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Representation of Ideal Magnetohydrodynamic modes

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Abstract

One of the most fundamental properties of ideal magnetohydrodynamics is the condition that plasma motion cannot change magnetic topology. The conventional representation of ideal magnetohydrodynamic modes by perturbing a toroidal equilibrium field through $\delta\vec{B} = \nabla\times(\vec{\xi}\times\vec{B})$ ensures that $\delta\vec{B}\cdot\nabla\psi = 0$ at a resonance, with ψ labelling an equilibrium flux surface. Also useful for the analysis of guiding center orbits in a perturbed field is the representation $\delta\vec{B} = \nabla\times\alpha\vec{B}$. These two representations are equivalent, but the vanishing of $\delta\vec{B}\cdot\nabla\psi$ at a resonance is necessary but not sufficient for the preservation of field line topology, and a indiscriminate use of either perturbation in fact destroys the original equilibrium flux topology. It is necessary to find the perturbed field to all orders in $\vec{\xi}$ to conserve the original topology. The effect of using linearized perturbations on stability and growth rate calculations is discussed.

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The most rapidly growing instabilities in a plasma are the ideal modes, including interchange, kink and ballooning modes. For this reason they are associated with some of the most violent events occuring in plasmas, both in laboratory fusion experiments and in astrophysical plasmas. There is a long history of theoretical analysis of stability to and the calculation of growth rates for, ideal magnetohydrodynamic (MHD) modes[1–8]. One of the most important properties of MHD plasmas is the condition that the magnetic field be frozen in the plasma, *ie* that there can be no changes in the topology of the magnetic flux surfaces. Instabilities are studied with two major methods, that of the use of an energy principle, whereby stability of a system is guaranteed provided the energy is increased by any arbitrary plasma displacement, and the initial value approach, whereby an initial perturbation is introduced and equations of motion developed to ascertain whether the instability will grow. We demonstrate in this paper that in fact in both these methods using the perturbation to lowest order destroys the frozen in condition, the magnetic topology of the initial state is changed. To ensure topological invariance of the original flux surfaces the effect of the perturbation must be calculated to all orders, or a full Lagrangian treatment used to find the perturbed field. But normally in analytical and numerical instability calculations or in the analysis of the effect of ideal modes on particle trajectories only the lowest order perturbed field is used.

There are two representations of ideal magnetohydrodynamic modes used to describe instabilities regularly observed in toroidal magnetic confinement devices, to carry out theoretical analyses of stability and growth rates, and to analyze the effect of these modes on high energy particle populations. The first is $\delta\vec{B} = \nabla \times (\vec{\xi} \times \vec{B})$ with $\vec{\xi}$ the plasma displacement, and the second $\delta\vec{B} = \nabla \times \alpha\vec{B}$ with α a scalar function of position and time, and where $\vec{B}$ is the equilibrium field. The second representation is particularly useful for the analysis of guiding center equations for charged particles in a perturbed field. In this paper we relate these two representations and show that they produce equivalent changes of the cross field perturbation, the most important for the analysis of resonances, but to lowest order both destroy the initial flux surface topology of the equilibrium.

The magnetic perturbation $\delta\vec{B} = \nabla \times \alpha\vec{B}$ with $\vec{B}$ the equilibrium field is convenient for representing modes in low β plasmas, where β is the ratio of plasma to magnetic energy. For resistive modes it exactly represents the cross field magnitude of the perturbation, responsible for producing magnetic islands and most important for the production of resonances between high energy particles and MHD perturbations, and the guiding center equations for charged particles in this field are readily derived[9]. The function α is also simply related to the ideal MHD displacement $\vec{\xi}$. Using a time dependent α to describe the perturbation introduces an electric field parallel to $\vec{B}$ proportional to the mode frequency. But the rapid mobility of the electrons shorts out the parallel electric field felt by the ions; nonzero parallel electric field is forbidden in ideal MHD, so a potential must be introduced to cancel this field if the α form is to represent an ideal perturbation. The equilibrium field in a toroidal axisymmetric equilibrium has covariant and contravariant representations, given by $\vec{B} = (\nabla\zeta \times \nabla\psi)/q(\psi) + \nabla\psi \times \nabla\theta = g\nabla\zeta + I\nabla\theta + \delta\nabla\psi_p$ with $q(\psi)$ the field line helicity, ψ the toroidal flux, ψ_p the poloidal flux, θ and ζ poloidal and toroidal coordinates and ψ, θ, and ζ forming a right handed coordinate system with Jacobian $1/\mathcal{J} = \nabla\psi \cdot (\nabla\theta \times \nabla\zeta)$. Contravariant bases for the coordinate system are given by $\vec{e}^{\beta} = \nabla\beta$ with $\beta = \psi$, θ, and ζ.[9]

