seen by differentiating Eq. (80) with respect to k with $\omega = \omega(k)$], which is commonly known as the linear group velocity. From Eqs. (78) and (81), one gets for linear waves that

$$v^{(\mathcal{I})} = v^{(\mathcal{W})} = v^{(\mathcal{P})} = v_g, \tag{83}$$

so the wave canonical energy and momentum propagate at the group velocity, just like the action; also,

$$\mathcal{W} = \omega \mathfrak{L}_\omega, \quad \mathcal{P} = k \mathfrak{L}_\omega, \quad \mathcal{P} = k \mathcal{W} / \omega. \tag{84}$$

Keep in mind that Eqs. (84) describe the *canonical*, or Minkowski energy-momentum. In application to electromagnetism, it matches [2] what often appears in textbooks as "the" wave energy-momentum [1, 62]. But, in addition, the so-called kinetic, or Abraham energy-momentum is also introduced sometimes, and its properties are quite different. For example, in isotropic fluid at rest, Abraham momentum can be expressed as $v_g \mathcal{W}/c^2$. (Here c is the speed of light, so, for electrostatic waves that we focus on in the present paper, Abraham momentum is relativistically small.) Contrary to a popular presumption, the latter result is independent of Maxwell's equations but rather flows from the Lorentz-transformation properties of Eq. (67). (For a discussion on the Abraham-Minkowski controversy, see Refs. [2, 8, 70].) Contrary to another presumption, the Poynting vector generally has little to do with either wave momentum, as it describes the electromagnetic *field* [35, Sec. 6.9] rather than the wave *per se*, which also includes medium oscillations. Notice, for instance, that Eqs. (84) do not rely on electromagnetism whatsoever.

C. Dynamics of modulations

Nonlinear group velocity. — By analogy with the linear case [Eq. (83)], the nonlinear group velocity, $\bar{v}_g$, is often defined as $v^{(\mathcal{W})}$ too, or as ω_k with the derivative taken at fixed $\mathfrak{L}/\omega$, since

$$(\omega_k)_{\mathfrak{L}/\omega} = -\frac{\partial_k(\mathfrak{L}/\omega)}{\partial_\omega(\mathfrak{L}/\omega)} = -\frac{\mathfrak{L}_k}{\mathfrak{L}_\omega - \mathfrak{L}/\omega} = v^{(\mathcal{W})}. \qquad (85)$$

However, this generalization is arbitrary, and other definitions, such as $\bar{v}_g = v^{(\mathcal{P})}$ or $\bar{v}_g = v^{(\mathcal{I})}$, would be equally justified. (In fact, the latter would be more fundamental, because the ACT holds also in nonstationary medium, unlike the energy conservation law.)

More consistently, the nonlinear group velocity is defined as the velocity of *information*. If a signal, or a modulation of a homogeneous wave, propagates at a fixed velocity $\bar{v}_g$, all the wave variables can be expressed through a single variable, $\zeta(t,x) \doteq x - \bar{x}(t)$, where $d_t\bar{x} = \bar{v}_g$ is called a characteristic, or GO ray. (Below we assume that the medium is homogeneous and stationary.) Then, $\partial_x = d_\zeta$ and $\partial_t = -\bar{v}_g\, d_\zeta$, so Eq. (69) yields

$$\bar{v}_g = \omega'/k' \qquad (86)$$

on this particular characteristic. (The primes denote d_ξ.) In this sense, $\bar{v}_g$ is the natural generalization of the group velocity v_g from the linear problem, where ω'/k' is the same on all rays [2], namely, $\omega'/k' = \omega_k(k)$.

To actually find $\bar{v}_g$, we proceed as follows. Using that

$$\partial_t \mathfrak{L}_\omega = \mathfrak{L}_{\omega a}\, \partial_t a + \mathfrak{L}_{\omega\omega}\, \partial_t \omega + \mathfrak{L}_{\omega k}\, \partial_t k, \qquad (87)$$

$$\partial_x \mathfrak{L}_k = \mathfrak{L}_{ka}\, \partial_x a + \mathfrak{L}_{kk}\, \partial_x k + \mathfrak{L}_{k\omega}\, \partial_x \omega, \qquad (88)$$

one can rewrite the ACT as

$$\bar{v}_g(\mathfrak{L}_{\omega a} a' + \mathfrak{L}_{\omega\omega}\omega' + \mathfrak{L}_{\omega k} k') + \mathfrak{L}_{ka} a' + \mathfrak{L}_{kk} k' + \mathfrak{L}_{k\omega}\omega' = 0. \qquad (89)$$

Yet a' can be derived from Eq. (68), after differentiating the latter with respect to ζ:

$$0 = d_\zeta \mathfrak{L}_a = \mathfrak{L}_{aa}\, a' + \mathfrak{L}_{a\omega}\omega' + \mathfrak{L}_{ak} k'. \qquad (90)$$

Specifically, one gets $a'/k' = -(\mathfrak{L}_{a\omega}\bar{v}_g + \mathfrak{L}_{ak})/\mathfrak{L}_{aa}$, so Eq. (89) rewrites as follows

$$p\bar{v}_g^2 + 2r\bar{v}_g + q = 0, \qquad (91)$$

where we introduced

$$p = \mathfrak{L}_{aa}\mathfrak{L}_{\omega\omega} - \mathfrak{L}_{\omega a}^2, \qquad (92)$$

$$r = \mathfrak{L}_{aa}\mathfrak{L}_{\omega k} - \mathfrak{L}_{\omega a}\mathfrak{L}_{ka}, \qquad (93)$$

$$q = \mathfrak{L}_{kk}\mathfrak{L}_{aa} - \mathfrak{L}_{ka}^2. \qquad (94)$$

Since Eq. (91) is a quadratic equation for $\bar{v}_g$, there are generally *two* group velocities different from each other,

$$\bar{v}_g = \left(-r \pm \sqrt{r^2 - pq}\,\right)/p, \qquad (95)$$

regardless of the type of nonlinearity. This means, for example, that a general modulation imposed on the wave profile eventually splits into two signals propagating with different velocities [19]. Each signal may then evolve further, if having a finite spread of a and thus of $\bar{v}_g$ too; however, such a signal will be comprised of characteristics that all correspond to the same sign in Eq. (95), so further splitting *per se* will not occur. Such pulse splitting is an inherent feature of all nonlinear waves, as well known in classical hydrodynamics [3, 71] and was also observed in plasma physics experiments [72, 73].

