Princeton Plasma Physics Laboratory

PPPL- 4639

PPPL- 4639

Neoclassical Drift of Circulating Oorbits Due to Toroidal Electric Field in Tokamaks

Hong Qin, Xiaoyin Guan and Nathaniel J. Fisch

July, 2011

Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

RAPID COMMUNICATION

Neoclassical Drift of Circulating Orbits Due to Toroidal Electric Field in Tokamaks

Hong Qin

Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China

Theory Center for Magnetic Fusion, Chinese Academy of Sciences, Hefei, Anhui 230026, China

Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543

Xiaoyin Guan

Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543

Nathaniel J. Fisch

Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543

Abstract. In tokamaks, Ware pinch is a well known neoclassical effect for trapped particles in response to a toroidal electric field. It is generally believed that there exists no similar neoclassical effect for circulating particles without collisions. However, this belief is erroneous, and misses an important effect. We show both analytically and numerically that under the influence of a toroidal electric field parallel to the current, the circulating orbits drift outward toward the outer wall with a characteristic velocity $O\left(\varepsilon^{-1}\right)$ larger than the $E \times B$ velocity, where ε is the inverse aspect-ratio of a tokamak. During a RF overdrive, the toroidal electric field is anti-parallel to the current. As a consequence, all charged particles, including backward runaway electrons, will drift inward towards the inner wall.

Keywords: Neoclassical drift, Tokamak, Circulating Particles

PACS numbers: 52.20.Dq, 52.55.Fa

Submitted to: *Nuclear Fusion*

In the magnetic field of a tokamak, a charged particle's trajectory can be either trapped or circulating. When there exists an electric field in the toroidal direction parallel to the current, the trapped orbits will be pinched inward radially. This is the well known Ware pinch [1]. The inward pinch velocity is

$$v_w = \frac{cE_\xi}{B_\theta}\,, \tag{1}$$

where E_ξ is the toroidal electric field in the co-current direction, and B_θ is the strength of the poloidal magnetic field. The Ware pinch velocity is larger than the standard $E \times B$ drift velocity

$$v_{E\times B} = \frac{cE_\xi B_\theta}{B^2} \tag{2}$$

by $O\left(\varepsilon^{-2}\right)$. Here B is the total magnetic field strength and ε is the inverse aspect ratio. The Wave pinch is caused by the neoclassical effect associated with the toroidal geometry of a tokamak. Because of the symmetry with respect to the toroidal direction, the canonical momentum $p_\xi = -e\psi/c + muB_\xi/B - eE_\xi R_0 t$ [Eq. (8)] is conserved. The momentum input due to the electric field cannot be reflected by the increase of the parallel velocity u, since the particle is trapped and the average parallel velocity cannot increase. Therefore, the magnetic flux ψ must decrease to compensate for the momentum input, which results in the inward pinch.

For a circulating particle, it is commonly believed that there is no such neoclassical drift [1–4], and the toroidal electric field only produces the standard $E \times B$ drift given by Eq. (2), which is in the negative radial direction. However, this belief is erroneous, and misses an important effect. We will show both analytically and numerically that in a large aspect ratio tokamak with nearly circular flux surfaces, the circulating orbit actually drifts outward toward the wall with a characteristic drift velocity

$$v_d \approx \frac{cqE_\xi}{B}, \tag{3}$$

where q is the safety factor of the tokamak magnetic field. This outward drift is larger than the standard $E \times B$ drift by $O\left(\varepsilon^{-1}\right)$. It is induced by toroidicity, as in the case of the Ware pinch for trapped particles. All circulating orbits drift with the same velocity, and the drift velocity is independent of charge, mass, and energy, as in the cases of the $E \times B$ drift and Ware pinch. Before giving a formal derivation of Eq. (3) in a general tokamak geometry with non-circular flux surfaces, let's first look at two simple physical pictures of this outward drift in a simplified tokamak geometry. It is well known that in a simplified tokamak geometry with circular concentric flux surfaces, the displacement d of a circulating orbit relative to the flux surfaces is proportional to the parallel velocity u,

