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Geometric Phase of the Gyromotion for Charged Particles in a Time-dependent Magnetic Field

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Geometric phase of the gyromotion for charged particles in a time-dependent magnetic field[a)]

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We study the dynamics of the gyrophase of a charged particle in a magnetic field which is uniform in space but changes slowly with time. As the magnetic field evolves slowly with time, the changing of the gyrophase is composed of two parts. The first part is the dynamical phase, which is the time integral of the instantaneous gyrofrequency. The second part, called geometric gyrophase, is more interesting, and it is an example of the geometric phase which has found many important applications in different branches of physics. If the magnetic field returns to the initial value after a loop in the parameter space, then the geometric gyrophase equals the solid angle spanned by the loop in the parameter space. This classical geometric gyrophase is compared with the geometric phase (the Berry phase) of the spin wave function of an electron placed in the same adiabatically changing magnetic field. Even though gyromotion is not the classical counterpart of the quantum spin, the similarities between the geometric phases of the two cases nevertheless reveal the similar geometric nature of the different physics laws governing these two physics phenomena.

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I. INTRODUCTION

A moving classical charged particle is exerted the Lorentz force in a magnetic field, and it follows the so-called gyromotion with a helical-like orbit. In a strongly magnetized plasma, the fast gyromotion of particles makes the study of magnetized plasmas cumbersome because of the mixture of different temporal and spacial scales. Magnetohydrodynamis and traditional gyrokinetics theories choose to remove the fast gyromotion, by averaging out the gyrophase variable, to simplify the problem for both analytical and numerical purposes. While yielding many important results, these ingenious methods ignore some physics carried by gyrophase, which is sometimes pivotal. The gyrophase survives in modern gyrokinetics, which rigorously decouples, instead of eliminates, the gyrophase from other slow components of the particle dynamics[1–3]. The gyrophase contributes its due part in many interesting phenomena such as the polarization density, the shear Alfvén waves, and the radio-frequency wave heating. In the gyrokinetic theory, we begin to pay attentions to the physics of the gyrophase, especially its responses on high frequency electro-magnetic field[4–6]. On the other hand, the gyrophase in slowly changing magnetic fields contains no little physical meanings. We will reveal an interesting physics of the gyrophase in an adiabatically changing magnetic field in this paper.

We study the dynamics of the gyrophase of a charged particle in a magnetic field which is uniform in space but changes slowly with time. If the magnetic field doesn't change with time, the particle will rotate with a constant angular velocity $\Omega = qB/m$ in the plane perpendicular to the magnetic field, where q is the electric charge carried by the particle, m is the mass of the particle, and B is the magnitude of the magnetic field. The gyrophase of the particle at time t can be easily written as $\theta(t) = qBt/m$, given $\theta = 0$ when $t = 0$. As the magnetic field evolves slowly with time, the changing of the gyrophase turns out to be composed of two parts. The first part is the dynamical phase $\theta_d(t) = \int_0^t \Omega_d(t')\,dt'$, which is simply the time integral of the instantaneous angular velocity $\Omega_d(t) = qB(t)/m$ resulting from the Lorentz force. Of course, this dynamical phase is of no surprise. What is interesting is that in addition to the dynamics phase, the gyrophase contains another part θ_g which is called the geometric phase. The name comes from its elegant geometric meaning and its geometric origin, the noncommutativity of the rotation operations[7]. If the evolution of the magnetic field $\mathbf{B}(t)$ forms a closed loop C in the parameter space composed of (B_x, B_y, B_z), the magnitude of the geometric phase equals to the solid angle α spanned by the loop C (see Fig. 1). Its value depends only on the closed path C instead of other physical ingredients such as the particle's mass, velocity, and the changing rate of the field, *etc.* This geometric phase associated with the gyrophase is an example of the general geometric phase in physics.

As early as in 1958, Lyman Spitzer anticipated the existence of geometric phase through studying rotation transforms and invented the Figure-8 stellarator[8]. In 1984, M. V. Berry studied the quantum adiabatic system[9]. According to the adiabatic theorem first introduced by M. Born and V. A. Fock[10], if the initial state of a system is an eigenstate of its initial Hamiltonian, it should stay at its corresponding eigenstate of the

