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Woltjer-Taylor state without Taylor's conjecture – plasma relaxation at all wavelengths

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Abstract

In astrophysical and laboratory plasmas, it has been discovered that plasmas relax towards the well-known Woltjer-Taylor state specified by $\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}$ for a constant α. To explain how such a relaxed state is reached, Taylor developed his famous relaxation theory based on the conjecture that the relaxation is dominated by short wavelength fluctuations. However, there is no conclusive experimental and numerical evidence to support Taylor's conjecture. A new theory is developed, which predicts that the system will evolve towards the Woltjer-Taylor state for an arbitrary fluctuation spectrum.

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In the study of astrophysical and laboratory plasmas [1–10], it has been observed that plasmas tend to evolve towards the final state satisfying

$$\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}, \quad \alpha \text{ is a constant.} \tag{1}$$

This is the well-known Woltjer-Taylor state. Theoretically, it is perplexing how and why such a state is reached. In 1958 Woltjer first showed that this state [11] is the state that minimizes the global magnetic energy

$$W \equiv \int_V \boldsymbol{B}^2 d^3\boldsymbol{x}, \tag{2}$$

while keeping the global magnetic helicity

$$H \equiv \int_V \boldsymbol{A} \cdot \boldsymbol{B} d^3\boldsymbol{x} \tag{3}$$

constant. In Eqs. (2) and (3) the integration domain V is the entire 3D volume surrounded by the perfectly-conducting wall of the vacuum vessel in an experiment. Woltjer [12] showed that in the ideal MHD (magnetohydrodynamics) model the global magnetic helicity H defined in Eq. (3) is a constant of motion, which justifies constraining H during the variation of W. A very simple variational calculation shows that such a minimizing state is indeed given by Eq. (1) with the constant α being a Lagrange multiplier. To be faithful to history, it is necessary to point out that such a force-free field with a constant α had been discussed earlier by Lust [13] and Chandrasekhar [14] in the context of astrophysics. But, no convincing justification was given as to why it should be the most interesting force free field. Woltjer's theory was the first theoretical attempt; however, it is not complete since it does not specify how this relaxed state can be reached. In the ideal MHD model, the global helicity defined in Eq. (3) is not the only invariant. For any given flux surface $\varphi = const.$, the helicity $H_\varphi \equiv \int_\varphi \boldsymbol{A} \cdot \boldsymbol{B} d^3\boldsymbol{x}$ of the volume enclosed by the flux surface is a conserved quantity, and the topological structures of the magnetic field are invariant with respect to the dynamics. As a consequence [15–17], for an arbitrary initial condition the final state specified by Eq. (1) is not be accessible.

To explain this puzzle, Taylor suggested that if the plasma is resistive, then the topological structure of the magnetic field will be destroyed, and the only invariant that may not subject to this destruction of flux surfaces is the global helicity H defined by Eq. (3). Taylor argued that in a slightly resistive plasma, the system will relax towards a final state which minimizes the global magnetic energy W while keeping the global helicity H constant, and the system will evolve towards this relaxed state for any given initial condition. Such a state is of course specified by Eq. (1). For this reason, we will refer this relaxed state as the Woltjer-Taylor state in this paper.

Taylor's theory is successful in terms of predicting the reversal of the toroidal field in a series of RFP (Reverse Field Pinch) experiments [1–10]. Even though it has been extended in various directions since the 1980s [17–26], Taylor's relaxation theory maintains its popularity because of its simplistic beauty, and is widely accepted as a fundamental theory with great importance in plasma physics. However, there is one unsatisfactory element in Theory's theory, *i.e.*, the so-called Taylor's conjecture. The relaxation in Taylor's theory is caused by resistivity. Rigoriously speaking, when resistivity is finite, *both* helicity H and magnetic energy W are not conserved quantities any more. In order to justify the variational procedure of minimizing W while keeping H constant, Taylor, along with other researchers [10, 15, 16, 27], observed that the decaying rates for H and W are