$$d \approx \frac{cqu}{emB},$$

where e is the charge and m is the mass [5]. If u is accelerated or decelerated by E_ξ, *i.e.* $m\Delta u = eE_\xi \Delta t$, then we see that the displacement moves as

$$\frac{\Delta d}{\Delta t} \approx \frac{cqE_\xi}{B},$$

which is same as Eq. (3). This is of course the neoclassical effect associated with the toroidal geometry. It may sound trivial that larger u corresponds to larger orbit displacement relative to the flux surfaces. But it is remarkable that when the increase of u is induced by a toroidal electric field, the resulting orbit displacement is $O(\varepsilon^{-1})$ larger than the $E \times B$ drift, which contradicts the common wisdom [1–4]. In this simple physical picture, we need to pay extra attention to the sign of the displacement, which can be either outward or inward depending on the signs of charge e and parallel velocity u. Under the influence of E_ξ, the absolute value of u can either decrease or increase. However, the drift of the orbit only depends on E_ξ. It always drifts outward for co-current E_ξ, even if the absolute value of u decreases. This fact shows that drift of the circulating orbit is in the same category of the $E \times B$ drift and Ware pinch. Another physical picture can be given using the argument of canonical momentum. On average, the momentum input by the electric field is balanced by the increase of the kinetic angular momentum. However, the kinetic angular momentum increase inside ($R < R_0$) is smaller than that outside ($R > R_0$). To conserve the canonical momentum, the circulating orbit drifts outward. These two physical pictures are constructed using a simplified tokamak geometries with circular concentric flux surfaces. In Ref. [6], we have demonstrated this effect for runaway electrons in this simplified geometry. In this paper, we show that this drift exists for all charged particles in more general geometries with flux surfaces whose non-circularity is $O(\varepsilon)$.

During the Ohmic discharge and global disruptions, a significant toroidal field will be generated in the co-current direction. The neoclassical behavior of circulating orbits under the influence of a toroidal electric field becomes important. For example, the outward drift of the circulating orbit provides a mechanism for the runaway

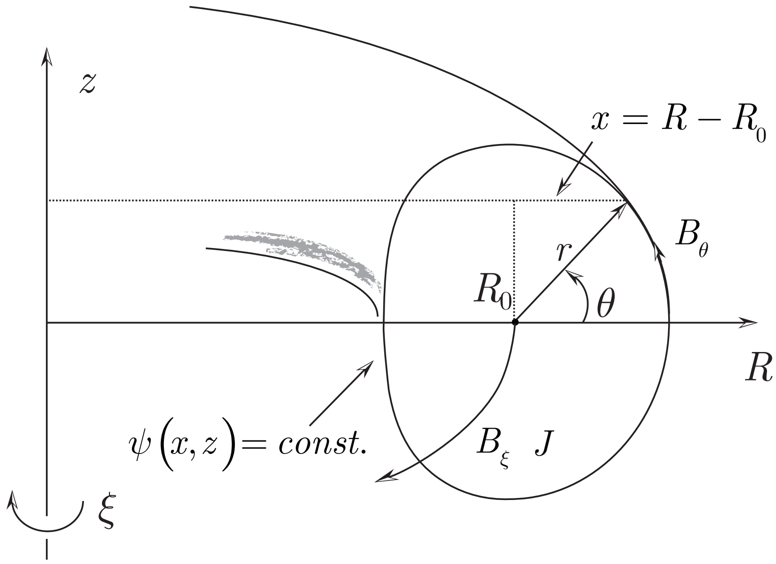

Figure 1. The magnetic geometry and coordinate systems for a large aspect-ratio tokamak.

electrons [7–11] to strike the first wall or limiter [6]. During lower hybrid current drive [12], the wave-particle interaction will selectively generate momentum input for resonant circulating (super-thermal) electrons, and these current-carrying electrons will drift outward according to our theory.