The exception is the linear regime, Eq. (79). In that case, it is convenient to use A instead of a; then the same equations hold, if ∂_a is replaced with ∂_A. On the other hand, $\mathfrak{L}_{AA} = 0$, so one obtains

$$\bar{v}_g^2 \mathfrak{L}_{\omega A}^2 + 2\bar{v}_g \mathfrak{L}_{\omega A}\mathfrak{L}_{kA} + \mathfrak{L}_{kA}^2 = 0, \qquad (96)$$

and the two roots coincide:

$$\bar{v}_g = -\mathfrak{L}_{kA}/\mathfrak{L}_{\omega A} = -\mathfrak{L}_k/\mathfrak{L}_\omega = v_g. \qquad (97)$$

Generalized Lighthill's criterion. — If $r^2 < pq$, there are no real solutions for $\bar{v}_g$, so no stable envelope is possible in this regime. This means that amplitude modulations will grow with time, i.e., the wave is modulationally unstable. This criterion can also be put in a different form, namely, as follows. Let us choose the action density, $\mathcal{I}$, to serve as an independent variable instead of a. Also, using the NDR, let us exclude ω from the list of independent variables. Hence,

$$\omega = \omega(k, \mathcal{I}), \quad a = a(k, \mathcal{I}), \quad \mathcal{J} = \mathcal{J}(k, \mathcal{I}), \qquad (98)$$

and $\mathfrak{L}$ takes the following form:

$$\mathfrak{L}(a(k,\mathcal{I}),\omega(k,\mathcal{I}),k) \mathrel{\underset{\cdot}{=}} \mathscr{L}(k,\mathcal{I}), \tag{99}$$

so, due to Eq. (68), we get $\mathscr{L}_k = \mathcal{I}\omega_k - \mathcal{J}$ and $\mathscr{L}_\mathcal{I} = \mathcal{I}\omega_\mathcal{I}$. Now Eqs. (69) and (70) on characteristics become

$$-\bar{v}_g k' + \omega_k k' + \omega_\mathcal{I}\mathcal{I}' = 0, \quad -\bar{v}_g\mathcal{I}' + \mathcal{J}_k k' + \mathcal{J}_\mathcal{I}\mathcal{I}' = 0.$$

From $\mathscr{L}_{k\mathcal{I}} = \mathscr{L}_{\mathcal{I}k}$, it follows that $\omega_k = \mathcal{J}_\mathcal{I}$, so one gets

$$\bar{v}_g = \omega_k \pm \sqrt{\omega_\mathcal{I}\mathcal{J}_k}. \tag{100}$$

(In a linear wave, $\omega_\mathcal{I} = 0$, so there is only one group velocity, $\bar{v}_g = v_g$.) Hence, the wave is stable if

$$\omega_\mathcal{I}\mathcal{J}_k > 0. \tag{101}$$

In case of a weakly nonlinear wave, when $\omega_\mathcal{I}$ is small, one can substitute the lowest-order approximation for $\mathcal{J}$ and take $\mathcal{I} \propto a^2$. Then Eq. (101) becomes

$$\frac{\partial\omega(k,a)}{\partial a}\,\frac{dv_g(k)}{dk} > 0, \tag{102}$$

which is known as Lighthill's stability criterion [74]. Contrary to a popular presumption, however, Eq. (102) is not universal. In Sec. VI, we will show that it can lead to incorrect predictions for WTP, yet Eq. (101), or generalized Lighthill's criterion, remains valid.

Keep in mind also that modulationally stable waves are not necessarily stable overall. This is seen from the fact that the model discussed here [based on Eq. (67)] describes only the self-action of a single GO wave, whereas multiple-wave interactions are missed but may lead, e.g., to resonant decay [75, 76]. Some information regarding resonant effects can be inferred from so-called Manley-Rowe relations, which serve as the ACT extension [2, 37, 77–80]. We leave a first-principle exposition of those to future publications, as explaining Manley-Rowe relations consistently requires a more fundamental approach to the physics of waves.

V. LINEAR EPW

Wave Lagrangian density. — Let us now discuss an EPW as an example, assuming, in this section, that the wave amplitude is small and that there are no resonant particles. In this case, Eq. (47) can be adopted for $\mathfrak{L}$, with $\mathcal{H}$ taken from Eq. (16), where we also add interaction with a quasistatic potential for completeness. This gives

$$\mathfrak{L} = \bar{\mathfrak{L}} + \frac{\mathscr{E}^2}{16\pi} - \int_{-\infty}^{\infty}\Phi(t,x,P)\,\mathsf{F}(t,x,P)\,dP, \tag{103}$$

assuming Eq. (66). Here we introduced

$$\bar{\mathfrak{L}} = \frac{(\partial_x\bar{\varphi})^2}{8\pi} - e\bar{\varphi}\int_{-\infty}^{\infty}\mathsf{F}(t,x,P)\,dP, \tag{104}$$

but $\bar{\mathfrak{L}}$ is independent of the wave variables ($\mathscr{E}$ and ξ), so we omit it below. With Φ from Eq. (17), one then gets

$$\mathfrak{L} = \epsilon(\omega,k)\mathscr{A}. \tag{105}$$

As usual, parametric dependence on (t,x) is assumed here, $\mathscr{A}$ is given by Eq. (66), and

$$\epsilon(\omega,k) \doteq 1 - \frac{4\pi e^2}{m}\int_{-\infty}^{\infty}\frac{\mathsf{F}(P)}{(\omega - kP/m)^2}\,dP. \tag{106}$$

Since $\mathsf{F}(P)$ is assumed to be zero in the resonance vicinity, the integrand in Eq. (106) is analytic, so the integral can be taken by parts. To put the result in a more familiar form (yet see Ref. [81]), let us introduce $f(v) \doteq m\mathsf{F}(mv)/n$, normalized as $\int_{-\infty}^{\infty} f(t,x,v)\,dv = 1$ [cf. Eq. (45)]. Then Eq. (106) becomes

$$\epsilon(\omega,k) = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{f'(v)}{v-\omega/k}\,dv, \tag{107}$$

where $\omega_p^2 \doteq 4\pi ne^2/m$. This $\epsilon(\omega,k)$ matches the linear dielectric function, Eq. (B13), except the resonance-pole contribution is now identically zero due to the assumed absence of resonant particles.

Dispersion relation. — The dispersion relation, which Eq. (105) yields via Eq. (68), is *linear*,

$$\epsilon(\omega,k) = 0 \tag{108}$$

[and also agrees with Eq. (B14)]. On the other hand, it is seen now to flow from the expression for the ponderomotive potential Φ, same that causes a *nonlinear* force imposed by the wave on plasma particles (Sec. II B). This ambiguity in separation of linear and nonlinear wave effects is exactly the root of the long-lasting controversy regarding the wave mechanical properties [2].

Action, energy, and momentum. — The wave group velocity is found from Eq. (82) in the form

$$v_g = \omega_k = -\epsilon_k/\epsilon_\omega. \tag{109}$$

The density of the wave action, energy, and momentum are obtained from Eqs. (61) and (84) and given by

$$\mathcal{I} = \epsilon_\omega\mathscr{A}, \quad \mathcal{W} = \omega\epsilon_\omega\mathscr{A}, \quad \mathcal{P} = k\epsilon_\omega\mathscr{A}. \tag{110}$$

But remember that these are *canonical* quantities that describe the wave alone, whereas OCs too can carry some energy-momentum proportional to $\mathscr{E}^2$. That originates from the ponderomotive momentum, ΔP [Eq. (15)]. Its density is given by $\Delta\mathcal{P} = -n\langle\langle\partial_V\Phi\rangle\rangle$, where $\langle\langle\ldots\rangle\rangle = \int_{-\infty}^{\infty}(\ldots)\,f(V)\,dV$ denotes ensemble-averaging; then it is easy to see that

$$\Delta\mathcal{P} = nk\langle\langle\Phi_\omega\rangle\rangle = -k\mathfrak{L}_\omega = -\mathcal{P}. \tag{111}$$

The overall momentum that is proportional to $\mathscr{E}^2$, called kinetic momentum, is thus $\mathcal{P} + \Delta\mathcal{P} = 0$ [2]. Therefore, the total momentum that accompanies the wave, if any, must be attributed to the OC average flow, $nm\langle\langle V\rangle\rangle$.