$$\frac{dH}{dt} \simeq -\frac{2Vc^2\eta}{4\pi} \sum_{\boldsymbol{k}} |\boldsymbol{k}| \boldsymbol{B}_{\boldsymbol{k}}^2 \tag{4}$$

$$\frac{dW}{dt} \simeq -\frac{2Vc^2\eta}{4\pi} \sum_{\boldsymbol{k}} \boldsymbol{k}^2 \boldsymbol{B}_{\boldsymbol{k}}^2 \tag{5}$$

when η is the resistivity, $\boldsymbol{k}$ is the wavenumber of the fluctuation and $\boldsymbol{B}_{\boldsymbol{k}}$ is the Fourier component of the magnetic field at $\boldsymbol{k}$. A detailed derivation of Eqs. (4) and (5) is given by Eqs. (13), (14) and (27), and the validity of the Eqs. (4) and (5) is discussed near the end of the paper. According to Eqs. (4) and (5) [10, 15, 16, 27], the dissipation of both the magnetic energy W and the helicity H are mainly due to the finite resistivity. However, the dissipation rate for W scales with $\boldsymbol{k}^2$, while that for H scales with $\boldsymbol{k}$. If the relaxation process is dominated by structures with wavelengths shorter than $\eta^{1/2}$ over the entire volume, then the dissipation rate of W is much larger than that of H. Taylor conjectured that this is indeed the case, and the justification of minimizing W with H fixed follows this conjecture naturally. "Unfortunately", as pointed out by Ortolani and Schnack in Ref. [10], "in the RFP there is no experimental evidence that relaxation is produced by small scale turbulence. The dominant magnetic fluctuations associated with the relaxation process appear to have global, long wavelength structure. This view is supported by extensive numerical simulations, which show that relaxation is produced by the nonlinear interaction of long wavelength instabilities. (Many of these results will be described in detail in Chapter 5)." It is at least fair to conclude that, through experimental [1–10] and theoretical [28–35] studies in the last 40 years, there is not conclusive evidence to support the conjecture that plasma relaxations are always dominated by short wavelength structures. Realizing this shortcoming in Taylor's theory, Bhattacharjee *et al.* [17, 18, 20–22] discovered that for a single helicity long wavelength resistive tearing mode, H is approximately an invariant along with an infinite set of other approximate invariants. A theory of relaxation has been developed using these invariants [17, 18, 20–22], and the relaxed state in general is different from the Woltjer-Taylor state.

In this paper, we present a new theory on how the Woltjer-Taylor state can be reached during the relaxation of a resistive plasma without invoking Taylor's conjecture. We do not assume that the fluctuation spectrum is dominated by short wavelength structures or that W decays faster than H. In our theory, the Woltjer-Taylor state is not reached by minimizing W with H fixed. We show that the Woltjer-Taylor state can be reached in a resistive MHD relaxation process for any fluctuation spectrum. We prove this fact as follows.

For any vector potential $\boldsymbol{A}$ and magnetic field $\boldsymbol{B}$, the well-known Cauchy-Schwartz inequality is

$$QW - H^2 \geq 0\,, \tag{6}$$

$$Q \equiv \int_V \boldsymbol{A}^2 d^3\boldsymbol{x}\,. \tag{7}$$

The equality is reached if and only if $\boldsymbol{B} = \alpha \boldsymbol{A}$ every where for a constant α. Amazingly, this equality condition is exactly Eq. (1), *i.e.*, the conditon for the Woltjer-Taylor state. We will further prove that in the resistive MHD model the difference between QW and H^2 decreases with time, *i.e.*,

$$\frac{d}{dt}\left(QW - H^2\right) \leq 0, \tag{8}$$

and that the equality in inequality (8) holds if and only if Eq. (1) is satisfied. From inequalities (6) and (8), it is evident that when the Woltjer-Taylor state is not reached, $QW - H^2$ is positive definite and decreases at a non-vanishing rate, *i.e.*, the system evolves towards the Woltjer-Taylor state. The non-negative value of $QW - H^2$ ceases to decrease only when it is zero, *i.e.*, when the Woltjor-Taylor state is reached. When the system is far away form the Woltjer-Taylor state, the change rate $d(QW - H^2)/dt$ can be significantly negative and system evolves towards the Woltjor-Taylor state at a fast pace. The dynamical behavior of H^2 and QW is illustrated in Fig. 1.