Expand α in a Fourier series of the form

$$\alpha = \sum_{m,n} \alpha_{m,n}(\psi) \sin(n\zeta - m\theta - \omega t), \tag{1}$$

with ω the mode frequency, giving

$$\delta\vec{B} \cdot \nabla\psi_p = \sum_{m,n} \frac{mg + nI}{\mathcal{J}} \alpha_{m,n} \cos(n\zeta - m\theta - \omega t). \tag{2}$$

Now add an electric potential Φ to cancel the parallel electric field induced by $d\vec{B}/dt$, with

$$\sum_{m,n} \omega B \alpha_{m,n} \sin(n\zeta - m\theta - \omega t) - \vec{B} \cdot \nabla\Phi/B = 0, \tag{3}$$

and make a similar Fourier expansion of the potential Φ, giving a very simple expression if one uses Boozer coordinates[10] with I independent of θ,

$$(gq + I)\omega\alpha_{mn} = (nq - m)\Phi_{mn}. \tag{4}$$

If coordinates are used in which I is a function of θ there is instead a coupling of different harmonics in this expression.

Now relate α to the ideal displacement $\vec{\xi}$. We require that the cross field component of the perturbation is exactly given by α,

$$\nabla\psi \cdot [\nabla \times (\vec{\xi} \times \vec{B})] = \nabla\psi \cdot [\nabla \times \alpha\vec{B}]. \tag{5}$$

Use the identity $\nabla \times (\vec{\xi} \times \vec{B}) = \vec{\xi}\nabla \cdot \vec{B} - \vec{B}\nabla \cdot \vec{\xi} + (\vec{B} \cdot \nabla)\vec{\xi} - (\vec{\xi} \cdot \nabla)\vec{B}$ and the fact that $\nabla \cdot \vec{B} = 0$. Now expand $\vec{B}$ and $\vec{\xi}$ in the covariant basis $\vec{\xi} = \xi^\alpha \vec{e}_\alpha$, $\vec{B} = B^\alpha \vec{e}_\alpha$, giving

$$\vec{e^\psi} \cdot \nabla \times (\vec{\xi} \times \vec{B}) = \vec{e^\psi} \cdot [B^\alpha \partial_\alpha (\xi^\beta \vec{e}_\beta) - \xi^\alpha \partial_\alpha (B^\beta \vec{e}_\beta)]. \tag{6}$$

All terms in $\partial_\alpha \vec{e}_\beta$ cancel. To see this write $\partial_\alpha \vec{e}_\beta = [\alpha\beta, \kappa] g^{\kappa\tau} \vec{e}_\tau$ with the Christoffel symbol

$$[\alpha\beta, \kappa] = \frac{1}{2}[\partial_\alpha g_{\beta,\kappa} + \partial_\beta g_{\alpha,\kappa} - \partial_\kappa g_{\alpha,\beta}], \tag{7}$$

from which we note the symmetry of this expression in α, β. These terms then become

$$[B^\alpha \xi^\beta - B^\beta \xi^\alpha][\alpha\beta, \kappa] g^{\kappa\psi} \tag{8}$$

which is zero by symmetry.

We then have

$$\vec{e^\psi} \cdot \nabla \times (\vec{\xi} \times \vec{B}) = \vec{e^\psi} \cdot \vec{e}_\beta [B^\alpha \partial_\alpha \xi^\beta - \xi^\alpha \partial_\alpha B^\beta], \tag{9}$$

and using $\vec{e^\psi} \cdot \vec{e}_\beta = \delta^\psi_\beta$ and $B^\psi = 0$ we have $\vec{e^\psi} \cdot \nabla \times (\vec{\xi} \times \vec{B}) = B^\alpha \partial_\alpha \xi^\psi$. The right hand side of Eq. 5 is simplified using $\nabla \times \vec{B} = \vec{j}$ and $\vec{j} \cdot \nabla\psi = 0$, giving finally

$$B^\alpha \partial_\alpha \xi^\psi = \nabla\alpha \cdot (\vec{B} \times \nabla\psi). \tag{10}$$