As a side note, each OC also carries energy proportional to $\mathscr{E}^2$, namely, $-V\partial_V\Phi$ [82]. Its density then can be written as $\Delta\mathcal{W} = -n\langle\langle V\partial_V\Phi\rangle\rangle$. Since

$$\begin{aligned}- V\partial_V\Phi &= -2kV\Phi/(\omega - kV)\\ &= 2[1-\omega/(\omega-kV)]\Phi = 2\Phi + \omega\Phi_\omega,\end{aligned}$$

and $n\langle\langle\Phi\rangle\rangle = \mathscr{A} - \mathfrak{L}(\mathscr{A},\omega,k)$, one obtains, due to Eq. (81), that

$$\Delta\mathcal{W} = 2\mathscr{A} - \omega\mathfrak{L}_\omega = 2\mathscr{A} - \mathcal{W}. \tag{112}$$

The overall energy proportional to $\mathscr{E}^2$ is thus $\mathcal{W}+\Delta\mathcal{W} = 2\mathscr{A}$, i.e., twice the electrostatic field average energy.

Action conservation. — Consider now the ACT, Eq. (70). For $\mathfrak{L}$ given by Eq. (105), it takes the form

$$\partial_t(\epsilon_\omega\mathscr{A}) - \partial_x(\epsilon_k\mathscr{A}) = 0. \tag{113}$$

For instance, a stationary wave satisfies $\epsilon_k\mathscr{A} = \text{const}$, or $v_g\mathcal{I} = \text{const}$, which matches what is inferred from the Wentzel-Kramers-Brillouin (WKB) approximation [1, Chap. 4]. Note that $\mathcal{I} = \mathcal{W}/\omega$, and, according to Eq. (69), $\omega = \text{const}$, as $\partial_t = 0$. Thus, $v_g\mathcal{W} = \text{const}$, which one could also get from Eq. (74), using Eq. (81).

Consider now the wave evolution in a plasma that is time-dependent, e.g., undergoes mechanical compression. If (correctly) derived within a fluid or Vlasov-Poisson approach, general wave equations are enormously complicated in this case; see Ref. [83] and references therein. The ACT, in contrast, has a simple and manifestly conservative form, showing that the total action is conserved; i.e., $\int \epsilon_\omega\mathscr{A}\,dx = \text{const}$. If both the plasma and the wave are homogeneous, that gives

$$\epsilon_\omega\mathscr{A}/n = \text{const}. \tag{114}$$

For simplicity, assume cold plasma (i.e., $kv_T/\omega \ll 1$, where v_T is the electron thermal speed). Then [84],

$$\epsilon(\omega,k) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2v_T^2}{\omega^2}\right), \tag{115}$$

and Eqs. (108) leads to

$$\omega = \omega_p + 3k^2v_T^2/(2\omega_p), \tag{116}$$

or, even simpler, $\omega \approx \omega_p$. This gives

$$\epsilon_\omega \approx 2/\omega_p \propto n^{-1/2}, \tag{117}$$

so one obtains

$$\mathscr{E}n^{-3/4} = \text{const}. \tag{118}$$

Equation (118) can be understood as the plasmon-gas adiabat. It shows, in particular, that plasma compression amplifies the wave, which effect was confirmed through particle-in-cell (PIC) simulations in Ref. [85]. The scaling (118) holds even if the plasma transformation is caused by cosmological metric expansion [51], but not at essentially nonadiabatic densification, such as through ionization or recombination. That said, even in the latter case the OC formalism significantly simplifies the derivation of the corresponding $\mathscr{E}(n)$, as shown in Ref. [86].

VI. NONLINEAR EPW

Let us now discuss nonlinear EPW, specifically, WTP with phase-mixed trapped-electron distributions. For simplicity, we will assume that the trapped-particle loading is homogeneous (Sec. III D) and also that the wave amplitude is small enough, so waves can be considered sinusoidal [58–60]. Such WTP satisfy the definition of simple waves, described in Sec. IV. We elaborate on some of their paradigmatic features below to illustrate the general theory. For details, see Refs. [17–21].

A. Wave Lagrangian density

First, let us write down $\mathfrak{L}$ explicitly by substituting Eqs. (21), (24), and (66) into Eq. (54). This yields

$$\begin{aligned}\mathfrak{L} = \mathscr{A} &- n^{(p)}\langle\langle\varepsilon + Pu - mu^2/2\rangle\rangle^{(p)}\\ &- n^{(t)}\langle\langle\varepsilon\rangle\rangle^{(t)} + n^{(t)}mu^2/2 - n^{(t)}e\bar{\varphi}.\end{aligned} \tag{119}$$

The indexes $^{(p)}$ and $^{(t)}$ denote passing and trapped particles, respectively, and $\langle\langle\ldots\rangle\rangle$ denotes ensemble averaging over the corresponding distributions. The combination $(\partial_x\bar{\varphi})^2/(8\pi) - n^{(p)}e\bar{\varphi}$ has been dropped for the same reason as in Sec. V. The term $n^{(t)}e\bar{\varphi}$, in contrast, cannot be dropped, as it depends on k. Specifically, $n^{(t)} = \ell k$ (Sec. III D), where ℓ is now a constant, so we will use

$$\begin{aligned}\mathfrak{L} = \mathscr{A} &- n^{(p)}\langle\langle\varepsilon + Pu - mu^2/2\rangle\rangle^{(p)}\\ &- \ell k\langle\langle\varepsilon\rangle\rangle^{(t)} + \ell m\omega^2/(2k) - \ell ke\bar{\varphi}.\end{aligned} \tag{120}$$

Keep in mind that derivatives of $\mathfrak{L}$ must be taken at fixed P-distribution of passing particles and at fixed J-distribution of trapped particles. Hence, the main wave equations are derived as follows.