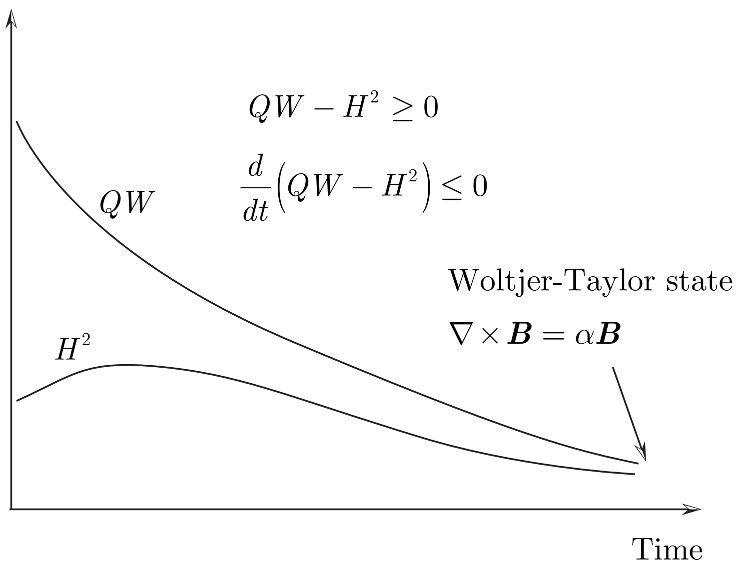

Figure 1: Before reaching the Woltjer-Taylor state, $QW - H^2$ is positive definite and decreases with time at a non-vanishing rate, and the system evolves towards the Woltjer-Taylor state. The system reaches the Woltjer-Taylor state when $QW - H^2 = 0$.

We now give the proof of inequality (8) and the fact that equality is reached if and only if at the Woltjor-Taylor state. As in previous studies, we will use the resistive MHD equations [27] and assume that the thermal energy and kinetic energy are much smaller than the magnetic energy. We will adopt the method of Fourier analysis. Let

$$(\boldsymbol{B}, \boldsymbol{J}, \boldsymbol{A}) = \sum (\boldsymbol{B}_{\boldsymbol{k}}, \boldsymbol{J}_{\boldsymbol{k}}, \boldsymbol{A}_{\boldsymbol{k}}) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \,, \tag{9}$$

then Ampére's law is expressed as

$$\boldsymbol{j}_{\boldsymbol{k}} = \frac{ic\boldsymbol{k} \times \boldsymbol{B}_{\boldsymbol{k}}}{4\pi} \,, \tag{10}$$

and $\nabla \times \boldsymbol{A} = \boldsymbol{B}$ is expressed as

$$i\boldsymbol{k} \times \boldsymbol{A}_{\boldsymbol{k}} = \boldsymbol{B}_{\boldsymbol{k}} \,. \tag{11}$$

For $\boldsymbol{k} = 0$, we will let $\boldsymbol{A}_0 = 0$, because the there is no magnetic field associated with $\boldsymbol{A}_0$ and we are free to choose an arbitrary constant for it. To solve for $\boldsymbol{A}_{\boldsymbol{k}} (\boldsymbol{k} \neq 0)$ in terms of $\boldsymbol{B}_{\boldsymbol{k}}$, we choose to work with the Coulomb gauge $\boldsymbol{k} \cdot \boldsymbol{A}_{\boldsymbol{k}} = 0$, which leads to

$$\boldsymbol{A}_{\boldsymbol{k}} = \frac{i\boldsymbol{k} \times \boldsymbol{B}_{\boldsymbol{k}}}{\boldsymbol{k}^2} \quad (\boldsymbol{k} \neq 0). \tag{12}$$