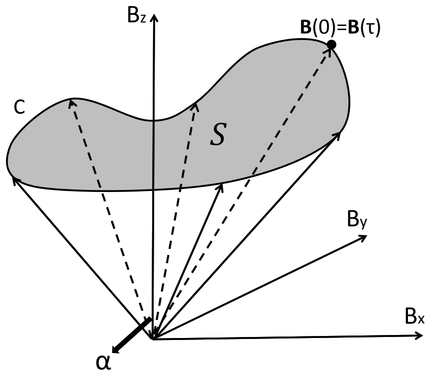

FIG. 1. The time-dependent adiabatic magnetic field vector $\mathbf{B}(t)$ goes back to its initial position after a period τ in the parameter space composed of (B_x, B_y, B_z). Its path creates a closed loop C, and S is a surface enclosed by C. The magnitude of the geometric phase associated with the gyrophase equals the solid angle α spanned by the loop C. The Berry phase of the electron spin wave function in the same magnetic field is half of the solid angle α.

instantaneous Hamiltonian during the adiabatic process. Berry found that an additional phase factor was required, apart from the dynamical phase, to make the instantaneous eigenfunction to satisfy the Schödinger equation. This additional phase factor is called the Berry phase, which is a quantum version of geometric phase[9,11,12]. Soon Hannay published his classical version of the Berry phase, known as the Hannay's angle[13]. Simon studied the geometric phase from the perspective of more abstract mathematics[14]. In 1988, Littlejohn studied the geometric phase associated with the gyromotion of a charged particle in a spatially inhomogeneous magnetic field[15]. In 1992, A. Bhattacharjee, *et al.* pointed out the deep connection between geometric phase and the variation of the longitudinal invariant among various types of guiding-center orbits[16]. All these geometric phases have clear geometric meanings in the space consisting of adiabatic environmental parameters. From then on the geometric phase has been intensively studied and began to play important roles in almost all branches of physics[17–19]. Hence it is not surprising for the geometric phase to find its way in plasma physics, or more specifically in the gyrophase.

To compare with the geometric phase associated with the gyromotion, we also study the geometric phase (the Berry phase) of the spin wave function of an electron placed in the same adiabatically changing magnetic field. We use the Schödinger equation and Pauli representation to derive the spin wave function of an electron. The phase factor of the wave function turns out to have a geometric part, the Berry phase, as expected besides its dynamical part. It is interesting to find out that the Berry phase in electron spin wave function shares many common properties with the geometric phase associated with the gyrophase. Its magnitude equals to half of the solid angle α spanned by the closed loop C in the parameter space. This Berry phase depends only on the evolution path of the magnetic field $\mathbf{B}(t)$ in the parameter space, as in the case of the gyromotion. These similarities as well as some distinctions reflect the links between the classical theory and the quantum theory and the profundity of the geometric phase.

This paper is organized as follows. In section II, we study the geometric phase associated with the classical gyromotion. Starting from the Newtonian equation, the gyrophase is rigorously defined using the gyrocenter transformation. After obtaining the expression of the geometric phase associated with the gyrophase, its interesting properties are studied. Its geometric properties in the parameter space, the physical observability, and the gauge choices are discussed. In section III, we derive the Berry phase of an electron with 1/2 spin in the same time-dependent adiabatic magnetic field. Though obeying totally different dynamical rules, the geometric phases associated with the gyromotion and the electron spin have many similarities, which are discussed in section IV.

II. GEOMETRIC PHASE IN GYROMOTION

We first consider a classical charged particle's motion in a time-dependent adiabatic magnetic field. The field is assumed to be uniform in space but change slowly with time. This simple model keeps all the crucial physics of interests here, while making the problem tractable. We start from the governing equation of the particle's motion

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v}\times\mathbf{B} \ , \tag{1}$$

where m and $\mathbf{v}$ are the particle's mass and velocity respectively, and $\mathbf{B}(t)$ is the time-dependent magnetic field.

In order to study the geometric phase associated with the gyrophase, we need to give a rigorous definition of the gyrophase. This can be achieved through the gyrocenter transformation, which transforms the particle coordinate $(\mathbf{x}, \mathbf{v})$ to the gyrocenter coordinate $(\mathbf{X}, v_\parallel, v_\perp, \theta)$. The position of the gyrocenter $\mathbf{X}$, the perpendicular velocity $v_\perp$, the parallel velocity $v_\parallel$, and the gyrophase θ are used to describe the particle's motion, replacing the position $\mathbf{x}$ and the velocity $\mathbf{v}$ of the particle. The coordinate transformation is given by

$$\mathbf{v} = v_\parallel\mathbf{b} + v_\perp\cos\theta\mathbf{e}_1 + v_\perp\sin\theta\mathbf{e}_2 \ , \tag{2}$$