B. Nonlinear dispersion

Basic notation. — For the purpose of this section (Sec. VI B), it is convenient to introduce the following dimensionless measure of the wave amplitude:

$$a \doteq \Omega_\star^2/\omega^2. \tag{121}$$

The area function, $J(\varepsilon)$, can then be expressed as

$$J(\varepsilon) = (m\omega/k^2)\sqrt{a}\,j(r), \tag{122}$$

where $r \doteq (\varepsilon + e\mathscr{E}/k)/(2e\mathscr{E}/k)$ is the normalized energy. The dimensionless action, $j(r)$, is understood as a (continuous) normalized area function and is presented explicitly in Ref. [18]. In particular, $j = 0$ for a particle resting at the bottom of a wave trough ($r = 0$), with the corresponding value at the separatrix ($r = 1$) being $j_* = 4/\pi$. Using the inverse function, $r(j)$, one also obtains another useful equality,

$$\varepsilon = mu^2\left[2ar(j) - a\right]. \tag{123}$$

General NDR. — The WTP dispersion is found by substituting Eq. (120) in Eq. (68), which yields

$$\partial_a \big[\mathscr{A} - n^{(p)} \langle\!\langle \varepsilon \rangle\!\rangle^{(p)} - n^{(t)} \langle\!\langle \varepsilon \rangle\!\rangle^{(t)}\big] = 0. \tag{124}$$

As the derivative with respect to a is taken at constant ω and k, keeping the passing-particle P fixed is the same as keeping its J fixed, so passing and trapped particles can be treated on the same footing. It is then convenient to introduce their common distribution, $F(J)$, in order to combine the latter terms in Eq. (124) into one,

$$n^{(p)} \langle\!\langle \varepsilon \rangle\!\rangle^{(p)} + n^{(t)} \langle\!\langle \varepsilon \rangle\!\rangle^{(t)} = n \langle\!\langle \varepsilon \rangle\!\rangle. \tag{125}$$

Here n is the total density, and the averaging on the right-hand side is defined as $\langle\!\langle \ldots \rangle\!\rangle \doteq \int_0^\infty (\ldots)\, F(J)\, dJ$, assuming $\int_0^\infty F(J)\, dJ = 1$. The function F relates to the canonical distribution F (Sec. III) as $nF = k\mathsf{F}$, both for passing and trapped particles. (The seemingly different units are explained by the fact that earlier we assumed unit area in the direction transverse to x axis.)

Equation (124) hence can be written as follows:

$$\omega^2 = (2\omega_p^2/a)\, \langle\!\langle G \rangle\!\rangle, \tag{126}$$

where we introduced a dimensionless function [18]

$$G \doteq \varepsilon_a(J, a)/(mu^2). \tag{127}$$

(Showing the action as a separate argument of ε is intended to emphasize that the derivative with respect to a is taken at fixed J.) Equation (126) is a master equation that accounts for *all* dispersive effects, both linear and nonlinear, to the extent that the sinusoidal-wave approximation applies. In particular, it shows that the contribution to ω^2 of particles with given J is determined solely by the dimensionless weight function G. The latter is finite ($|G| \le 1$) and continuous, albeit nonanalytic, and can be expressed in terms of the normalized area function, $j(r)$; namely, $G(j) = g(r(j))$, where [18]

$$g(r) \doteq 2r - 1 - j(r)/j'(r). \tag{128}$$

Yet, sufficient for practical purposes (except for evaluating numerical coefficients) are its asymptotics [87],

$$G(j) = \begin{cases} -1 + j + \ldots, & j \ll 1, \\ 1, & j = 4/\pi, \\ \frac{1}{2j^2} + \frac{5}{16 j^6} + \ldots, & j \gg 1. \end{cases} \tag{129}$$

Smooth distribution. — Among various limits that can be inferred from Eq. (126) [18], let us discuss the case when $F(J)$ has a scale much larger than J_*, so it is smooth compared to G. Then, Eq. (126) leads to [18]

$$\bar{\epsilon}(\omega, k) + \frac{\omega_p^2}{2k^2}\, C_1 \ln a + \frac{\omega \omega_p^2}{k^3}\, \varkappa C_2 \sqrt{a} = 0, \tag{130}$$

where we introduced

$$\bar{\epsilon}(\omega, k) \doteq 1 - \frac{m^2 \omega_p^2}{k^4} \times \int_0^\infty \left[F'(J) - F'(0)\, Q\!\left(\frac{k^2 J}{m\omega}\right)\right] \frac{dJ}{J}. \tag{131}$$

Here $Q(j) \doteq 1 + 2j \int_0^j G(\jmath)\, d\jmath$, which, at small, j behaves as $Q(j) \approx 1 - j^2$, so the integral in Eq. (131) is absolutely converging; also, $\varkappa \doteq \int_0^\infty Q(j)\, dj \approx 0.544$, and

$$C_1 \doteq (m/k)^2 F'(0), \quad C_2 \doteq (m/k)^3 F''(0). \tag{132}$$

The nonlinear terms in Eq. (130) must be small, for otherwise the assumed sinusoidal-wave approximation would not apply. Let us thus search for a solution in the form $\omega = \omega_{\rm L} + \omega_{\rm NL}$, such that $\omega_{\rm NL}$ is small, and $\omega_{\rm L}$ satisfies $\bar{\epsilon}(\omega_{\rm L}, k) = 0$. The logarithmic term does not appear in literature because standard calculations imply $C_1 = 0$ [47, 59, 88–94]. That one aside, Eq. (131) hence approximately [95] leads to the nonlinear frequency shift

$$\omega_{\rm NL} = -\frac{\varkappa \omega_p^2\, C_2}{k^2 \bar{\epsilon}_\omega(\omega_{\rm L}, k)} \sqrt{\frac{e\mathscr{E}}{mk}}. \tag{133}$$

Equation (133) precisely reproduces the well known result of Ref. [88] if one substitutes [18]

$$F(J) = (k/m)\, [f_0(u + kJ/m) + f_0(u - kJ/m)]\,. \tag{134}$$

This meets the assumption made in (the relevant part of) Ref. [88] that the wave is adiabatically excited at fixed u, so all J are conserved; in particular, $f_0(v)$ is understood as the initial velocity distribution, and $\bar{\epsilon}[F(J)]$ then turns into the linear dielectric function, $\epsilon[f_0(v)]$, given by Eq. (107). That said, the assumption of fixed u is problematic, as $\omega_{\rm NL}$ evolves together with the amplitude during amplification. Solving the *initial-value problem* [as opposed to just finding the NDR for a given plasma state, $F(J)$] more accurately thus requires accounting for frequency sweeping; then, passing particles would conserve their P but not J, and Eq. (134) would no longer apply. This may explain the discrepancies found in Ref. [95] between the analytical prediction of Ref. [88] and results of numerical simulations.