Because $(\boldsymbol{B}, \boldsymbol{J}, \boldsymbol{A})$ are real, we have $(\boldsymbol{B}^*_{\boldsymbol{k}}, \boldsymbol{J}^*_{\boldsymbol{k}}, \boldsymbol{A}^*_{\boldsymbol{k}}) = (\boldsymbol{B}_{-\boldsymbol{k}}, \boldsymbol{J}_{-\boldsymbol{k}}, \boldsymbol{A}_{-\boldsymbol{k}})$. Here $\boldsymbol{u}^*$ denotes the complex conjugate of $\boldsymbol{u}$, and we adopt the notation $\boldsymbol{u}^2 \equiv \boldsymbol{u} \cdot \boldsymbol{u}^* = |\boldsymbol{u}|^2$ for a vector $\boldsymbol{u}$. For resistive MHD, the rate of change of H is given by

$$\frac{dH}{dt} = -2c\int_V \eta \boldsymbol{j}\cdot\boldsymbol{B}d^3\boldsymbol{x}\,, \tag{13}$$

where the Ohm's law $\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}/c=\eta\boldsymbol{j}$ has been used. Note that rate of change of H can be both positive and negative. The integral in Eq. (13) can be evaluated using the Fourier components,

$$\int_V \boldsymbol{j}\cdot\boldsymbol{B}d^3\boldsymbol{x} = \sum_{\boldsymbol{k},\boldsymbol{l}}\boldsymbol{j}_{\boldsymbol{k}}\cdot\boldsymbol{B}_{\boldsymbol{l}}\int_V \exp[i(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{x}]d^3\boldsymbol{x} = V\sum_{\boldsymbol{k}}\boldsymbol{j}_{\boldsymbol{k}}\cdot\boldsymbol{B}_{-\boldsymbol{k}}$$
$$= V\sum_{\boldsymbol{k}}\boldsymbol{j}_{\boldsymbol{k}}\cdot\boldsymbol{B}^*_{\boldsymbol{k}} = V\sum_{\boldsymbol{k}}\frac{ic\boldsymbol{B}_{\boldsymbol{k}}\times\boldsymbol{B}^*_{\boldsymbol{k}}}{4\pi}\cdot\boldsymbol{k} = V\sum_{\boldsymbol{k}\neq 0}\frac{2c\boldsymbol{B}_{\boldsymbol{k}R}\times\boldsymbol{B}_{\boldsymbol{k}I}}{4\pi}\cdot\boldsymbol{k}\ , \tag{14}$$

where V is the volume of the system, $\boldsymbol{B}_{\boldsymbol{k}R}$ and $\boldsymbol{B}_{\boldsymbol{k}I}$ are real and imaginary parts of $\boldsymbol{B}_{\boldsymbol{k}}$, and use is made of the following identities

$$\int_V \exp[i(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{x}]d^3\boldsymbol{x} = \begin{cases} 0, & \boldsymbol{k}\neq -\boldsymbol{l}\,, \\ V, & \boldsymbol{k}=-\boldsymbol{l}\,, \end{cases} \tag{15}$$

$$\boldsymbol{B}_{\boldsymbol{k}}\times\boldsymbol{B}^*_{\boldsymbol{k}} = -2i\boldsymbol{B}_{\boldsymbol{k}R}\times\boldsymbol{B}_{\boldsymbol{k}I}\,. \tag{16}$$