$$\mathbf{x} = \mathbf{X} - \frac{m\mathbf{v}\times\mathbf{b}}{qB} \ . \tag{3}$$

Here, $\mathbf{b} = \mathbf{B}/B$ is a unit vector parallel to the magnetic field, $\mathbf{e}_1$ and $\mathbf{e}_2$ are two unit vectors satisfying $\mathbf{e}_1\times\mathbf{e}_2 = \mathbf{b}$. It is easy to see these three unit vectors are perpendicular to each other and hence form a set of local orthogonal frames. With Eq. (2) we can rewrite Eq. (1) in the gyrocenter coordinate as

$$(\dot{v}_\parallel \mathbf{b} + v_\parallel \dot{\mathbf{b}}) + \left(\dot{v}_\perp \cos\theta - v_\perp \sin\theta\ \dot{\theta} - \frac{qB}{m} v_\perp \sin\theta\right) \mathbf{e}_1 + \left(\dot{v}_\perp \sin\theta + v_\perp \cos\theta\ \dot{\theta} + \frac{qB}{m} v_\perp \cos\theta\right) \mathbf{e}_2$$
$$+ v_\perp \cos\theta\ \dot{\mathbf{e}}_1 + v_\perp \sin\theta\ \dot{\mathbf{e}}_2 = 0\ , \tag{4}$$

where the dots above the variables denote the time derivatives.

We note that in Eq. (2), there is still a freedom left to determine $\mathbf{e_1}$ and $\mathbf{e_2}$. They can rotate within the plane perpendicular to $\mathbf{b}$ freely. Once a direction of $\mathbf{e_1}$ in this plane is chosen, $\mathbf{e}_2$ is fixed correspondingly. This is actually a choice of gauge which gives different definitions of the gyrophase, while all these gauges are equivalent to each other for describing real physical processes.

We now choose a gauge as follows

$$\mathbf{e_1} = \frac{B_z}{B\sqrt{B_x^2 + B_y^2}} \left(B_x \mathbf{e}_x + B_y \mathbf{e}_y - \frac{B_x^2 + B_y^2}{B_z} \mathbf{e}_z\right), \tag{5}$$

$$\mathbf{e_2} = \frac{1}{\sqrt{B_x^2 + B_y^2}} \left(-B_y \mathbf{e}_x + B_x \mathbf{e}_y\right)\ , \tag{6}$$

where (B_x, B_y, B_z) are three components of the magnetic field, $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are unit vectors in $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ directions respectively, and $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$ is the magnitude of the magnetic field. This gauge keeps $\mathbf{e}_2$ within **x-y** plane and leads to the the following dynamic equations for the gyrophase,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \Omega_d + \omega_g + \omega_a\ , \tag{7}$$

$$\Omega_d = -\frac{qB}{m}\ , \tag{8}$$

$$\omega_g = -\frac{B_z}{B(B_x^2 + B_y^2)} (B_x \dot{B}_y - B_y \dot{B}_x)\ , \tag{9}$$

$$\omega_a = -\frac{v_\parallel}{v_\perp B^2 \sqrt{B_x^2 + B_y^2}} [-B_z(B_x \dot{B}_x + B_y \dot{B}_y + B_z \dot{B}_z) \sin\theta + \dot{B}_z B^2 \sin\theta + B(B_x \dot{B}_y - B_y \dot{B}_x) \cos\theta]\ . \tag{10}$$

Integrating Eq. (7) over time, we obtain the expression of the gyrophase

$$\Delta\theta = \theta(t) - \theta(t=0) = \theta_d(t) + \theta_g(t) + \theta_a(t)\ , \tag{11}$$

where

$$\theta_d(t) = \int_0^t \Omega_d\, \mathrm{d}t'\ , \tag{12}$$

$$\theta_g(t) = \int_0^t \omega_g\, \mathrm{d}t'\ , \tag{13}$$

$$\theta_a(t) = \int_0^t \omega_a\, \mathrm{d}t'\ . \tag{14}$$

The gyrophase consists of three terms. The first term $\theta_d(t) = -\int_0^t [qB(t)/m]\, \mathrm{d}t'$ is the so-called dynamical phase, which is simply the time integral of the instantaneous gyro-frequency. And the negative sign comes from the definition that the counterclockwise direction is the positive direction of rotation. The dynamical phase is determined directly by the Lorentz force exerted on the particle and has a clear physical meaning. The assumption of adiabatic time-dependence implies that the time scale of the system evolution is much larger than the gyro-period,

$$\frac{1}{\Omega_d} \frac{\dot{B}_i}{B} \sim \epsilon \ll 1 \quad (i = x, y, z)\ . \tag{15}$$