We begin our study from the Lagrangian of the guiding center dynamics in a magnetized plasma [13]

$$L = \left(\frac{e}{c}\mathbf{A} + mu\mathbf{b}\right)\cdot\dot{\mathbf{x}} - \left(\mu B + \frac{1}{2}mu^2\right) , \tag{4}$$

where $\mathbf{A}$ is the vector potential, $\mathbf{b}$ is the unit vector in the direction of the magnetic field, u is the parallel velocity of the guiding center, $\mathbf{x}$ is the guiding center position, μ is the magnetic moment, e is the charge, and m is the mass. In our study, the magnetic field is that of a large aspect-ratio tokamak with toroidal symmetry. The magnetic geometry and coordinate systems are illustrated in Fig. 1. The vector potential in a tokamak with toroidal symmetry is

$$\mathbf{A} = \psi_T\nabla\theta - \psi\nabla\xi\,. \tag{5}$$

The corresponding magnetic field is

$$\mathbf{B} = \nabla\psi_T\times\nabla\theta - \nabla\psi\times\nabla\xi\,.$$

Here θ and ξ are properly chosen poloidal and toroidal angles for the flux coordinates. Functions ψ_T and ψ are the toroidal flux and poloidal flux respectively. The covariant representations for $\mathbf{B}$ is

$$\begin{aligned}\mathbf{B} &= B_\psi \nabla\psi + B_\theta \nabla\theta + B_\xi \nabla\xi\,,\\ B_\psi &= \frac{1}{J}\left(\nabla\theta\times\nabla\xi\right)\cdot\mathbf{B}\,,\\ B_\theta &= \frac{1}{J}(\nabla\xi\times\nabla\psi\cdot\mathbf{B}\,,\\ B_\xi &= \frac{1}{J}\left(\nabla\psi\times\nabla\theta\right)\cdot\mathbf{B}\,.\end{aligned}$$

To include a ξ-independent toroidal electric field $\mathbf{E}_\xi$, we let

$$\mathbf{A}_1 = -cE_\xi R_0 t\nabla\xi\,,$$

which corresponds to an inductive field due to a loop voltage induced by the the primary winding. Let

$$\dot{\psi}\equiv\dot{\mathbf{x}}\cdot\nabla\psi\,,\ \dot{\theta}\equiv\dot{\mathbf{x}}\cdot\nabla\theta\,,\ \dot{\xi}\equiv\dot{\mathbf{x}}\cdot\nabla\xi\,, \tag{6}$$

representing the contravariant components of $\dot{\mathbf{x}}$ in the flux coordinate (ψ,θ,ξ) . Then the Lagrangian acquires the form

$$L = \frac{mu}{B}B_\psi\dot{\psi} + \left(\frac{e}{c}\psi_T + \frac{mu}{B}B_\theta\right)\dot{\theta} + \left(-\frac{e}{c}\psi + \frac{mu}{B}B_\xi - eE_\xi R_0 t\right)\dot{\xi} - \left(\frac{m}{2}u^2 + \mu B\right)\,. \tag{7}$$

Because $\partial L/\partial\xi = 0$, the canonical moment is conserved, *i.e.*,

$$\begin{aligned}p_\xi = \frac{\partial L}{\partial\dot{\xi}} &= -\frac{e}{c}\psi + \frac{mu}{B}B_\xi - eE_\xi R_0 t\\ &= -\frac{e}{c}\psi + muR - eE_\xi R_0 t + O\left(\varepsilon^2\right) = const.\,,\end{aligned} \tag{8}$$

where we have made use of the fact that $B_\xi/B = R + O\left(\varepsilon^2\right)$. Since our purpose is to show that the outward drift of circulating orbit is $O\left(\varepsilon\right)$, we will neglect terms of $O\left(\varepsilon^2\right)$ hereinafter. After the parallel velocity u is known, Eq. (8) determines the projection of the trajectory on the poloidal plane. The parallel velocity can be solved from the energy equation

$$H\left(t_1\right) - H\left(t_0\right) = e\int_{t_0}^{t_1} E_\xi R\dot{\xi}\,dt\,. \tag{9}$$