Make a similar Fourier decomposition of $\vec{\xi}$ through $\xi^\psi = \sum_{mn} \xi^\psi_{mn} sin(Q)$, $\xi^\theta = \sum_{mn} \xi^\theta_{mn} cos(Q)$, and $\xi^\zeta = \sum_{mn} \xi^\zeta_{mn} cos(Q)$ with $Q = n\zeta - m\theta - \omega t$. Again using Eq. 1 and $\vec{B} \cdot \nabla = \frac{1}{q\mathcal{J}}(\partial_\theta + q\partial_\zeta)$ and the covariant representation for $\vec{B}$, we have

$$\alpha_{mn} = \frac{(m/q - n)}{(mg + nI)} \xi^\psi_{mn}. \tag{11}$$

Thus alpha is simply related to the cross field component of the ideal displacement, and is zero at the rational surface $q = m/n$.

Magnetic islands are produced by resonant perturbations of $\vec{B}$ directed across the flux surfaces of an equilibrium, so the vanishing of α_{mn} at the rational surface and Eq. 2 is thought to exclude island formation. But the condition $\delta\vec{B} \cdot \nabla\psi = 0$ is a necessary but not a sufficient condition to exclude the formation of islands. In Fig. 1 is an example of the modification of magnetic flux surfaces by a single mode, showing islands appearing adjacent to the rational surface $r/a = 0.7$. We have used a circular equilibrium, with major radius

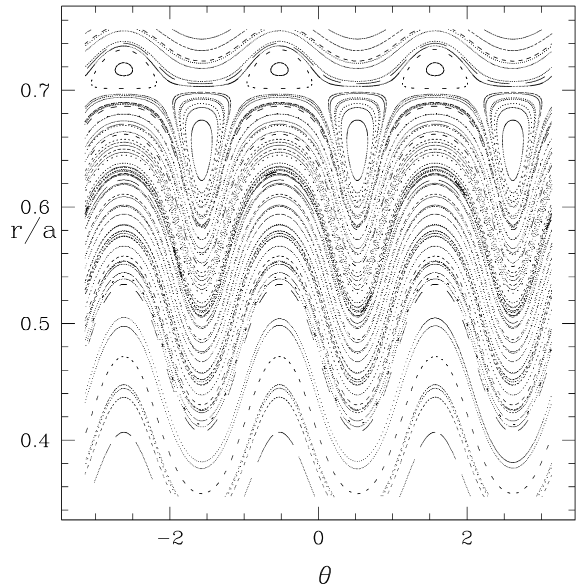

FIG. 1: A Poincaré plot of field lines for a linear ideal perturbation with $m/n = 3/2$, with $\alpha_{m,n}(r_s) = 0$ at the rational surface $r_s/a = 0.7$ where $q = 3/2$. No island elliptic points form at the rational surface, but hyperbolic points do, and islands form on each side of the surface.

$R = 1$, minor radius $a = 1/4$, and $B = 1$ at the magnetic axis, so the toroidal flux is $\psi = r^2/2$. The q profile is quadratic in minor radius r. The perturbation has $m/n = 3/2$, and $\alpha_{32}(\psi)$ is zero at the rational surface. The functional form taken was $\alpha_{32}(\psi) = A\psi(\psi_w - \psi)(m - nq)$ with ψ_w the value at the edge of the plasma. This function is smooth and square integrable.

We can find the nature of the magnetic field lines analytically. Let $Q = n\zeta - m\theta$, use the same circular equilibrium for which $g = 1$ and neglect I, which is second order in r/R. $d\psi = \vec{B} \cdot \nabla\psi ds$, $d\theta = \vec{B} \cdot \nabla\theta ds$, and $d\zeta = \vec{B} \cdot \nabla\zeta ds$, with s a parameterization of distance along the field line give

$$\frac{d\psi}{ds} = -m\alpha_{mn}\cos Q, \qquad \frac{dQ}{ds} = n - m/q + m\alpha'_{mn}\sin Q. \tag{12}$$

with $\alpha'_{mn} = d\alpha_{mn}/d\psi$. Fixed points are given by $d\psi = 0$, $dQ = 0$, or

$$\alpha_{mn}\cos Q = 0, \qquad n - m/q + m\alpha'_{mn}\sin Q = 0. \tag{13}$$

The first equation has two solutions. First consider $\alpha_{mn} = 0$, *i.e.* $\psi = \psi_r$, the rational surface. The second equation then gives $\sin Q = 0$. Expand $\psi = \psi_r + x$, and $\sin Q = \pm y$,

$\cos Q = \pm(1-y^2/2)$, with $x, y << 1$ and drop terms of third order in Eq. 12 . We then find

$$ydx + xdy \mp \frac{q'}{q^2\alpha'_{mn}} xdx = 0 \tag{14}$$

or

$$\frac{q'x^2}{q^2\alpha'_{mn}} \mp xy = constant \tag{15}$$

so these are hyperbolic points, located on the rational surface at $m\theta = k\pi$ with k integer, and with one separatrix lying along the x axis.