C. Nonlinear action conservation

Now let us briefly address some paradigmatic dynamics of WTP. Like in Sec. V, the wave evolution is inferred from the ACT, but $\mathcal{I}$ and $\mathcal{J}$ are different from those in linear EPW. In particular, differentiating Eq. (120) with respect to ω leads to [17]

$$k\mathcal{I} = n^{(p)} \langle\!\langle mV - P \rangle\!\rangle^{(p)} + \ell m\omega, \tag{135}$$

where we used that $\varepsilon^{(p)} = \varepsilon^{(p)}(J(P, \omega, k), \mathscr{E}, k)$ and $\varepsilon^{(t)} = \varepsilon^{(t)}(J, \mathscr{E}, k)$. It is seen from Eq. (135) that the shape of the trapped-particle distribution has no effect on $\mathcal{I}$, except indirectly (and weakly) through the NDR. Notice also that the contribution of each trapped particle to $\mathcal{I}$ is $\mathcal{O}(\mathscr{E}^0)$, whereas the contribution of each passing particle [ranging from $\mathcal{O}(\mathscr{E}^2)$ to $\mathcal{O}(\mathscr{E}^{1/2})$] vanishes at small $\mathscr{E}$. It is then often enough to use the linear-response approximation for the latter, as the main nonlinear effect (on $\mathcal{I}$) comes from the former. This gives

$$\mathcal{I} \approx \epsilon_\omega \mathscr{A} + \ell m u. \tag{136}$$

For the same reason, the term $\ell m u$ can become comparable to $\epsilon_\omega \mathscr{A}$, even when $n^{(t)}$ is small; then, the adiabat (118) is altered. Consider a WTP in a plasma undergoing compression transverse to the wave vector [20]. (For WTP parallel compression, see Ref. [96].) Then k remains fixed, so the ACT yields $\mathcal{I}/n = \text{const}$, or

$$\frac{\mathscr{E}^2}{8\pi\omega_p n} + \left(\frac{n^{(t)}}{n}\right)\frac{mu}{k} = \text{const}, \tag{137}$$

where we assumed that $\omega_{\rm NL}$ is small and adopted Eqs. (115)-(117) again. Keeping only the first term would reproduce Eq. (118). However, as $\omega \sim \omega_p(t)$ increases, the second term in Eq. (137) eventually can become dominant and halt amplification. This effect was confirmed through PIC simulations in Ref. [20].

D. Model system

More insight into the WTP dynamics can be inferred if one completely ignores the nonlinearity of the passing-particle response and assumes that trapped particles are trapped *deeply* [19, 97, 98]; i.e.,

$$F(J) = \delta(J). \tag{138}$$

[Strictly speaking, the delta function here must be understood as the limit of $\delta(J - J_c)$ at $J_c \to 0$.] The Lagrangian density (120) then becomes particularly simple,

$$\begin{aligned}\mathfrak{L}(\mathscr{E}, \omega, k) = \epsilon(\omega, k)\,\mathscr{E}^2/(16\pi)\\ + \ell e \mathscr{E} + \ell m \omega^2/(2k) - \ell k e \bar{\varphi}.\end{aligned} \tag{139}$$

The corresponding dispersion relation can either be inferred from Sec. VI B [18, 21] or, even more easily, from Eqs. (68). In agreement with Refs. [60, 98], one gets

$$\epsilon(\omega, k) + 2\omega_b^2/\Omega_\star^2 = 0, \tag{140}$$

where $\omega_b^2 \doteq 4\pi n^{(t)} e^2/m$. Expanding $\epsilon(\omega, k)$ around the linear frequency, for which we adopt Eqs. (115)-(117), one then obtains

$$\omega \approx \omega_p\left(1 - \frac{\omega_b^2}{\Omega_\star^2}\right) + \frac{3k^2 v_T^2}{2\omega_p}. \tag{141}$$

Equations (61), in turn, become

$$\mathcal{I} = \epsilon_\omega \mathscr{A} + \ell m u, \tag{142}$$

$$\mathcal{J} = -\epsilon_k \mathscr{A} + \ell m u^2/2 + \ell e \bar{\varphi}. \tag{143}$$

Hence, the ACT [Eq. (70)] takes the form

$$\partial_t(\epsilon_\omega \mathscr{A} + \ell m u) + \partial_x(-\epsilon_k \mathscr{A} + \ell m u^2/2) = \ell e \bar{E}, \tag{144}$$

where $\bar{E} \doteq -\partial_x \bar{\varphi}$. A number of paradigmatic effects can be derived from here, namely, as follows.

Amplification by a dc field. — First of all, it is seen that a WTP can be manipulated by a dc field. Equation (144) predicts, for instance, that a homogeneous wave with small enough ℓ satisfies

$$\dot{\mathcal{W}}_{\rm L} = n^{(t)} e u \bar{E}, \tag{145}$$

where we used $\omega \approx \omega_p$, and $\mathcal{W}_{\rm L} \doteq 2\mathscr{A}$ is approximately the linear-wave energy density [cf. Eqs. (115)-(117)]. If the wave travels *along* the dc force on trapped electrons ($eu\bar{E} > 0$), Eq. (145) shows that the wave is amplified. This is because the field $\bar{E}$ performs work on those electrons, yet they cannot change their velocity and thus serve as mediators, channeling the gained energy to the wave field. The effect was also reported, e.g., in Refs. [99–101], but by means of a different machinery.

Nonlinear tunneling. — For a stationary wave ($\partial_t = 0$), Eqs. (141) and (144) predict that WTP can penetrate overcritical plasma [102–104]. See Ref. [19] for details.

Trapped-particle modulational instability. — One can also use Eqs. (141) and (144), together with Eq. (69), to study the modulational stability of a WTP as described in Sec. IV C. Here is a simple way to proceed. Assuming $\bar{E} = 0$ and $\ell m u \ll \epsilon_\omega \mathscr{A}$ (unlike in Sec. VI C), let us approximate Eqs. (142) and (143) as follows [19]:

$$\mathcal{I} \approx \frac{2\mathscr{A}}{\omega_p}, \quad \mathcal{J} \approx \frac{6kv_T^2 \mathscr{A}}{\omega_p^2} + \frac{\ell m \omega_p^2}{2k^2}, \tag{146}$$

assuming Eqs. (115)-(117), as usual. Note that, while $\mathcal{I}$ is then the same as for a linear wave, $\mathcal{J}$ is $(1 + \mathbb{S})$ times bigger than its linear-wave value, where

$$\mathbb{S} \doteq \frac{n^{(t)} m u^3/2}{6kv_T^2 \mathscr{A}/\omega} \tag{147}$$

can be understood as the ratio of the energy flux carried by trapped particles and that carried by passing particles.

From Eqs. (146), one gets $\omega_{\mathcal{I}}(k, \mathcal{I}) \propto \omega_{\mathscr{E}}(k, \mathscr{E})$ and $\mathcal{J}_k(k, \mathcal{I}) \propto \mathcal{J}_k(k, \mathcal{E})$. Then, according to Eq. (101), the wave is stable if

$$(1/2 - \mathbb{S})\,\omega_{\mathscr{E}} > 0. \tag{148}$$

In our case, $\omega_{\mathscr{E}} > 0$, so having $\mathbb{S} > 1/2$ results in what is known as the trapped-particle modulational instability (TPMI). Note that the standard Lighthill's criterion of wave stability, Eq. (102), misses the $\mathbb{S}$-dependent threshold, the reason being that it relies on the assumption

$\mathscr{J} \propto \mathscr{A}$, invalid for WTP. The traditional models of the TPMI [73, 93, 105, 106] fall short at large $\mathbb{S}$ for the same reason and would mistakenly predict that the specific distribution (138) is always modulationally stable [19]. In addition, remember that WTP can also exhibit more general, nonadiabatic instabilities such as the coalescence instability [96, 107–109] and the sideband instability in its various manifestations [42].