To avoid small but rapid magnetic field changes, we also assume

$$\frac{1}{\Omega_d^2} \frac{\ddot{B}_i}{B} \sim \epsilon^2 \quad (i = x, y, z)\ . \tag{16}$$

From Eqs. (15) and (16), we can prove that θ_a is much smaller than θ_d and θ_g for $t \gg 1/\Omega_d$. The biggest change of θ_a within one gyro-period may be the same order as θ_g, which means that $\Delta\theta_a/\Delta\theta_g \sim 1$ may hold within one gyro-period. But their averaged changes over one gyro-period differ greatly, *i.e.*, $\langle\Delta\theta_a\rangle/\langle\Delta\theta_g\rangle \sim \epsilon$. This fact is proved in Appendix A. That means after one complete gyro-period, the order of the three parts are $\Delta\theta_d \sim O(1)$, $\Delta\theta_g \sim O(\epsilon)$, and $\Delta\theta_a \sim O(\epsilon^2)$ respectively. So θ_a can be neglected in Eq. (11).

The term θ_g in Eq. (11) is the geometric phase, which is consistent with the results from modern gyrokinetics. Equation (9) is equivalent to the term $-R_0$ in Eq. (4.43) of Reference[1], where $R_0 = \partial \mathbf{e}_1/\partial t \cdot \mathbf{e}_2$. According to Eq. (9), particle's physical characters, such as the charge and the mass, have no influences on the geometric phase. Changing the sign of the electric charge will make the particle rotate in different directions, but the particle still have the same geometric phase. To show the properties of the geometric phase more clearly, we rewrite θ_g as

$$\theta_g(t) = -\int_0^t \frac{B_z}{B(B_x^2+B_y^2)}(B_x\dot{B}_y - B_y\dot{B}_x)\,\mathrm{d}t' = -\int_{\mathbf{B}(0)}^{\mathbf{B}(t)} \frac{B_z(B_x\mathrm{d}B_y - B_y\mathrm{d}B_x)}{B(B_x^2+B_y^2)} \ . \quad (17)$$

This is an integral in the parameter space consisting of the three components of the magnetic field (B_x, B_y, B_z). Its value depends only on the integral path in the parameter space. On the other hand, the changing rate of the field does not enter the geometric phase as long as the field changes adiabatically. If the path in the parameter space is closed, that is $\mathbf{B}(\tau) = \mathbf{B}(0)$, we can apply Stokes theorem to obtain

$$\begin{aligned}\theta_g(\tau) &= -\oint_C \frac{B_z(B_x\mathrm{d}B_y - B_y\mathrm{d}B_x)}{B(B_x^2+B_y^2)} \\ &= -\iint_S \frac{B_x\mathrm{d}B_y\mathrm{d}B_z + B_y\mathrm{d}B_z\mathrm{d}B_x + B_z\mathrm{d}B_x\mathrm{d}B_y}{B^3} \\ &= -\iint_S \frac{\mathbf{B}\cdot\mathrm{d}\mathbf{S}_B}{B^3} \ , \end{aligned} \quad (18)$$

where C is the closed curve and S is a surface enclosed by C in (B_x, B_y, B_z) space (see Fig. 1). In the configuration space, a solid angle is defined as

$$\alpha = \iint \frac{\mathbf{r}\cdot\mathrm{d}\mathbf{S}}{r^3} \ .$$

Therefore the geometric phase in Eq. (18) is exactly the negative of the solid angle spanned by the field's changing path in the parameter space. As a real physical quantity, the geometric phase should provide an observable physical quantity and does not rely on any gauge choice. In gyromotion, we can calculate the difference between two gyrophases in two different perpendicular planes. But the value is not absolute, and it depends on the choice of gauge. To examine the changes of gyrophases, we can restrict them to the same plane. This can be achieve by letting the magnetic field go back to its initial value $\mathbf{B}(\tau) = \mathbf{B}(0)$. Alternatively, we can also compare the gyrophases carried by two particles with two different paths in the parameter space, but having the same starting and ending points. The difference between their geometric phases can be calculated as follows

$$\begin{aligned}\theta_{g1} - \theta_{g2} &= \int_{C_1} - \int_{C_2} \frac{B_z(B_x\mathrm{d}B_y - B_y\mathrm{d}B_x)}{B(B_x^2+B_y^2)} \\ &= \int_{C_1} + \int_{-C_2} \frac{B_z(B_x\mathrm{d}B_y - B_y\mathrm{d}B_x)}{B(B_x^2+B_y^2)} \\ &= \oint_C \frac{B_z(B_x\mathrm{d}B_y - B_y\mathrm{d}B_x)}{B(B_x^2+B_y^2)} \ , \end{aligned} \quad (19)$$

where C_1 and C_2 are two curves in the parameter space with the same starting and ending point. The curves C_1 and $-C_2$ form a closed curve C (see Fig. 2).