If there is no toroidal electric field, *i.e.*, $E_\xi = 0$, then Eq. (9) states that the energy is conserved, and it can be solved for u in terms of μB as $u = \sqrt{(2H - \mu B)/m}$. Substituting this expression into Eq. (8), we obtain an equation determines the closed curve for the projection of the trajectory in the poloidal plane. When E_ξ is not zero, u increases with time for circulating particles, and the orbit is no longer closed in the poloidal plane. However, the increase of u induced by the electric field in one poloidal transition is a small quantity, because the energy increase due to the loop voltage eV_{loop} is always much smaller than the parallel kinetic energy. We can thus consider the small variation of the otherwise closed trajectory in the poloidal plane due to the inductive field E_ξ. In the time interval of one poloidal period centered at $t = t_0$, the trajectory is given by

$$p_\xi = f_0\left(x, z\right) = -\frac{e}{c}\psi + mu(x + R_0) - eE_\xi R_0 t_0\,. \tag{10}$$

Around $t = t_0 + \Delta t$, where Δt is a large multiple of one poloidal transit time, the parallel velocity will increase by Δu, and the trajectory is given by

$$p_\xi = f_1\left(x, z\right) = f_0\left(x, z\right) + mx\Delta u + (m\Delta u - eE_\xi \Delta t)R_0\,. \tag{11}$$

From Eq. (11), we see that the orbit is perturbed at $t_0 + \Delta t$ in two ways. The term $mx\Delta u$ will cause the orbit to drift outward in the x-direction, as we will show later. The term $(m\Delta u - eE_\xi \Delta t)R_0$ will induce radial expansion or contraction for the orbit, because it does not depend on x and z for Δt much larger than one poloidal period, and its effect is equivalent to modifying the conserved p_ξ. But, we can show that $m\Delta u - eE_\xi \Delta t = O\left(\varepsilon^2\right)$, if the non-circularity of the flux surfaces is $O\left(\varepsilon\right)$. From Eq. (9), we obtain

$$mu\Delta u + \mu\Delta B = \int_{t_0}^{t_0+\Delta t} eR\dot{\xi}E_\xi \, dt\,. \tag{12}$$

Because the orbit displacement is small, the integration can be calculated along the unperturbed orbit, and the $\mu\Delta B$ term can be ignored, because it will be clear and verified later that

$$\mu\Delta B \sim \mu\frac{\partial B}{\partial x}\Delta x \ll mu\Delta u\,. \tag{13}$$

For flux surfaces whose non-circularity is $O\left(\varepsilon\right)$, we expect that

$$R = R_0 + c_1\left(\psi\right)\cos\theta + c_2\left(\psi\right)\sin\theta + O\left(\varepsilon^2\right)\,,$$
$$\dot{\xi} = \dot{\xi}^{(0)} + c_3\left(\psi\right)\cos\theta + c_4\left(\psi\right)\sin\theta + O\left(\varepsilon^2\right)\,,$$
$$\dot{\theta} = \dot{\theta}^{(0)} + c_5\left(\psi\right)\cos\theta + c_6\left(\psi\right)\sin\theta + O\left(\varepsilon^2\right)\,,$$

where

$$\dot{\xi}^{(0)} = \frac{u}{R_0},\ \dot{\theta}^{(0)} = \frac{u}{qR_0},$$

and terms proportional to $\sin\theta$ or $\cos\theta$ are of $O\left(\varepsilon\right)$. Therefore

$$\int_{t_0}^{t_1} R\dot{\xi}\,dt = \int_{t_0}^{t_1} \frac{R\dot{\xi}}{\dot{\theta}}d\theta = \int_{t_0}^{t_1} \frac{R_0\dot{\xi}^{(0)}}{\dot{\theta}^{(0)}}d\theta + O\left(\varepsilon^2\right) = u\Delta t + O\left(\varepsilon^2\right)\ , \tag{14}$$

which implies

$$m\Delta u = eE_\xi\Delta t + O\left(\varepsilon^2\right)\ . \tag{15}$$

Returning to Eq. (11) with the result of Eq. (15), we can see immediately that the curve determined by Eq. (11) can be written as

$$p_\xi = f_1\left(x, z\right) = f_0\left(x - \Delta x, z\right)\ , \tag{16}$$

$$\Delta x \equiv \frac{-x\Delta u}{\partial f_0/\partial x} = \frac{cxE_\xi\Delta t}{\partial\psi/\partial x} + O\left(\varepsilon^2\right). \tag{17}$$