Similarly,

$$H = V\sum_{\boldsymbol{k}\neq 0}\frac{i\boldsymbol{B}_{\boldsymbol{k}}\times\boldsymbol{B}^*_{\boldsymbol{k}}}{\boldsymbol{k}^2}\cdot\boldsymbol{k} = V\sum_{\boldsymbol{k}\neq 0}\frac{2\boldsymbol{B}_{\boldsymbol{k}R}\times\boldsymbol{B}_{\boldsymbol{k}I}}{\boldsymbol{k}^2}\cdot\boldsymbol{k}\,. \tag{17}$$

From Eqs. (13), (14), and (17), we have

$$\frac{dH^2}{dt} = -\frac{V^2c^2\eta}{\pi}\sum_{\boldsymbol{k}}\boldsymbol{B}_{\boldsymbol{k}R}\times\boldsymbol{B}_{\boldsymbol{k}I}\cdot\boldsymbol{k}\sum_{\boldsymbol{k}\neq 0}\frac{\boldsymbol{B}_{\boldsymbol{k}R}\times\boldsymbol{B}_{\boldsymbol{k}I}}{\boldsymbol{k}^2}\cdot\boldsymbol{k}\,. \tag{18}$$

The rate of change of H^2 given by Eq. (18) will be compared with that of QW,

$$\frac{d(QW)}{dt} = Q\frac{dW}{dt} + W\frac{dQ}{dt}\ , \tag{19}$$

where W and Q can be expressed in terms of the Fourier components of $\boldsymbol{B}$,

$$W \equiv \int_V \boldsymbol{B}^2 d^3\boldsymbol{x} = V\sum_{\boldsymbol{k}}\boldsymbol{B}_{\boldsymbol{k}}\cdot\boldsymbol{B}^*_{\boldsymbol{k}}, \tag{20}$$

$$Q \equiv \int_V \boldsymbol{A}^2 d^3\boldsymbol{x} = V\sum_{\boldsymbol{k}\neq 0}\frac{\boldsymbol{B}_{\boldsymbol{k}}\cdot\boldsymbol{B}^*_{\boldsymbol{k}}}{\boldsymbol{k}^2}\,. \tag{21}$$

To calculate the rate of change of Q and W, we need to know $d\boldsymbol{B_k}/dt \cdot \boldsymbol{B_k^*}$, which can be calculated from Faraday's law $d\boldsymbol{B_k}/dt = -ic\boldsymbol{k} \times \boldsymbol{E_k}$ and Ohm's law as

$$\frac{d\boldsymbol{B_k}}{dt} \cdot \boldsymbol{B_k^*} = -ic\eta\boldsymbol{k} \times \boldsymbol{j_k} \cdot \boldsymbol{B_k^*} + i\boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B})_{\boldsymbol{k}} \cdot \boldsymbol{B_k^*}\,. \tag{22}$$

The second term on the right-hand-side of Eq. (22) can be expressed in terms of the current

$$i\boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B})_{\boldsymbol{k}} \cdot \boldsymbol{B_k^*} = \frac{4\pi}{c}\sum_{\boldsymbol{l}} \boldsymbol{B_l} \times \boldsymbol{j_{-k}} \cdot \boldsymbol{v_{k-l}} \tag{23}$$

which is associated with the variation of kinetic energy due to the Lorentz force. To see this, we observe that the energy conservation law in the low-β limit takes the form of

$$\frac{\partial}{\partial t}\left(\rho\frac{\boldsymbol{v}^2}{2}\right) + \nabla\cdot\left(\rho\boldsymbol{v}\frac{\boldsymbol{v}^2}{2}\right) - \frac{\boldsymbol{v}}{c}\cdot(\boldsymbol{j} \times \boldsymbol{B}) = 0\,. \tag{24}$$