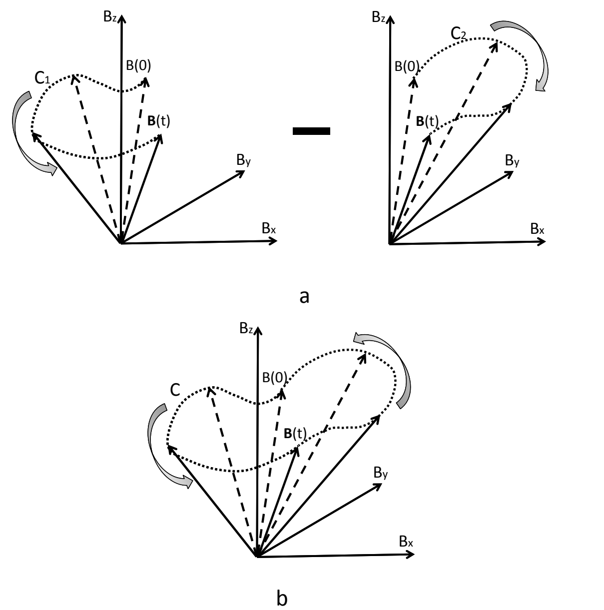

FIG. 2. Two charged particles undergo different external magnetic fields, both of which have the same initial and final values. The two fields' changes follow the paths C_1 and C_2 respectively. The solid angle spanned by the closed curve $C = C_1 - C_2$ gives the difference between the two geometric gyrophases which is measurable. The arrows indicate the directions of curves.

We note that the geometric phase is not a gauge invariant. However, we can assume that the frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$ is smoothly changing with $\mathbf{B}(t)$, *i.e.*, the three unit vectors are all single-valued smooth functions of $\mathbf{B}$. This restriction can help to resolve the ambiguity when splitting the phase between the dynamical and geometric parts, which was discussed by Anandan[20], Berry and Hannay[21], and Bhattacharjee[22]. It puts the effects of rotation transformation in the gyrophase instead of in the choice of the local frames. With this restriction, the frames $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$ returns to itself as $\mathbf{B}(t)$ returns to itself after a loop, and then we can show that the differences in the geometric phases for different gauges can only be integer multiples of 2π. To prove this fact, let's assume a gauge $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{b})$ different from Eqs. (5) and (6) is chosen. The new frame can be specified by a function $\theta_f(t)$, which measures the rotation of $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{b})$ relative to $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$ at time t. Then the geometric phase changes to

$$\theta_g'(t) = \theta_g(t) + \theta_f(t) \ , \quad (20)$$

where θ_g' is the geometric phase in the new gauge $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{b})$, and θ_g' is the geometric phase in the old gauge $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$ determined by Eq. (17) (see Fig. 3). For a closed integral path C in the parameter space, *i.e.*, $\mathbf{B}(\tau) = \mathbf{B}(0)$, all frames return to their initial positions at time $t = \tau$. This implies $\theta_f(\tau) = 2n\pi$, where n is an integer, and hence we have the general expression of the

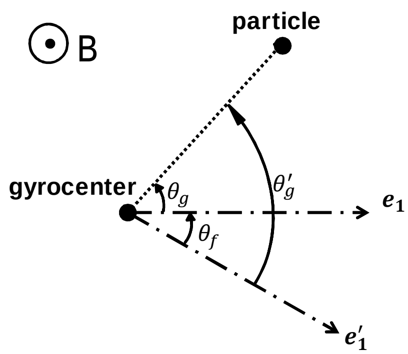

FIG. 3. This figure demonstrates the relationship between θ'_g, θ_g, and θ_f in the plane perpendicular to $\mathbf{B}$. The direction of $\mathbf{B}$ points out off the paper. The two black dots denote the position of gyrocenter and the position of the charged particle respectively, and $\mathbf{e}_1$ and $\mathbf{e}'_1$ are two different gauges.

geometric phase in any gauge

$$\begin{aligned} \theta'_g(\tau) =& \theta_g(\tau) + 2n\pi = -\iint_S \frac{\mathbf{B}\cdot \mathrm{d}\mathbf{S}_B}{B^3} + 2n\pi \ , \\ & (n = 0, \pm 1, \pm 2, \cdots) \ . \end{aligned} \tag{21}$$

This $2n\pi$ difference of the geometric phases carries no physical importance.