Obviously, Eq. (16) represents a curve around $t_0{+}\Delta t$, shifted by Δx in the x–direction relative to the curve around t_0 represented by Eq. (10). The drift velocity is

$$v_d = \frac{\Delta x}{\Delta t} = \frac{cxE_\xi}{\partial\psi/\partial x} + O\left(\varepsilon^2\right) \sim \frac{cqE_\xi}{B}, \tag{18}$$

where q is the safety factor of the magnetic field. We emphasize the v_d in Eq. (18) is $O\left(\varepsilon^{-1}\right)$ larger then the $E{\times}B$ drift velocity. Using Eq. (17), it is easy to verify that the inequality (13) is indeed satisfied. When the flux surfaces deviate significantly from circular shapes, the toroidal electric field will also result in expansion or shrinking of the orbit on the poloidal plane.

We now use the following model tokamak equilibrium field with circular concentric flux surfaces to demonstrate the drift effect for circulating orbits,

$$\mathbf{B} = \frac{B_0 r}{qR}\mathbf{e}_\theta + \frac{B_0R_0}{R}\mathbf{e}_\xi\ , \tag{19}$$

$$\mathbf{A}{=}\frac{B_0r^2}{2Rq}\mathbf{e}_\xi - \ln\left(\frac{R}{R_0}\right)\frac{R_0B_0}{2}\mathbf{e}_z + \frac{B_0R_0Z}{2R}\mathbf{e}_R\,. \tag{20}$$

The poloidal flux function is

$$\psi = \frac{B_0r^2}{2q}\,. \tag{21}$$

According to Eq. (18), the outward drift of the circulating orbit is

$$v_d = \frac{cqE_\xi}{B_0} + O\left(\varepsilon^2\right) . \tag{22}$$

To numerically test this prediction, we now carry out numerical calculations for a typical set of tokamak parameters with $B_0 = 2\text{T}$, $R_0 = 1.7\text{m}$, $a = 0.45\text{m}$, and $v_{loop} = 2.5\text{V}$. We adopted a variational symplectic integration algorithm [14, 15], which directly discretizes the Lagrangian in Eq. (7) instead of the corresponding differential equations, to ensure the energy error of the numerical solution is globally bounded by a small number for all time-steps.

For a H^+ ion with 4KeV parallel kinetic energy and 1KeV perpendicular kinetic energy, the numerical solution plotted in Fig. 2(a) indeed shows that the circulating orbit is drifting outward toward the wall in the x-direction. In addition, the measured outward drift is plotted in Fig. 2(b) against the analytical result given by Eq. (22). It is clear that they agree very well.

Now let's go back to Eq. (11) to understand why the circulating orbit drifts outward. According to Eq. (15), the momentum increase imposed by E_ξ is balanced by the increase of Δu on average, *i.e.*, $(m\Delta u - eE_\xi \Delta t)R_0 = O\left(\varepsilon^2\right)$. However, because of the toroidicity, the kinetic angular momentum increase inside ($x < 0$) is smaller than that outside ($x > 0$), reflected by the $m\Delta ux$ term. As a consequence, the orbit shifts outward to compensate this variation, in order to conserve the total canonical momentum p_ξ at every point on the orbit. We can obtain another heuristic explanation of this effect by studying the poloidal trajectory without E_ξ. According to Eq. (8), when $E_\xi = 0$, the poloidal trajectory is given by

$$p_\xi = -\frac{e}{c}\psi + muR = const. , \tag{23}$$

which describes a curve shifted by $\Delta x = -muxc/(e\partial\psi/\partial x)$ in the x-direction relative the flux surface determined by $-e\psi/c + muR_0 = const.$. Assuming this orbit estimate is still correct when u is accelerated slowly by a small E_ξ, then the increase of u will induce an increase in Δx. If $m\Delta u = eE_\xi \Delta t$, as predicted by Eq. (15), we then obtain $\Delta x = -cxE_\xi \Delta t/(\partial\psi/\partial x)$, which is the same as Eq. (17). From the guiding center point of view, the drift of flux surfaces is caused by the variation of the curvature drift induced by the parallel acceleration under the influence of electric field. The fact that this drift is larger than the $E \times B$ drift suggests that when calculating