VII. CONCLUSIONS

In this paper, we reviewed the variational approach to adiabatic collisionless plasma waves that was originally introduced in Ref. [2, 17–21] and some earlier publications [22]. The focus was made, for clarity, on electrostatic EPW, both linear and nonlinear. We showed how to calculate the Lagrangian density for such waves explicitly, emphasizing the contribution of resonantly trapped particles. As it turns out, the problem can be reduced to finding the OC energies of individual particles in a strictly periodic field, and that is readily done by a quadrature both for passing and trapped trajectories. Then a variety of wave effects can be deduced straightforwardly, much along the lines of Whitham's theory and, notably, often without appealing to the VM system. We discussed some paradigmatic physics of EPW in this regard, for illustration purposes, but the reader is encouraged to refer to the aforecited literature for details.

The work was supported by the U.S. DOE through Contract No. DE-AC02-09CH11466, by the NNSA SSAA Program through DOE Research Grant No. DE274-FG52-08NA28553, and by the U.S. DTRA through Research Grant No. HDTRA1-11-1-0037.

APPENDIX A: PONDEROMOTIVE FORCE FROM AVERAGING THE MOTION EQUATION

In this appendix, we briefly restate the traditional derivation [110] of the ponderomotive force on a single particle in nonmagnetized [111] plasma, under the conditions when the effect of the wave vector on the particle polarizability $\hat{\boldsymbol{\alpha}}$ [34] is negligible. (For the longitudinal polarizability, $\alpha_\parallel$, such as in Sec. II B, this implies negligibly small k, whereas the transverse polarizability, $\alpha_\perp$, is more forgiving and does not depend on k except at relativistic velocities [29].) We proceed with the same reservations as in Sec. II B, except now we allow for 3D motion and include the effect of both the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ of the wave.

Let us separate the particle motion into the slow motion in variables $(\mathbf{X}, \mathbf{V})$ and the quiver motion $(\tilde{\mathbf{x}}, \tilde{\mathbf{v}})$ as in Eq. (4). By Taylor-expanding the Lorentz force and dropping nonlinear terms, we then arrive at

$$m(\ddot{\mathbf{X}} + \ddot{\tilde{\mathbf{x}}}) \approx e[\mathbf{E}(t,\mathbf{X}) + (\tilde{\mathbf{x}}\cdot\nabla)\,\mathbf{E}(t,\mathbf{X}) + (\tilde{\mathbf{v}}/c)\times\mathbf{B}(t,\mathbf{X}) + \mathbf{R}]. \quad \text{(A1)}$$

Here c is the speed of light, and the two terms in $\mathbf{R} \doteq (\mathbf{V}/c)\times[\mathbf{B}(t,\mathbf{X}) + (\tilde{\mathbf{x}}\cdot\nabla)\mathbf{B}(t,\mathbf{X})]$ are negligible on the score of being relativistically small compared to the first and second terms on the right-hand side of Eq. (A1), correspondingly. Then, extracting the part that oscillates at the wave first harmonic yields $m\ddot{\tilde{\mathbf{x}}} = e\mathbf{E}(t,\mathbf{X})$, or

$$e\tilde{\mathbf{x}} = \alpha\mathbf{E}(t,\mathbf{X}), \quad e\tilde{\mathbf{v}} = -i\omega\alpha\mathbf{E}(t,\mathbf{X}), \quad \text{(A2)}$$

where $\alpha = -e^2/(m\omega^2)$. (To infer the oscillations at the second harmonic accurately, one would need to include nonlinear terms that we have neglected.) Note that, to the extent that the polarizability is independent of k, one need not distinguish between $\alpha_\parallel$ and $\alpha_\perp$, so we consider longitudinal and transverse waves on the same footing.

The average part of Eq. (A1) can be expressed as

$$m\ddot{\mathbf{X}} = \langle (e\tilde{\mathbf{x}}\cdot\nabla)\,\mathbf{E}(t,\mathbf{X})\rangle + \langle (e\tilde{\mathbf{v}}/c)\times\mathbf{B}(t,\mathbf{X})\rangle. \quad \text{(A3)}$$

Equations (A2) and Faraday's law, $\mathbf{B} = -i(c/\omega)\nabla\times\mathbf{E}$, yield

$$\langle (e\tilde{\mathbf{x}}\cdot\nabla)\,\mathbf{E}\rangle = (\alpha/4)\,[(\mathbf{E}^*\cdot\nabla)\,\mathbf{E} + (\mathbf{E}\cdot\nabla)\,\mathbf{E}^*],$$
$$\langle (e\tilde{\mathbf{v}}/c)\times\mathbf{B}\rangle = (\alpha/4)\,[\mathbf{E}^*\times(\nabla\times\mathbf{E}) + \mathbf{E}\times(\nabla\times\mathbf{E})^*],$$

and we can substitute [112]

$$(\mathbf{E}^*\cdot\nabla)\,\mathbf{E} + (\mathbf{E}\cdot\nabla)\,\mathbf{E}^* = -\mathbf{E}^*\times(\nabla\times\mathbf{E}) - \mathbf{E}\times(\nabla\times\mathbf{E})^* + \nabla|\mathbf{E}|^2. \quad \text{(A4)}$$

Then, in agreement with Sec. II B, Eq. (A3) becomes

$$m\ddot{\mathbf{X}} = -\nabla\Phi(t,\mathbf{X}), \quad \Phi \doteq -\alpha|E|^2/4. \quad \text{(A5)}$$

APPENDIX B: SAMPLE PROBLEMS

In this appendix, we put forth canonical variable transformations and equations of the single particle dynamics as a replacement for the Vlasov theory in some paradigmatic calculations for 1D electrostatic plasma waves.

1. Distribution function

First of all, let us introduce some notation and, particularly, discuss how the distribution function formalism is affected by variable transformations. Consider two arbitrary (not necessarily canonical) sets of phase space coordinates, z_0 and z, mapped via some

$$\bar{z} : \, z_0 \mapsto z. \quad \text{(B1)}$$

The phase space density corresponding to a single particle located at z_0 is $\delta(z - \bar{z}(z_0))$, so the distribution $\varrho(z)$ for an ensemble is obtained by averaging over z_0; i.e.,

$$\varrho(z) = \int \delta(z - \bar{z}(z_0))\,\varrho_0(z_0)\,dz_0, \quad \text{(B2)}$$

where ϱ_0 is the distribution over z_0. This yields

$$\varrho(z) = \varrho_0(\bar{z}^{-1}(z))\, \bar{J}(z), \tag{B3}$$

where $\bar{z}^{-1}$ is the mapping inverse to $\bar{z}$, and $\bar{J} \doteq |\partial z_0/\partial z|$ is the Jacobian of the variable transformation (B1).