Similarly if $\cos Q = 0$ we have $Q = \pm\pi/2$, $\sin Q = \pm 1$, and $n - m/q = \pm m\alpha'_{mn}$ defines surfaces $\psi_\pm$ located away from the rational surface. Again expanding $\psi = \psi_\pm + x$ and $Q = \pm\pi/2 + y$ gives

$$xdx + \frac{\alpha_{mn}(\psi_\pm)q^2}{q'} ydy = 0 \tag{16}$$

or

$$x^2 + \frac{\alpha_{mn}(\psi_\pm)q^2}{q'} y^2 = constant \tag{17}$$

so these are elliptic points located at $\psi_\pm$ and $m\theta = \pm\pi/2$, clearly seen in Fig. 1. Since one separatrix lies along the rational surface we can estimate the island width to be twice the distance from the separatrix to the O-point, giving

$$\Delta\psi \simeq \frac{2\alpha'_{mn}q^2}{q'}, \tag{18}$$

and thus these islands have a width proportional to the perturbation amplitude, not the square root of the amplitude as do even parity resistive modes, so are much smaller. Ideal MHD does not allow reconnection to occur, so evolution of a mode from an infinitesimal initial state should not allow islands to evolve, but this condition is apparently not respected by a linearized ideal perturbation of an equilibrium state. Note that this analysis is linear in the perturbation, the obtained structure scales with the magnitude of the perturbation but is essentially unchanged. For this $m = 3$ perturbation there appear three elliptic points on each side of the separatrix.

Note that the equilibrium field has a continuum of fixed points along the rational surface. Every point on this line satisfies $dQ = d\psi = 0$.

Now consider the field lines defined by the usual ideal MHD perturbation $\delta\vec{B} = \nabla \times (\vec{\xi} \times \vec{B})$. Use a simple large aspect ratio equilibrium with coordinates ψ, θ, ϕ with $\vec{e}^{\beta} = \nabla\beta$ and $\mathcal{J} = 1$, and $\vec{B} = \vec{e}_{\phi} + \frac{1}{q}\vec{e}_{\theta}$. Write $\vec{\xi} = \xi^{\psi}\vec{e}_{\psi} + \xi^{\theta}\vec{e}_{\theta} + \xi^{\phi}\vec{e}_{\phi}$ and $\xi^{\psi} = \xi^{\psi}_{mn} sin(Q)$, $\xi^{\theta} = \xi^{\theta}_{mn} cos(Q)$, $\xi^{\phi} = \xi^{\phi}_{mn} cos(Q)$, with $Q = n\phi - m\theta$. Then using $\vec{e}_{\alpha} \times \vec{e}_{\beta} = \vec{e}^{\gamma}$ with α, β, γ cyclic we find

$$\vec{\xi} \times \vec{B} = (\xi^{\theta} - \xi^{\phi}/q)\nabla\psi - \xi^{\psi}\nabla\theta + (\xi^{\psi}/q)\nabla\phi \tag{19}$$

and

$$\begin{aligned}
\nabla\psi \cdot [\nabla \times (\vec{\xi} \times \vec{B})] &= \partial_{\phi}\xi^{\psi} + \partial_{\theta}\xi^{\psi}/q \\
\nabla\theta \cdot [\nabla \times (\vec{\xi} \times \vec{B})] &= \partial_{\phi}(\xi^{\theta} - \xi^{\phi}/q) - \partial_{\psi}(\xi^{\psi}/q) \\
\nabla\phi \cdot [\nabla \times (\vec{\xi} \times \vec{B})] &= -\partial_{\theta}(\xi^{\theta} - \xi^{\phi}/q) - \partial_{\psi}\xi^{\psi}
\end{aligned} \tag{20}$$

Then the field lines satisfy

$$\frac{d\psi}{ds} = -(m/q - n)\xi^{\psi}_{mn}\cos Q, \qquad \frac{dQ}{ds} = n - m/q + [(m/q - n)\xi^{\psi}_{mn}]'\sin Q. \tag{21}$$

but using $\alpha_{mn} = (1/q - n/m)\xi^{\psi}_{mn}$ we again obtain Eq. 12, leading to the islands located away from the rational surface.