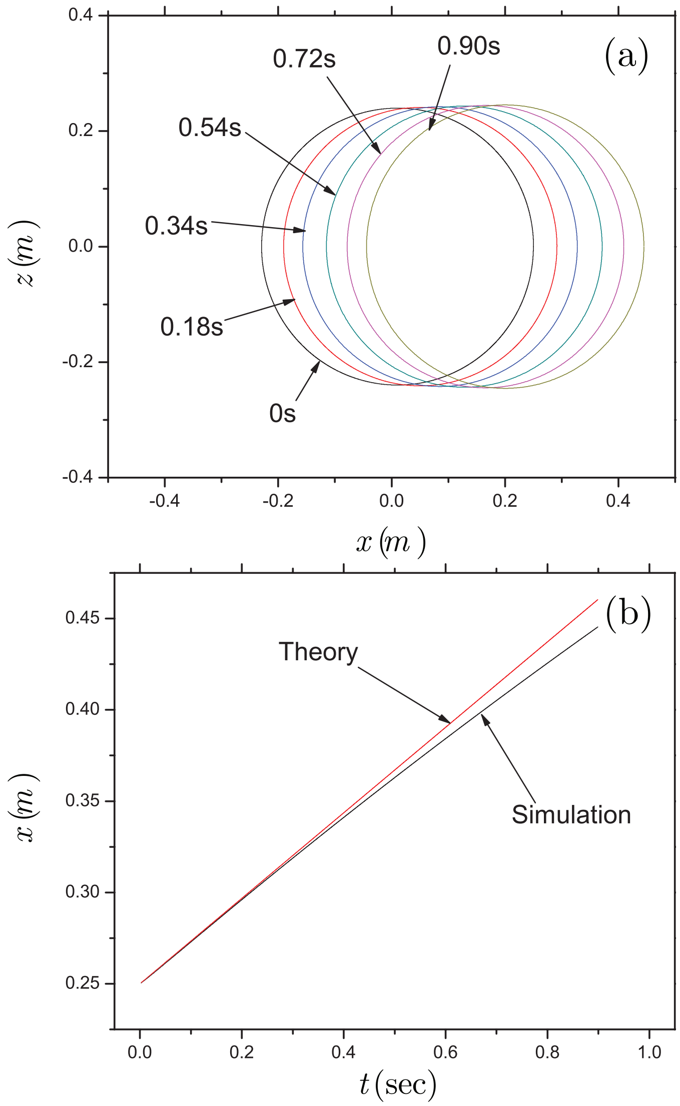

Figure 2. Numerical solution shows that the circulating orbit drifts outward toward the wall in the x-direction (a). The measured outward drift agrees with the analytical result given by Eq. (22) (b).

the turbulence induced anomalous transport, the curvature drift should be included in the calculation of the flux, *i.e.*, $\Gamma = \left\langle \tilde{n} \left(v_{\tilde{E}\times B} + \tilde{v}_c \right) \right\rangle$, where $v_{\tilde{E}\times B}$ is the $E \times B$ drift and $\tilde{v}_c$ denotes the curvature drift, which fluctuates with the fluctuating electric field $\tilde{E}$ through the parallel acceleration. This effect associated with the fluctuating curvature drift $\tilde{v}_c$ has been neglected so far [16].

Another prediction our theory is that during a RF overdrive [12, 17], the toroidal electric field is anti-parallel to the current. As a consequence, all charged particles, including backward runaway electrons, will drift inward towards the inner wall.