III. GEOMETRIC PHASE IN ELECTRON SPIN

For comparison, we now calculate the Berry phase (the geometric phase of the wave function) of an electron with 1/2 spin in the same time-dependent magnetic field. The two-state spin Hamiltonian problem and related Berry phase problem have been studied by many researchers, including Berry himself[9], and discussed in many textbooks[18,23,24]. Here, we will follow their methods to give a brief derivation as a reminder, and then compare the result with the geometric phase associated with the classical gyromotion. The wave function of the electron in a magnetic field satisfies the time-dependent Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial}{\partial t} |\phi\rangle = \hat{H}(\mathbf{B}(t)) |\phi\rangle \ , \tag{22}$$

where the Hamiltonian is a function of $\mathbf{B}(t)$ which changes with time adiabatically. Following Berry's method and using the adiabatic theorem, the Berry phase can be expressed as

$$\theta_g = \beta_n = \oint \mathbf{A}_n(\mathbf{B}) \cdot d\mathbf{B} \ , \tag{23}$$

where

$$\mathbf{A}_n(\mathbf{B}) = \mathrm{i} \langle \phi_n(\mathbf{B})| \frac{\partial}{\partial \mathbf{B}} |\phi_n(\mathbf{B})\rangle \ , \tag{24}$$

and $|\phi_n(\mathbf{B})\rangle$ is the nth eigenstate of the system with Hamiltonian $\hat{H}(\mathbf{B})$. It can be proved that $\mathbf{A}_n(\mathbf{B})$ is a real number. The electron spin operators can be expressed as Pauli Matrices,

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \ , \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{25}$$

The corresponding Hamiltonian is

$$\begin{aligned} \hat{H} &= -\mu \hat{\sigma} \cdot \mathbf{B} = -\mu(\hat{\sigma}_x B_x + \hat{\sigma}_y B_y + \hat{\sigma}_z B_z) \\ &= \mu \begin{pmatrix} -B_z & -B_x + \mathrm{i}B_y \\ -B_x - \mathrm{i}B_y & B_z \end{pmatrix} \ . \end{aligned} \tag{26}$$

Substituting it into the eigen-equation,

$$\hat{H} |\phi\rangle = E |\phi\rangle \ , \tag{27}$$

we obtain the eigenvalue of the Hamiltonian

$$E_\pm = \mp \mu B \ , \tag{28}$$

and corresponding eigen-states

$$|\phi_+\rangle = \frac{e^{i\theta}}{\sqrt{2B(B+B_z)}} \begin{pmatrix} B + B_z \\ B_x + \mathrm{i}B_y \end{pmatrix} \ , \tag{29}$$

$$|\phi_-\rangle = \frac{e^{i\theta}}{\sqrt{2B(B+B_z)}} \begin{pmatrix} -B_x + \mathrm{i}B_y \\ B + B_z \end{pmatrix} \ . \tag{30}$$

The two states demonstrate two different spin directions along or opposite to the direction of the magnetic field. Applying them to Eq. (23), the Berry phase can be calculated as

$$\beta_\pm = \pm \oint_C \frac{B_y \mathrm{d}B_x - B_x \mathrm{d}B_y}{2B(B+B_z)} = \mp \frac{1}{2} \iint_S \frac{\mathbf{B}\cdot \mathrm{d}\mathbf{S}_B}{B^3} \ . \tag{31}$$

We find that the magnitude of the geometric phase of the electron spin wave function here is half of the solid angle α spanned by the closed path C in the parameter space. The Berry phases have different signs for the two different eigenstates.

There is a gauge freedom for θ in Eq. (29) and Eq. (30) as well. The normalization condition of the wave function only limits the its amplitude while leaving this phase factor undetermined. We can choose a gauge for it. Similar to the gyrophase studied in section II, if θ changes smoothly with $\mathbf{B}$, then the gauge will have no influences on the final results.