To demonstrate how the topology is maintained provided one goes to all orders in the perturbation magnitude, consider a fluid perturbation of velocity $\vec{v}$, constant in time. Then

$$\frac{\partial\vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \tag{22}$$

but one must iterate this equation to find successive orders (in ξ) of the perturbed field with

$$\frac{\partial\vec{B}_{n+1}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}_{n}) \tag{23}$$

and at each order the fixed points for the map move closer and closer to the resonance surface, and the resonance surface itself is perturbed in a sinusoidal fashion.

A better way to envision what happens is to use the ideal MHD condition that the field is frozen into the plasma. Use the Lundquist identity[11, 12]

$$\vec{B}(\vec{r}, t) = \vec{B}_0(\vec{r}_0, 0) + (\vec{B}_0(\vec{r}_0, 0) \cdot \nabla_0)\vec{\xi}(\vec{r}_0, t), \tag{24}$$

where $\vec{r} = \vec{r}_0 + \vec{\xi}$. Use a simple circular large aspect ratio equilibrium field

$$\vec{B}_0(\vec{r}_0, 0) = \hat{\phi} + \frac{r_0}{q(r_0)}\hat{\theta} \tag{25}$$

and take $\vec{\xi} = \vec{\xi}_0 \sin Q$ with $\vec{\xi}_0$ a constant and $Q = n\phi - m\theta$. We then have

$$\vec{B}(\vec{r}_0, t) = \vec{B}_0(\vec{r}_0 - \vec{\xi}, 0) + [\vec{B}_0(\vec{r}_0 - \vec{\xi}, 0) \cdot \nabla_0]\vec{\xi}(\vec{r}_0, t) \tag{26}$$

or

$$\vec{B}(\vec{r}_0, t) = \hat{\phi} + \frac{(r_0 - \xi_0^r \sin Q)}{q(r_0 - \xi_0^r \sin Q)}\hat{\theta} + \left(n - \frac{m}{q(r_0 - \xi_0^r \sin Q)}\right)\vec{\xi}_0 \cos Q. \tag{27}$$

We then have for the field line equations, using also $\nabla_0\theta = (\hat{\theta}/r_0)d/d\theta$,

$$\frac{dr}{d\phi} = \left(n - \frac{m}{q(r_0 - \xi_0^r \sin Q)}\right)\xi_0^r \cos Q$$
$$\frac{dQ}{d\phi} = n - \frac{m}{q(r_0 - \xi_0^r \sin Q)} \tag{28}$$

where we have used $\nabla \cdot \vec{\xi} = 0$ and $\vec{\xi}_0$ constant. The only fixed points are the degenerate ones, lying on the displaced rational surface $q(r_0 - \xi_0^r \sin Q) = m/n$, there are no additional fixed points in the field.

Two things are notable about the field. First, expanding Eq. 27 in powers of ξ one obtains to lowest order the expressions leading to breaking of the topology. Secondly, if $\vec{\xi}$ is divergence free the field line equations involve only ξ^r, and thus can also be expressed in terms of α, meaning that these representations reproduce the same field to all orders.

It is possible that the small islands produced by the indiscriminate use of a perturbed field in either representation have an effect on instability analysis or on the effect of ideal modes on high energy particles. Numerical stability and growth rate codes which are linear in $\vec{\xi}$ make use of the perturbed magnetic field to calculate magnetic energy and plasma flow fields to calculate kinetic energy, and these are in error near the rational surface. If the modification of the flow near this surface contributes significantly to the kinetic energy some error could be present. It has been pointed out by Goedbloed and Dagazian [13] that the use of the perturbed field $\nabla \times (\vec{\xi} \times \vec{B})$ rather than the plasma displacement $\vec{\xi}$ can lead to errors concerning plasma stability, in particular arising from regions near the singular surface $q = m/n$ giving singularities in the integration of δW. This may be related to the present work since the singular surface is improperly treated by a linear perturbation, leading to the change in the magnetic topology near it. However no mention is made in this work of a modification of magnetic field topology and it is not clear whether it is related to the present result. The problem of the effect of the modification of the field line topology on plasma stability will be examined in detail in a future publication.

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