Canonical coordinates, z, form a special class among all possible z. Transformations (B1) between them, called canonical transformations, have unit Jacobians [27, Sec. 46], and, according to Eq. (B3), thereby leave the distribution function unaffected. One can hence introduce *the* canonical distribution, f, independently of the specific choice of z. The system evolution, considered as mapping $\mathsf{z}_t : \mathsf{z}_0 \mapsto \mathsf{z}$, is a canonical transformation too [27, Sec. 45] (here t is a time label, and z_0 is the initial location), so we can write

$$\mathsf{f}(t, \mathsf{z}) = \mathsf{f}_0(\mathsf{z}_t^{-1}(\mathsf{z})) \equiv \mathsf{f}_0(\mathsf{z}_0). \tag{B4}$$

In other words, not only is the canonical probability distribution f unique, but it is also time-independent.

Equation (B4), often unjustly considered as an inference from the Vlasov equation (as opposed to its root), is by itself enough to *replace* that equation; what it takes is only to find $\mathsf{z}_t^{-1}(\mathsf{z})$ by solving for the single particle motion. This is sometimes called the method of characteristics in the Vlasov theory [1, Sec. 10.4], but note that a PDE for f may not need to be introduced in the first place. Some traditional paradigmatic calculations will now be revisited in this context, for didactic purposes. Keep in mind, however, that the consistent OC formalism that we discuss in the main text is often more convenient, since it simplifies $\mathsf{z}_t^{-1}(\mathsf{z})$ and thus renders ensemble-averaging particularly simple.

2. Linear dielectric function

First, let us address calculating the plasma linear response, such as the conductivity. For spatially monochromatic fields with given real k, the conductivity $\sigma(t, k)$ of given species is introduced via [62, Sec. 77]

$$\Delta j(t, x) = \int_0^\infty \sigma(t', k)\, E(t - t', x)\, dt', \tag{B5}$$

where $\Delta j(t, x)$ is the driven increment of the current density. The concept of an *increment* is undefined for processes that are strictly monochromatic in time (as there is no initial state), so let us adopt that the field is turned on at, say, $t = 0$. Assume also that it hence oscillates at a fixed frequency ω; i.e.,

$$E(t) = E_0 e^{ikx - i\omega t}\Theta(t), \tag{B6}$$

where $\Theta(t)$ is the Heaviside step function. Then, $j(t, x) = \Sigma(t, \omega, k)\, E(t, x)$, where $\Sigma(t, \omega, k) \doteq \int_0^t \sigma(t', k)\, e^{i\omega t'}\, dt'$, assuming $t > 0$. In the limit $t \to \infty$, we thus get $j(t, x) = \sigma(\omega, k)\, E(t, x)$, where

$$\sigma(\omega, k) \doteq \int_0^\infty \sigma(t', k)\, e^{i\omega t'}\, dt' \tag{B7}$$

is the frequency-domain representation of σ. Notice, however, that the integral [related to the Laplace transform of $\sigma(t, k)$] may diverge at too small $\operatorname{Im}\omega$. Since $\sigma(t, k)$ exists nonetheless, the general $\sigma(\omega, k)$ is hence *defined* via analytic continuation of Eq. (B7).

Explicitly, $\sigma(\omega, k)$ is found as follows. Note first that the current density, j, is the sum over the individual-particle current densities, $ev_i\delta(x - x_i)$. Then,

$$j(t, x) = en_0 \int v(t, \mathsf{z})\, \delta(x - x(t, \mathsf{z}))\, \mathsf{f}(t, \mathsf{z})\, d\mathsf{z}, \tag{B8}$$

where n_0 is the average density, and f is normalized such that $\int \mathsf{f}(t, \mathsf{z})\, d\mathsf{z}$ equals the system length. It is convenient to adopt x as the canonical coordinate here, so $\mathsf{z} = (x, mv)$; this yields $j = en_0 \int v f(t, x, v)\, dv$, where $f(t, x, v) = m\mathsf{f}(t, x, mv)$. Since f is conserved [Eq. (B4)], and $f_0(x_0, v_0)$ is independent of x_0 due to assumed inhomogeneity of the plasma [so $\int f_0(v)\, dv = 1$], we get

$$j(t, x) = en_0 \int_{-\infty}^{\infty} v f_0(v_0(t, x, v))\, dv. \tag{B9}$$

The velocity v_0 is found from the conservation of V, or $v - \tilde{v} = v_0 - \tilde{v}_0$, where $\tilde{v}_0 \doteq \tilde{v}(0, X, V) \approx \tilde{v}(0, x, v)$, so

$$j(t, x) = en_0 \int_{-\infty}^{\infty} v f_0(v - [\tilde{v}(t, x, v) - \tilde{v}_0(x, v)])\, dv. \tag{B10}$$

Expanding the latter to the first order in the field amplitude yields $j - en_0 \int v f_0(v)\, dv \doteq \Delta j$ in the form $\Delta j = -en_0 \int v f_0'(v)\, (\tilde{v} - \tilde{v}_0)\, dv$. Then, with $\tilde{v}$ taken from Eq. (9), we get

$$\Delta j(t, x) = -\frac{in_0 e^2}{m}\, E(t, x)(1 - e^{i\omega t}) \int_{-\infty}^{\infty} \frac{v f_0'(v)}{\omega - kv}\, dv. \tag{B11}$$

For the limit $\Delta j/E$ to exist at large t, we must require $\operatorname{Im}\omega > 0$, simply by definition of $\sigma(\omega, k)$ (see above). Also use $v/(\omega - kv) = -(1 + u/(v - u))/k$, where $u \doteq \omega/k$, and introduce $\omega_p^2 \doteq 4\pi ne^2/m$; then,

$$\sigma(\omega, k) = -\frac{\omega}{4\pi i}\, \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{f_0'(v)}{v - \omega/k}\, dv. \tag{B12}$$

Generalization to $\operatorname{Im}\omega \le 0$ is performed via analytic continuation of Eq. (B12). As usual, this can be done by replacing the integration contour with the Landau contour L that goes below the resonance pole at $k > 0$ and above it at $k < 0$ [1, Chap. 8]. Finally, summing over the conductivities (B12) of all species s also yields the dielectric function, $\epsilon = 1 + \sum_s 4\pi i\sigma_s/\omega$, in the form

$$\epsilon(\omega, k) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{\mathsf{L}} \frac{f_{0s}'(v)}{v - \omega/k}\, dv. \tag{B13}$$

Eigenwaves can hence be found by Laplace-transforming Gauss's law, $\epsilon(\omega, k)E = 4\pi\rho_{\text{ext}}$. Here ρ_{ext} is an external charge density of the same form as the field in Eq. (B6), and $t \to \infty$. At vanishing $\rho_{\rm ext}$, nonzero E is possible if

$$\epsilon(\omega, k) = 0, \tag{B14}$$

which then represents the sought dispersion relation.