Integrating over the entire volume gives

$$\frac{\partial}{\partial t}\left(\int_V \rho\frac{\boldsymbol{v}^2}{2}d^3\boldsymbol{x}\right) = \int_V \frac{\boldsymbol{v}}{c}\cdot(\boldsymbol{j} \times \boldsymbol{B})\,d^3\boldsymbol{x} = V\sum_{\boldsymbol{k},\boldsymbol{l}} \frac{1}{c}\boldsymbol{v_{k-l}}\cdot\boldsymbol{j_{-k}}\times\boldsymbol{B_l} = -\frac{V}{4\pi}\sum_{\boldsymbol{k}} i\boldsymbol{k}\times(\boldsymbol{v}\times\boldsymbol{B})_{\boldsymbol{k}}\cdot\boldsymbol{B_k^*}\,, \tag{25}$$

which indicates that the work done by the Lorentz force is converted into the variation of the kinetic energy. Since we have assumed that the kinetic energy is much smaller than the magnetic energy, *i.e.*, $\int_V \rho\boldsymbol{v}^2 d\boldsymbol{x}/2 \ll \int_V \boldsymbol{B}^2 d^3\boldsymbol{x}/8\pi$, it is therefore clear that the work due to the Lorentz force can only produce a small variation of the magnetic energy, *i.e.*,

$$i\sum_{\boldsymbol{k}} \boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B})_{\boldsymbol{k}} \cdot \boldsymbol{B_k^*} \ll \sum_{\boldsymbol{k}} \frac{d\boldsymbol{B_k}}{dt} \cdot \boldsymbol{B_k^*}, \tag{26}$$

and the variation of the magnetic energy is mainly due to the finite resistivity,

$$\frac{dW}{dt} = \int_V 2\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}d^3\boldsymbol{x} = 2V\sum_{\boldsymbol{k}} \boldsymbol{B_k}\cdot d\boldsymbol{B_k^*}/dt = -\frac{2Vc^2\eta}{4\pi}\sum_{\boldsymbol{k}} \boldsymbol{k}^2\boldsymbol{B_k}\cdot\boldsymbol{B_k^*}\,. \tag{27}$$

We further assume that the condition that the kinetic energy is much smaller than the magnetic energy holds for large scales, and thus when each term in Eq. (26) is weighted by $1/\boldsymbol{k}^2$, the total contribution from the Lorentz force term is still small, *i.e.*,

$$i\sum_{\boldsymbol{k}} \frac{1}{\boldsymbol{k}^2}\boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B})_{\boldsymbol{k}} \cdot \boldsymbol{B_k^*} \ll \sum_{\boldsymbol{k}} \frac{1}{\boldsymbol{k}^2}\frac{d\boldsymbol{B_k}}{dt} \cdot \boldsymbol{B_k^*}. \tag{28}$$

Under these conditions, the rate of change of Q can be expressed in terms of $\boldsymbol{B_k}\cdot\boldsymbol{B_k^*}$ as well,

$$\frac{dQ}{dt} = \int_V 2\boldsymbol{A}\cdot\frac{\partial\boldsymbol{A}}{\partial t}d^3\boldsymbol{x} = 2V\sum_{k\neq 0} \frac{\boldsymbol{B_k}\cdot d\boldsymbol{B_k^*}/dt}{\boldsymbol{k}^2} = -\frac{2Vc^2\eta}{4\pi}\sum_{k\neq 0} \boldsymbol{B_k}\cdot\boldsymbol{B_k^*}\,. \tag{29}$$

We will give a more detailed discussion on the validity of Eqs. (5), (27) and (29) near the end of the paper. Equations (21), (20), (27), and (29) can be assembled together to give