IV. DISCUSSIONS

It is interesting to compare the geometric phases from the classical electron gyromotion and the electron spin wave function under the same adiabatic variation of the magnetic field, but different physical rules. The gyromotion is a pure classical motion ruled by classical mechanics, and the electron spin is a thoroughly quantum property without any classical counterpart. But the geometric phases in these two cases show many common characters. The magnitude of the geometric phase associated with the gyrophase equals to the solid angle spanned by the closed path in the parameter space, while the Berry phase in the spin wave function is half of that value. In the gyromotion the perpendicular plane are obliged to change with the time-dependent magnetic field, while for the electron spin the eigenstates have to adjust correspondingly. On the other hand, there are also some interesting distinctions. The magnitude of the Berry phase for the electron spin is half the solid angle, and there are 1/2 factors in the commutation relations of the electron spin operators too. There are two electron spin eigenstates. The Berry phases of them have opposite signs. However, the geometric phases associated with the gyrophase have the same value for both positive and negative charged particles, and their magnetic moments are in the same direction though carrying opposite electric charges.

Even though the gyromotion is not the classical counterpart of the quantum spin, the similarities between the geometric phases for the gyromotion and the spin wave function nevertheless show the common geometric nature of the laws governing these two different physics phenomena. Actually, the geometric phases in the both cases come from the same origin, the noncommutativity of the rotation operations.While the quantum geometric phases have found many important applications in optics, atomic and molecular physics, chemistry, laser physics, material science, quantum information, cosmology, *etc.*[25–34], the importance and implications of the geometric phase in plasma physics are still unexplored. But we do expect interesting physics associated with the geometric phase to exist. For example, as a L-wave and a R-wave travel along the magnetic field line, because the difference in the phase velocity, the combined linearly polarized wave will rotate as the wave propagates. This is the Faraday rotation. However, if the magnetic field is not homogeneous, as the wave propagating, we will observe a geometric component in the Faraday rotation, which could dominate the non-geometric Faraday rotation in a typical laboratory plasma. This will be one of the topics of our future investigation.

Appendix A

In this appendix we will prove that the change of θ_a is much smaller than the change of θ_g after one complete gyro-period, *i.e.*, $\langle\Delta\theta_a\rangle/\langle\Delta\theta_g\rangle \sim \epsilon$, and so θ_a is much smaller than θ_g for $t \gg 1/|\Omega_d|$.

For a general strong magnetic field, we can choose an appropriate Cartesian coordinate to keep $B \sim B_i$, where $i = x, y, z$. According to Eqs. (9) and (10), if $v_\parallel \ll v_\perp$, it is obvious that $\omega_a \ll \omega_g$ always holds, and hence we have $\theta_a \ll \theta_g$. If $v_\parallel \sim v_\perp$, $\omega_a \sim \omega_g$ may hold, and we may have $\Delta\theta_a \sim \Delta\theta_g$ within a short time interval. To examine their longtime behavior, we need to study the phase changes after a complete gyro-period.

The change of θ_g after one complete gyro-period is

$$\langle\Delta\theta_g\rangle = \int_t^{t+T} \omega_g(t')\,\mathrm{d}t' \ , \tag{A1}$$

where T is the gyro-period satisfying $\theta(T+t)-\theta(t) = 2\pi$. After expanding $\omega_g(t')$ we have

$$\begin{aligned}\langle\Delta\theta_g\rangle &= \int_t^{t+T} [\omega_g(t) + \dot{\omega}_g(t)(t'-t) \\ &\qquad + \frac{1}{2}\ddot{\omega}_g(t)(t'-t)^2 + \cdots]\,\mathrm{d}t' \\ &= T\omega_g(t) + \int_t^{t+T} [\dot{\omega}_g(t)(t'-t) \\ &\qquad + \frac{1}{2}\ddot{\omega}_g(t)(t'-t)^2 + \cdots]\,\mathrm{d}t' \ .\end{aligned} \tag{A2}$$

According to Eq. (15) we have

$$T\omega_g(t) \approx 2\pi\frac{\omega_g(t)}{\Omega_d(t)} \sim \epsilon \ . \tag{A3}$$

While $\dot{\omega}_g$ contains terms such as

$$\frac{B_z\dot{B}_x\dot{B}_y}{B(B_x^2+B_y^2)} \quad \text{and} \quad \frac{B_zB_x\ddot{B}_y}{B(B_x^2+B_y^2)} \ ,$$

according to Eqs. (15) and (16) we have

$$\frac{1}{\Omega_d^2}\frac{B_z\dot{B}_x\dot{B}_y}{B(B_x^2+B_y^2)} \sim \frac{1}{\Omega_d^2}\frac{B_zB_x\ddot{B}_y}{B(B_x^2+B_y^2)} \sim \epsilon^2 \ ,$$

and

$$\int_t^{t+T} [\dot{\omega}_g(t)(t'-t)]\,\mathrm{d}t' = \frac{1}{2}\dot{\omega}_g(t)T^2 \approx \frac{1}{2\Omega_d^2}\dot{\omega}_g(t) \sim \epsilon^2 \ .$$