A comment is due here pertaining to possible extensions of the above results. In case of a "phase-mixed" initial state, when f_0 is a function of V_0 rather than of v_0, a similar derivation yields Eq. (B10) without the term $\tilde{v}_0$. This eliminates the term $e^{i\omega t}$ from Eq. (B11), so $\Delta j/E = (in_0 e^2/m) \int v[f_0'(v)/(v-u)]\, dv$. Unlike in Eq. (B11), the right-hand side here is independent of time. The existence of $\sigma(\omega, k)$ is hence determined entirely by convergence of the velocity integral, which may be an issue at $\operatorname{Im}\omega = 0$. If the integral converges [113], then Eq. (B12) is valid at all ω; in particular, $\sigma(\omega, k)$ is purely imaginary at real ω and thus supports nondissipative waves. Otherwise, the concept of linear $\sigma(\omega, k)$ is, strictly speaking, ill-defined. (There is no reason to expect *a priori* that the essentially nonlinear dynamics of resonant particles *always* can be swept under the rug by the Landau rule.) To avoid this problem, general phase-mixed waves must be considered within a nonlinear theory, which we will now discuss; also see Sec. VI B.

3. BGK waves

Basic equations. — Here, we briefly revisit BGK waves [16], i.e., *stationary* nondissipative nonlinear waves which are allowed to contain both passing and trapped particles (Sec. II C). We will focus on periodic electron modes in their rest frame, assuming ions to form a homogeneous background. Then, Eq. (27) can be written as

$$\varphi''(x) = -4\pi e[n(x) - n_0], \tag{B15}$$

where $n(x)$ is the electron local density,

$$n(x) = n_0 \lambda \int \delta(x - x(\mathsf{z}))\, \mathsf{f}(\mathsf{z})\, d\mathsf{z}, \tag{B16}$$

and $\mathsf{f}(\mathsf{z})$ is normalized such that $\int \mathsf{f}(\mathsf{z})\, d\mathsf{z} = 1$. We will assume that there is only one minimum of the potential energy per period and adopt the action-angle variables $(\theta, J) \mathrel{=\!\!\cdot} \mathsf{z}$ *as they are defined in Sec. II C;* passing and trapped particles hence can be treated on the same footing. Since J are conserved, Eq. (B4) requires $\mathsf{f}(\mathsf{z}) = \mathsf{f}(J)$ for a wave to be stationary. (Such distributions are called phase-mixed.) Then,

$$n(x) = n_0 \int_0^\infty \mathsf{f}(J)\, \mathcal{G}(x, J)\, dJ, \tag{B17}$$

where we introduced a dimensionless function

$$\mathcal{G}(x, J) \doteq \lambda \int_0^{2\pi} \delta(x - x(\theta, J))\, d\theta$$
$$= \lambda \Omega(J) \int_0^{2\pi/\Omega(J)} \delta(x - x(t, J))\, dt. \tag{B18}$$

The argument of the delta function turns to zero at most once during the canonical period as we defined it, both for passing and trapped particles. One hence gets

$$\mathcal{G}(x, J) = \frac{\lambda \Omega(J)}{|w(x, J)|}\, \Theta(\varepsilon - e\varphi(x)). \tag{B19}$$

Here $w(x, J)$ is the instantaneous velocity w at location x on an orbit with a given J, and ε is given by Eq. (18), so

$$|w(x, J)| = \sqrt{(2/m)(\varepsilon - e\varphi)} \mathrel{=\!\!\cdot} \bar{w}(\varphi, \varepsilon). \tag{B20}$$

Using Eq. (22), one then obtains [114]

$$n = \lambda n_0 \int_{e\varphi}^\infty \frac{\mathsf{f}(\varepsilon)}{\bar{w}(\varphi, \varepsilon)}\, d\varepsilon, \tag{B21}$$

where $\mathsf{f}(\varepsilon) \doteq \mathsf{f}(J(\varepsilon))$ is normalized such that

$$\int_0^\infty 2\pi \Omega(\varepsilon)\, \mathsf{f}(\varepsilon)\, d\varepsilon = 1. \tag{B22}$$

Finding $F(J)$ from given $\varphi(x)$. — In conjunction with Poisson's equation, Eq. (B21) can be used to infer the particle distribution from $\varphi(x)$, namely, as follows. If $\varphi(x)$ is known, then $n(x)$ is also known via Eq. (B15), and thus we have $n(\varphi)$ too. Hence, Eq. (B21) leads to

$$\int_\phi^\infty \frac{\mathsf{f}(\varepsilon)}{\sqrt{\varepsilon - \phi}}\, d\varepsilon = \mathsf{g}(\phi), \tag{B23}$$

where $\phi \doteq e\varphi$, and $\mathsf{g}(\phi) = [n(\phi)/(\lambda n_0)]\sqrt{2/m}$. Using the inverse Abel transform then gives

$$\mathsf{f}(\varepsilon) = -\frac{1}{\pi} \int_\varepsilon^\infty \frac{\mathsf{g}'(\phi)}{\sqrt{\phi - \varepsilon}}\, d\phi \tag{B24}$$

for any given $\mathsf{g}(\phi)$. Therefore, one can find a distribution that realizes (almost) any given potential.

Finding $\varphi(x)$ from given $F(J)$. — Suppose now that $\mathsf{f}(\varepsilon)$ is given and search for $\varphi(x)$. Multiply Eq. (B15) by $\varphi'(x)$ and integrate the result over x. This gives

$$(\varphi')^2/(8\pi) + \mathcal{V}(\varphi) = \text{const}, \tag{B25}$$

where, up to an insignificant integration constant,

$$\mathcal{V}(\varphi) = -n_0 e\varphi + e \int^\varphi n(\varphi')\, d\varphi'. \tag{B26}$$

Equation (B26) leads to an equation of a nonlinear oscillator, $\varphi''(x) = -4\pi \mathcal{V}'(\varphi(x))$, with $4\pi\mathcal{V}$ acting at the effective potential energy; hence $\varphi(x)$ can be found (by a quadrature) for any given $\mathsf{f}(J)$. One can also calculate $\mathcal{V}(\varphi)$ explicitly using Eq. (B21) (cf. Refs. [16, 115]),

$$\mathcal{V}(\varphi) = -n_0 e\varphi - n_0 \lambda m \int_{e\varphi}^\infty \bar{w}(\varphi, \varepsilon)\, \mathsf{f}(\varepsilon)\, d\varepsilon.$$

Finally, multiple species s can be accommodated straightforwardly too, as their contributions are additive. In particular, if *all* species are phase-mixed, one gets

$$\mathcal{V}(\varphi) = -\lambda \sum_s n_{0s} m_s \int_{e_s\varphi}^\infty \bar{w}_s(\varphi, \varepsilon)\, f_s(\varepsilon)\, d\varepsilon, \tag{B27}$$

since $\sum_s n_{0s} e_s = 0$ due to plasma neutrality.

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[111] A dc magnetic field could also be included [126], but the derivation becomes close to intractable then [127], so one is referred instead to a much simpler, and more general, calculation within the Lagrangian approach [29].

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[113] Strictly speaking, we need $f_0(v)$ to be zero in the trapping-width vicinity of the resonance. This will make meaningful our Taylor expansion of $f_0(v-\tilde{v})$ and ensure that the velocity integral equals its principal value.

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