$$\frac{d(QW)}{dt} = -\frac{2c^2V^2\eta}{4\pi}\left[\boldsymbol{B}_0^2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2 + \left(\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2\right)^2 + \sum_{\boldsymbol{k}\neq 0}\frac{\boldsymbol{B}_{\boldsymbol{k}}^2}{\boldsymbol{k}^2}\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2\boldsymbol{k}^2\right]$$
$$= -\frac{2c^2V^2\eta}{4\pi}\left[\boldsymbol{B}_0^2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2 + 2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^4 + \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{l}\neq 0}^{\boldsymbol{k}\neq\boldsymbol{l}}\left(2+\frac{\boldsymbol{l}^2}{\boldsymbol{k}^2}+\frac{\boldsymbol{k}^2}{\boldsymbol{l}^2}\right)\boldsymbol{B}_{\boldsymbol{k}}^2\boldsymbol{B}_{\boldsymbol{l}}^2\right]$$
$$\leq -\frac{2c^2V^2\eta}{4\pi}\left[\boldsymbol{B}_0^2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2 + 2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^4 + \sum_{\boldsymbol{k},\boldsymbol{l}\neq 0}^{\boldsymbol{k}\neq\boldsymbol{l}}\left(\frac{|\,\boldsymbol{l}\,|}{|\,\boldsymbol{k}\,|}+\frac{|\,\boldsymbol{k}\,|}{|\,\boldsymbol{l}\,|}\right)\boldsymbol{B}_{\boldsymbol{k}}^2\boldsymbol{B}_{\boldsymbol{l}}^2\right], \quad (30)$$

where the following inequality has been used

$$\left(2+\frac{\boldsymbol{l}^2}{\boldsymbol{k}^2}+\frac{\boldsymbol{k}^2}{\boldsymbol{l}^2}\right) = \left(\frac{|\,\boldsymbol{l}\,|}{|\,\boldsymbol{k}\,|}+\frac{|\,\boldsymbol{k}\,|}{|\,\boldsymbol{l}\,|}\right)^2 \geq 2\left(\frac{|\,\boldsymbol{l}\,|}{|\,\boldsymbol{k}\,|}+\frac{|\,\boldsymbol{k}\,|}{|\,\boldsymbol{l}\,|}\right). \quad (31)$$

On the other hand, from Eq. (18),

$$-\frac{dH^2}{dt} \leq \frac{V^2c^2\eta}{\pi}\sum_{\boldsymbol{k}} 2\,|\,\boldsymbol{B}_{\boldsymbol{k}R}\,||\,\boldsymbol{B}_{\boldsymbol{k}I}\,||\,\boldsymbol{k}\,|\sum_{\boldsymbol{k}\neq 0} 2\,|\,\boldsymbol{B}_{\boldsymbol{k}R}\,||\,\boldsymbol{B}_{\boldsymbol{k}I}\,|\,/\,|\,\boldsymbol{k}\,|$$
$$\leq \frac{V^2c^2\eta}{\pi}\sum_{\boldsymbol{k}}\boldsymbol{B}_{\boldsymbol{k}}^2\,|\,\boldsymbol{k}\,|\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2/\,|\,\boldsymbol{k}\,| = \frac{V^2c^2\eta}{\pi}\left[\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^4 + \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{l}\neq 0}^{\boldsymbol{k}\neq\boldsymbol{l}}\left(\frac{|\,\boldsymbol{l}\,|}{|\,\boldsymbol{k}\,|}+\frac{|\,\boldsymbol{k}\,|}{|\,\boldsymbol{l}\,|}\right)\boldsymbol{B}_{\boldsymbol{k}}^2\boldsymbol{B}_{\boldsymbol{l}}^2\right]. \quad (32)$$

Combining inequalities (30) and (32), we have inequality (8). Examining inequalities (30) and (32) shows that the equality in (8) is reached when the following four conditions are satisfied (i) $\boldsymbol{B}_0 = 0$, (ii) $\boldsymbol{B}_{\boldsymbol{k}R}$, $\boldsymbol{B}_{\boldsymbol{k}I}$, and $\boldsymbol{k}$ are perpendicular to each other, (iii) $|\boldsymbol{B}_{\boldsymbol{k}R}| = |\boldsymbol{B}_{\boldsymbol{k}I}|$, and (iv) all of the non-zero components will have the same $|\,\boldsymbol{k}\,|$,*i.e.*, $|\,\boldsymbol{k}\,| = \alpha$. By applying the Fourier analysis to Eq. (1), it is easy to verify that these four conditions are necessary and sufficient for Eq. (1) to be satisfied. This completes the proof of $d\left(QW - H^2\right)/dt \leq 0$ and the fact that the equality holds only at the Woltjer-Taylor state.

In the derivation of Eqs. (5), (27) and (29), we have assumed that the variation of the magnetic energy due to the Lorentz force is small compared with that due to the resistivity. This approximation has been essentially adopted by Taylor and other researchers [10, 15, 16, 27] when Eq. (5) is used. Here we discuss the validity of this assumption. First of all, it is reasonable to argue that the Lorentz force term $i\boldsymbol{k}\times(\boldsymbol{v}\times\boldsymbol{B})_{\boldsymbol{k}}\cdot\boldsymbol{B}_{\boldsymbol{k}}^* = \frac{4\pi}{c}\sum_{\boldsymbol{l}}\boldsymbol{B}_{\boldsymbol{l}}\times\boldsymbol{j}_{-\boldsymbol{k}}\cdot\boldsymbol{v}_{\boldsymbol{k}-\boldsymbol{l}}$ in Eq. (22) is a nonlinear term responsible for generating the turbulence and therefore should not be ignored. On the other hand, if we consider the energy equation (25) for the kinetic energy in the low β limit, it is clear that the part of the magnetic energy variation due to the

Lorentz force is completely converted into the kinetic energy. This is of course not surprising. If we assume that the kinetic energy in the system is smaller than the magnetic energy for all time, then the work done by the Lorentz force, *i.e.*, the energy exchange between the magnetic energy and kinetic energy, has to be smaller than the magnetic energy. Thus any substantial magnetic energy variation has to be caused by the resistivity. In the paper, we further assume that this fact is true at large scales. We emphasize that this assumption does not imply that the nonlinearity of the Lorentz force is not as important as other nonlinearities, such as the convection term $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$, in the relaxation process. It only means that the total magnetic energy *dissipation* is mainly due to the resistivity. This assumption is actually independent from other assumptions that one may wish to adopt in developing a theory for plasma relaxation, and we believe that this assumption is a crucial component of Taylor's theory [*e.g.*, Eq. (5)] as well as our theory. This further corroborates the view that the Woltjer-Taylor state will not be that to which a plasma relaxes when β is high or the kinetic energy is comparable to the magnetic energy [23–26].

It is also necessary to point out that as a theoretical model, certain theoretical simplifications have been adopted in order to make progress. For example, as in previous theoretical analysis carried out by Taylor [15, 16], Schnack [27], and Bhattacharjee [17, 18, 20–22], we have assumed that resistivity is a constant, even though the resistivity in laboratory discharge experiments varies significantly between the center and the edge of the plasma. Nevertheless, we believe the theoretical understanding enabled by these theoretical simplifications do provide valuable insights into the complex plasma relaxation process. To bring our understanding to the next level with the effect of inhomogeneous resistivity, along with other effects such as the pressure and density gradients, further investigations are certainly necessary, probably with new and refined theoretical tools and methods.

To summarize, in our new theory for plasma relaxation, the relaxed Woltjer-Taylor state is reached as the non-negative quantity $QW - H^2$ evolves towards zero. In contrast to Taylor's theory, which is only valid for relaxation dominated by short wavelength fluctuations, our theory is valid for an arbitrary perturbation spectrum. The new theory can be tested by experiments and numerical simulations. The prediction of the new theory, specifically, the inequality (8) and the variations of H^2 and QW illustrated in Fig. 1 can be verified using magnetic fluctuation spectrum data. Testing the validity of this new relaxation theory is one of the scientific objectives of the Keda Toroidal eXperiment (KTX), a RFP device that is being constructed at the University of Science and Technology of China.

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