We can examine the higher-order terms similarly and obtain

$$\int_t^{t+T} [\dot{\omega}_g(t)(t'-t) + \frac{1}{2}\ddot{\omega}_g(t)(t'-t)^2 + \cdots]\,\mathrm{d}t' \sim O(\epsilon^2) \ , \tag{A4}$$

and then

$$\langle\Delta\theta_g\rangle \sim \epsilon \ . \tag{A5}$$

Next we apply the same technique to θ_a. Observing from Eq. (10) that ω_a contains terms with $\sin\theta$ or $\cos\theta$, we rewrite it as

$$\omega_a = \omega_{a1}\sin\theta + \omega_{a2}\cos\theta \ , \tag{A6}$$

where ω_{a1} and ω_{a2} have similar forms to terms in ω_g, and then we have

$$\langle\Delta\theta_g\rangle = \int_t^{t+T} \omega_{a1}(t')\sin\theta\,\mathrm{d}t' + \int_t^{t+T} \omega_{a2}(t')\cos\theta\,\mathrm{d}t' \ , \tag{A7}$$

where

$$\begin{aligned}
&\int_t^{t+T} \omega_{a1}(t') \sin\theta \, \mathrm{d}t' \\
&= \int_t^{t+T} [\omega_{a1}(t) + \dot{\omega}_{a1}(t)(t'-t) + \cdots] \sin\theta \, \mathrm{d}t' \\
&= \omega_{a1}(t) \int_t^{t+T} \sin\theta \, \mathrm{d}t' \\
&\quad + \int_t^{t+T} [\dot{\omega}_{a1}(t)(t'-t) + \cdots] \sin\theta \, \mathrm{d}t' \ , \qquad \text{(A8)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^{t+T} \omega_{a2}(t') \cos\theta \, \mathrm{d}t' \\
&= \int_t^{t+T} [\omega_{a2}(t) + \dot{\omega}_{a2}(t)(t'-t) + \cdots] \cos\theta \, \mathrm{d}t' \\
&= \omega_{a2}(t) \int_t^{t+T} \cos\theta \, \mathrm{d}t' \\
&\quad + \int_t^{t+T} [\dot{\omega}_{a2}(t)(t'-t) + \cdots] \cos\theta \, \mathrm{d}t' \ . \qquad \text{(A9)}
\end{aligned}$$

The first term in Eq. (A8)

$$\begin{aligned}
&\omega_{a1}(t) \int_t^{t+T} \sin\theta \, \mathrm{d}t' \\
&= \omega_{a1}(t) \int_\theta^{\theta+2\pi} \frac{\sin\theta}{\Omega_d(\theta') + \omega_g(\theta') + \omega_a(\theta')} \, \mathrm{d}\theta' \\
&= \frac{\omega_{a1}(t)}{\Omega_d(t)} \left[\int_\theta^{\theta+2\pi} \sin\theta \, d\theta' + O(\epsilon) \right] \\
&= \frac{\omega_{a1}(t)}{\Omega_d(t)} O(\epsilon) = O(\epsilon^2) \ , \qquad \text{(A10)}
\end{aligned}$$

together with the higher-order terms

$$\int_t^{t+T} [\dot{\omega}_{a1}(t)(t'-t) + \cdots] \sin\theta \, \mathrm{d}t' = O(\epsilon^2) \ , \qquad \text{(A11)}$$

give the relation

$$\int_t^{t+T} \omega_{a1}(t') \sin\theta \, \mathrm{d}t' = O(\epsilon^2) \ . \qquad \text{(A12)}$$

Similarly we also have

$$\int_t^{t+T} \omega_{a2}(t') \cos\theta \, \mathrm{d}t' = O(\epsilon^2) \ , \qquad \text{(A13)}$$

and hence

$$\langle \Delta\theta_a \rangle \sim \epsilon^2 \ . \qquad \text{(A14)}$$

Equation (A14) shows that after a long time $t \sim T/\epsilon$, the change of θ_a is the same order as ϵ, which is still much smaller than 2π. So θ_a is an adiabatical invariant. According to Eqs. (A5) and (A14) we finally prove the relation $\langle \Delta\theta_a \rangle / \langle \Delta\theta_g \rangle \sim \epsilon \ll 1$, which implies $|\theta_a| \ll |\theta_g|$ holds after a long time $t \gg 1/|\Omega_d|$.

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