In summary, we have shown both analytically and numerically that there is a nontrivial neoclassical effect of the circulating orbits under the influence of a tokamak electric field. The circulating orbits drift outward toward the wall in the x-direction with a characteristic velocity cqE_ξ/B. Even though this drift is $O(\varepsilon)$ smaller than the well known Ware pinch effect for trapped particles, it is nevertheless $O(\varepsilon^{-1})$ large than the standard $E \times B$ drift. It is therefore an important neoclassical effect. For example, it provides a mechanism for runaway electrons to strike the first wall or limiter. In this paper, we have ignored collisions and only investigated the neoclassical orbit effects associated with the toroidal geometry. This is especially valid for runaway electrons. Rutherford *et al* [18] pointed out that that to study neoclassical transport effect, collisions need to be considered, and the collisional friction between particles perturbs the orbit in a similar way to the electric field. The conclusion was that for the banana regime, the inward flux is roughly the Ware pinch velocity times the density of trapped electrons. The neoclassical orbit effect associated with the circulating particles discussed in the current study was not included in the study by Rutherford *et al*. This will be a topic of our future investigations.

Acknowledgments

This research was supported by the ITER-China Program (2010GB107001) and the U.S. Department of Energy (DE-AC02-09CH11466).

References

[1] A A Ware. *Physical Review Letters*, 25:15, 1970.

[2] F. L. Hinton and R. D. Hazeltine. Theory of plasma transport in toroidal confinement systems. *Reviews of Modern Physics*, 48:239–308, 1976.

[3] R. B. White. *Theory of Toroidally Confined Plasmas*, page 295. Imperial College Press, London, 2001.

[4] P. Helander and D. J. Sigmar. *Collisional Transport in Magnetized Plasmas*, page 203. Cambridge University Press, Cambridge, 2002.

[5] J Wesson. *Tokamaks*, page 127. Clarendon Press - Oxford, 3rd edition, 2004.

[6] X. Guan, H. Qin, and N. J. Fisch. Phase-space dynamics of runaway electrons in tokamaks. *Physics of Plasmas*, 17:092502, 2010.

[7] H. Dreicer. Electron and ion runaway in a fully ionized gas. i. *Physical Review*, 115:238, 1959.

[8] R. M. Kulsrud, Y. C. Sun, N. K. Winsor, and H. A. Fallon. Runaway electrons in a plasma. *Physical Review Letters*, 31:690–693, 1973.

[9] M. N. Rosenbluth and S. V. Putvinski. Theory for avalanche of runaway electrons in tokamaks. *Nuclear Fusion*, 37:1355, 1997.

[10] C. Liu, Y. Mok, K. Papadopoulos, F. Engelmann, and M. Bornatici. Nonlinear Dynamics of Runaway Electrons and Their Interaction with Tokamak Liners. *Physical Review Letters*, 39(11):701–704, September 1977.

[11] C. Liu and Y. Mok. Nonlinear Evolution of Runaway-Electron Distribution and Time-Dependent Synchrotron Emission from Tokamaks. *Physical Review Letters*, 38(4):162–165, January 1977.

[12] N. Fisch and C. Karney. Conversion of wave energy to magnetic field energy in a plasma torus. *Physical Review Letters*, 54(9):897–900, March 1985.

[13] R. G. Littlejohn. Variational-principles of guiding center motion. *Journal of Plasma Physics*, 29(2):111–125, 1983.

[14] H. Qin and X. Guan. A variational symplectic integrator for the guiding center motion of charged particles for long time simulations in general magnetic fields. *Physical Review Letters*, 100:035006, 2008.

[15] H Qin, X. Guan, and W. M. Tang. Variational symplectic algorithm for guiding center dynamics and its application in tokamak geometry. *Physics of Plasmas*, 16:042510, 2009.

[16] W. Wang, 2010. Private communication.

[17] Charles F. F. Karney and Nathaniel J. Fisch. Current in wave-driven plasmas. *Physics of Fluids*, 29(1):180, 1986.

[18] P. Rutherford, L. Kovrizhnikh, M. Rosenbluth, and F. Hinton. Effect of Longitudinal Electric Field on Toroidal Diffusion. *Physical Review Letters*, 25(16):1090–1093, October 1970.

The Princeton Plasma Physics Laboratory is operated
by Princeton University under contract
with the U.S. Department of Energy.

Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2245
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov