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Abstract Nonlinear gyrokinetics is the major formalism used for both the analytical and numerical description of low-frequency microturbulence in magnetized plasmas. Its derivation from noncanonical Lagrangian methods and field-theoretic variational principles is summarized. Basic properties of gyrokinetic physics are discussed, including polarization and the concept of the gyrokinetic vacuum, equilibrium statistical mechanics, and the two fundamental constituents of gyrokinetic turbulence, namely drift waves and zonal flows. Numerical techniques are described briefly, and illustrative simulation results are presented. Advanced topics include the transition to turbulence, nonlinear saturation of turbulence by coupling to damped gyrokinetic eigenmodes, phase-space cascades, subcritical turbulence, and momentum conservation.

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1 INTRODUCTION

It has been more than two decades since an article focused on plasma physics appeared in the Annual Review of Fluid Mechanics; see the review on "Plasma Turbulence" by Similon & Sudan (1990). Since then, enormous progress has been made in all facets of the field, particularly in the theoretical, numerical, and experimental exploration of the consequences of microturbulence for the magnetic confinement of fusion plasmas, but also more recently in astrophysical contexts. These accomplishments speak to a sea change in the method of attack. Whereas early theoretical work on plasma turbulence (beginning in the 1960's) employed analytical methods of statistical closure theory (Krommes 2002, see the Sidebar) applied to simple models, beginning in the mid-1980's the principle tool used in the theoretical and numerical studies became the nonlinear gyrokinetic formalism. Gyrokinetics, appropriate for the description of low-frequency fluctuations in magnetized plasmas, was mentioned only briefly by Similon & Sudan, whose article appeared eight years after the seminal derivation by Frieman & Chen (1982) of a nonlinear gyrokinetic equation (GKE). The present review attempts to provide a modern perspective. It is specifically intended for an audience of non-plasma physicists, so it does not delve deeply into the morass of fusion-related phenomenology. However, it does emphasize that gyrokinetics is evolving into a quantitatively predictive tool that has already enjoyed significant successful comparisons with experimental data. After decades of development, the field has become one of the major success stories in plasma-physics research.

The need for gyrokinetics arises from the enormous range of time and space scales present in many plasma configurations (in both the laboratory and space). Consider, for example, the ITER research device (ITER 2009), presently under construction by an international consortium. ITER is a large tokamak, intended for the study of burning plasmas, with (Green 2003, Table 1) a toroidal magnetic field B of 5.3 Tesla. The gyrofrequency of deuterium ions in that field is $\omega_{ci} \doteq q_i B/m_i c \approx 2.5\times10^8\,\mathrm{s}^{-1}$ ($\doteq$ is used for definitions, q_i and m_i are the ion charge and mass, and c is the speed of light), about 750 times larger than the characteristic turbulence frequency (see Sec. 3.3) $\omega_* \approx 3.3 \times 10^5\,\mathrm{s}^{-1}$. An even more dramatic comparison is with the discharge pulse length $\tau_{\mathrm{pulse}} \approx 400\,\mathrm{s}$: $\tau_{\mathrm{pulse}}/(2\pi/\omega_{ci}) \approx 10^{10}$. Gyrokinetics copes with such dramatic scale separations by analytically removing the details of the gyromotion and other high-frequency dynamics from consideration. That eliminates many physical processes that are not believed to be important for the problem of turbulent transport, and it leads to enormous savings in computational resources. However, although the underlying idea is very simple, both the modern theoretical formalism and its numerical implementations are sophisticated. I will touch on both of those facets, but because of the article's stringent length constraint I can only hint at the enormous volume of excellent work that has been done. More details can be found in some longer review articles. Brizard & Hahm (2007) focus on the analytical underpinnings, while Garbet et al. (2010) are mostly concerned with numerical simulations and their comparison with fusion experiments. A forthcoming review by G.L. Hammett (in preparation) also addresses those latter topics. Cary & Brizard (2009) review the modern theory of the closely related problem of guiding-center (zero-gyroradius) motion. Pedagogical background material on modern issues in plasma turbulence can be found in some articles by Krommes (2006a,b, 2009b,c).

2 GYROKINETIC FORMALISM

2.1 Fundamental particle kinetic equation

The generalization of Boltzmann's equation to include long-ranged electromagnetic interactions is the plasma kinetic equation [for the probability density function (PDF) f for particles of species s at position $\boldsymbol{x}$ and velocity $\boldsymbol{v}$]

$$\frac{\partial f_s(\boldsymbol{x},\boldsymbol{v},t)}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_s + \left(\frac{q}{m}\right)_s (\boldsymbol{E} + c^{-1}\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = -C_s[f], \tag{1}$$

where $C[f]$ denotes a positive-semidefinite collision operator functionally dependent on f (usually the Landau form is used). Maxwell's equations must be adjoined to calculate the self-consistent electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$.

It is assumed that $\boldsymbol{B}$ comprises a large part $\boldsymbol{B}_0$, arising from external coils and (for tokamaks) externally driven plasma currents, and an electromagnetic correction $\delta\boldsymbol{B}$. Here I shall mostly consider the electrostatic approximation, in which $\delta\boldsymbol{B}$ is neglected; thus $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$, where φ is the electrostatic potential obtained from Poisson's equation $-\nabla^2\varphi(\boldsymbol{x},t) = 4\pi\rho = 4\pi\sum_s(\overline{n}q)_s\int d\boldsymbol{v}\, f_s(\boldsymbol{x},\boldsymbol{v},t)$. ($\overline{n}$ is the mean density.) That is inadequate for detailed studies of modern high-pressure devices and some astrophysical situations, but still contains an ample amount of important physics.

2.2 Basic gyrokinetic equations

For strong $\boldsymbol{B}_0$, the rapid gyration arising from the Lorentz $\boldsymbol{v}\times\boldsymbol{B}_0$ term, the 6D nature of Eq. (1), and the high-frequency collective oscillations it supports render it intractable for studies of low-frequency nonlinear physics on turbulence time scales. Thus one turns to some sort of averaging procedure that removes the fast gyration time scale and reduces the 6D kinetic equation to a 5D one. Together with a particular low-frequency closure of Poisson's equation that eliminates high-frequency collective dynamics, this defines 'low-frequency gyrokinetics,' which I simply call 'gyrokinetics' in this article. More generally, the gyrocenter motion can merely be segregated (Kolesnikov et al. 2007, Qin et al. 2000, 1999), allowing one to discuss high-frequency gyrokinetics, useful for studies of plasma heating. However, lack of space precludes discussion here.

For low-frequency motions, a crucial quantity is the magnetic moment or first adiabatic invariant $\mu \approx \mu^{(0)} \doteq \frac{1}{2}v_\perp^2/\omega_c(\boldsymbol{x})$ associated with the rapid gyration of a charged particle around a magnetic field line (Cary & Brizard 2009, Northrop 1963). General results from the theory of almost-cyclic systems (Kruskal 1962, Lichtenberg & Lieberman 1992) show that μ should be asymptotically conserved even in the presence of weak magnetic inhomogeneities and slowly varying fields. If 'weak' and 'slow' are indicated by an ordering parameter ϵ, then μ can in principle be determined as an asymptotic expansion through all orders in ϵ, with $\mu^{(0)}$ being the lowest-order term. In this regard, a seminal calculation was by Taylor (1967), who found the first-order correction to $\mu^{(0)}$ due to the presence of a slowly varying electrostatic wave of arbitrary wavelength.

Note that μ is not an exact invariant. Dragt & Finn (1976) discussed μ conservation within a Hamiltonian formulation, relating it to the existence of a Kolmogorov–Arnold–Moser surface. They found stochastic regions in the motion of a charged particle in a dipolar magnetic field. Dubin & Krommes (1982) discussed the interaction of rapid gyration with high harmonics of periodic motion on much longer time scales (such as the bounce motion associated with the second or longitudinal invariant J). They found stochastic layers whose widths scale as $\exp(-b/\epsilon)$, where ϵ is the ratio of the small bounce frequency and large gyrofrequency and b is a constant; since $\exp(-b/\epsilon)$ is asymptotic to zero, stochastic wandering in the layers is overlooked by the asymptotic construction of a 'conserved' μ. Lichtenberg & Lieberman (1992) describe in more detail the situation, which is related to Arnold diffusion (Chirikov 1979). One must always keep in mind the possibility that the adiabatic invariance of μ can be broken. Related concerns were recently expressed by Sugiyama (2008); for discussion, see Krommes (2009a) and Sugiyama (2009). In the material to follow, μ conservation will be assumed; that is an excellent approximation for the situations of interest.

Gyrokinetics amounts to the determination of a change of variables (Catto 1978) from the particle phase space $\{\boldsymbol{x}, \boldsymbol{v}\} \equiv z^i$ to gyrocenter phase space $\{\overline{\boldsymbol{X}}, \overline{U}, \overline{\mu}, \overline{\zeta}\} \equiv \overline{z}^i$, together with a closure approximation to be described. (Note that a mere change of variables cannot alter the physical content of the kinetic equation.) Here $\overline{\boldsymbol{X}}$ is the gyrocenter position, $\overline{U}$ is the gyrocenter velocity along $\boldsymbol{B}$, and $\overline{\zeta}$ is the gyration phase; the overline (subsequently omitted) signifies development as an asymptotic series whose lowest-order form corresponds with that for circular motion. In standard gyrokinetics, the ordering parameter is $\epsilon \sim \omega/\omega_{ci} \sim k_\parallel/k_\perp \sim V_E/v_t$, where $\boldsymbol{k}$ is a typical fluctuation wavevector, parallel and perpendicular are with respect to $\boldsymbol{B}$, $\boldsymbol{V}_E$ is the $\boldsymbol{E}\times\boldsymbol{B}$ velocity, and v_t is the thermal velocity. [Other orderings are required in some situations (Brizard & Hahm 2007, GYP 2010).] The gyrocenter PDF $\widetilde{F}$ thus obeys the 6D equation

$$\frac{\partial \widetilde{F}_s(\boldsymbol{X}, U, \mu, \zeta, t)}{\partial t} + \dot{\boldsymbol{X}}\cdot\boldsymbol{\nabla}\widetilde{F} + \dot{U}\frac{\partial \widetilde{F}}{\partial U} + \dot{\zeta}\frac{\partial \widetilde{F}}{\partial \zeta} = -\mathcal{C}[\widetilde{F}]. \tag{2}$$

Here the tilde denotes a ζ-dependent quantity; $\mathcal{C}[\widetilde{F}]$ is the transformation of $C[f]$. No derivative with respect to μ appears because it is (adiabatically) conserved by construction. The gyrocenter drifts $\dot{\boldsymbol{X}}$, $\dot{U}$, and $\dot{\zeta}$ follow from the theory; importantly, they are constructed to be independent of ζ (see below). The gyrophase average of Eq. (2), denoted by $\langle \ldots \rangle_\zeta$, then eliminates the gyration term $\partial(\ldots)/\partial\zeta$, leaving one with the conventional GKE for the 5D $F \doteq \overline{\widetilde{F}}$:

$$\frac{\partial F}{\partial t} + \dot{\boldsymbol{X}}\cdot\boldsymbol{\nabla}F + \dot{U}\frac{\partial F}{\partial U} = -\langle \mathcal{C}[\widetilde{F}]\rangle_\zeta. \tag{3}$$

In the form usually implemented in simulations, the drifts are

$$\dot{\boldsymbol{X}} = B_*^{-1}(\boldsymbol{B}_* U + cq^{-1}\widehat{\boldsymbol{b}}\times\boldsymbol{\nabla}\overline{H}^{(1)}), \quad \dot{U} = -(mB_*)^{-1}\boldsymbol{B}_*\cdot\boldsymbol{\nabla}\overline{H}^{(1)}. \tag{4}$$

Here $\widehat{\boldsymbol{b}} \doteq \boldsymbol{B}/B$, $B_* \doteq \widehat{\boldsymbol{b}}\cdot\boldsymbol{B}_*$, $\boldsymbol{B}_* \doteq \boldsymbol{B} + (mc/q)U\boldsymbol{\nabla}\times\widehat{\boldsymbol{b}}$, $\overline{H}^{(1)} \doteq q\langle\varphi\rangle_\zeta$, and

$$\langle\varphi\rangle_\zeta(\boldsymbol{X}, \mu) \doteq (2\pi)^{-1}\int_0^{2\pi} d\zeta\, \varphi(\boldsymbol{X} + \boldsymbol{\rho}(\zeta)), \tag{5}$$

where $\boldsymbol{\rho} \doteq \omega_c^{-1}\widehat{\boldsymbol{b}} \times \boldsymbol{v}_\perp$ is the gyroradius vector. If periodicity is assumed, Fourier transformation is useful; one finds $\langle\varphi\rangle_{\zeta,\boldsymbol{k}} = J_0(k_\perp v_\perp/\omega_c)\varphi_{\boldsymbol{k}}$. [Thus gyrokinetics is able to handle wavelengths comparable to or smaller than the ion gyroradius ρ_i; nominally, $k_\perp\rho_i = O(1)$.] Physically, the Bessel function describes the $v_\perp$- or μ-dependent reduction in effective potential that arises because during one gyroperiod a particle samples different phases of a slowly time-varying fluctuation. For more discussion of the resulting nonlinear phase mixing, see Sec. 5.3. The drifts include the effective $\boldsymbol{E} \times \boldsymbol{B}$ velocity $\langle\boldsymbol{V}_E\rangle_\zeta \doteq c\widehat{\boldsymbol{b}} \times \boldsymbol{\nabla}\langle\varphi\rangle_\zeta/B_*$ and the magnetic $\boldsymbol{\nabla}B$ and curvature drifts.

If collisions are neglected, Eq. (3) appears to be closed in terms of F, but that is illusory because the drifts involve self-consistent electromagnetic fields. The theory is not complete until the relevant Maxwell equations are solved. Now the charge ρ and current $\boldsymbol{j}$ are given as momentum integrals over the particle PDF f; however, the GKE evolves the gyrocenter PDF $\widetilde{F}$, and the 'pull-back' transformation $\widetilde{T}$ from $\widetilde{F}$ to f, $f = \widetilde{T}\widetilde{F}$, is nontrivial. If one writes $\widetilde{F} = F + \delta\widetilde{F}$, then a closed system is obtained by neglecting the ζ-dependent $\delta\widetilde{F}$: $f \approx \widetilde{T}F$. This can be justified for collisionless plasmas (Dubin et al. 1983) in terms of a projection procedure and an assumption of ζ-independent initial conditions; the latter precludes the possibility of exciting high-frequency fluctuations. However, neglect of collisions is problematical because they represent the only true dissipative effect in the problem; see further discussion in Sec. 5.3. When $\mathcal{C}$ is retained, Eq. (3) is not closed in terms of F because the $\mathcal{C}$ in Eq. (2) (which describes the physics of collisions in the particle, not gyrocenter, phase space) drives a $\delta\widetilde{F}$; however, that correction is small and is frequently neglected. Discussion of the formal gyrokinetic collision operator $\langle\mathcal{C}\rangle_\zeta$ is given by Brizard (2004). Simplifications such as the model by Abel et al. (2008) are often used in practice.

In summary, the asymptotic construction of a conserved μ and the closure $f \approx \widetilde{T}F$ disconnect gyrokinetics from the Vlasov description, leaving one with a reduced dynamical system, describing the self-consistent collective interactions of gyrocenters, that is appropriate for the description of the low-frequency fluctuations that are believed to be important for turbulence in magnetized plasmas.

2.3 Noncanonical Lagrangian methods

In principle, the gyrocenter drifts should include corrections through all orders in ϵ, as should the pull-back $\widetilde{T}$. How to achieve a consistent practical truncation has been a source of confusion, and some issues remain controversial (see Sec. 5.5). One therefore requires a systematic methodology, which has undergone substantial development. Because Brizard & Hahm (2007) and Cary & Brizard (2009) treat the formalism in detail, I shall be brief. Early workers on linear gyrokinetics, e.g. Catto (1978) and Antonsen & Lane (1980), analyzed the Vlasov equation perturbatively by treating the magnetic Lorentz force term as large; gyrokinetic equations then arose as solvability conditions. That method was also followed by Frieman & Chen (1982), who derived a nonlinear GKE (including $\boldsymbol{E} \times \boldsymbol{B}$ advection). Their demonstration that a workable gyro-averaged equation emerges even in the face of nonlinearity was a nontrivial and significant result. However, the resulting equations were not in characteristic form, making them unusable for numerical simulation by the particle-in-cell (PIC) method discussed in Sec. 4.1.1. Lee (1983) derived suitable equations in characteristic form by using a recursive perturbation procedure, also showing how ion polarization effects appear most naturally in the gyrokinetic Poisson equation (see Sec. 3.1) instead of the kinetic equation; his initial simulations fathered an enormous effort on PIC gyrokinetics that continues to the present. But years earlier Littlejohn (1979, 1981, 1982) had begun emphasizing the advantages of Hamiltonian techniques and Lie perturbation methods, and his seminal work laid the foundations for many of the modern developments. In particular, it inspired Dubin et al. (1983) to give a Hamiltonian formulation of self-consistent gyrocenter dynamics in an electrostatic slab (constant $\boldsymbol{B}_0$), clearly spelling out the basic closure procedure for deriving the low-frequency gyrokinetic Poisson equation.

Further work by Littlejohn (1983) and Cary & Littlejohn (1983) emphasized the utility of Lagrangian methods, in which the equations of motion are derived from an action principle expressed in terms of noncanonical variables (the use of which leads to considerable technical simplifications). Those developments brought to the attention of the plasma physicists various notions of differential geometry (Fecko 2006, Misner et al. 1973), including the use of differential forms. The particle action can be written as $S = \int \gamma$, where the fundamental one-form is $\gamma \doteq \boldsymbol{p}\cdot d\boldsymbol{q} - H\, dt$, H being the single-particle Hamiltonian. Since S is indifferent to the particular variables used in its evaluation, γ can be transformed to any convenient set of variables (e.g., gyrocenter coordinates), which need not be canonical but whose equations of motion nevertheless still follow from the variational principle $\delta S = 0$. One constructs the change of variables perturbatively (possibly after a preparatory transformation to lowest-order gyrocenter variables) in such a way that μ is conserved order by order. This methodology relies on a form of Noether's theorem (Noether 1918), which states (Cary & Brizard 2009, Cary & Littlejohn 1983) that if all of the coefficients of γ are independent of ζ, then the coefficient of $d\zeta$ is conserved. That coefficient is precisely μ. Hahm (1988) used the one-form method to derive a nonlinear electrostatic GKE in the presence of magnetic inhomogeneities, generalizing the slab results of Dubin *et al.* Virtually all subsequent work on modern gyrokinetics, including electromagnetic corrections (Hahm et al. 1988), uses some variant of the one-form method. Technically, the perturbation expansions are performed with the aid of Lie transformations (Brizard & Hahm 2007, Cary 1981, Kaufman 1978).

The noncanonical Lagrangian methods focus on the derivation of the (ζ-independent) gyrocenter Hamiltonian $\overline{H}$. For example, through the first three orders of expansion (ϵ^{-1}, ϵ^0, ϵ^1) one finds electrostatically (with $\boldsymbol{B}_0 = \boldsymbol{\nabla}\times\boldsymbol{A}_0$) that

$$\gamma = [\underbrace{c^{-1}q\boldsymbol{A}_0(\boldsymbol{X})}_{O(\epsilon^{-1})} + \underbrace{mU\widehat{\boldsymbol{b}}}_{O(1)} - \underbrace{\mu\boldsymbol{K}_*}_{O(\epsilon)}]\cdot d\boldsymbol{X} + \underbrace{\mu\, d\zeta}_{O(\epsilon)} - (\underbrace{\overline{H}^{(0)}}_{O(1)} + \underbrace{\overline{H}^{(1)}}_{O(\epsilon)})dt, \tag{6}$$

where $\boldsymbol{K}_* \doteq \boldsymbol{K} + \frac{1}{2}\widehat{\boldsymbol{b}}(\widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\times\widehat{\boldsymbol{b}})$, $\boldsymbol{K} \doteq (\boldsymbol{\nabla}\widehat{\boldsymbol{e}}_1)\cdot\widehat{\boldsymbol{e}}_2$, and $\overline{H}^{(0)} \doteq \mu\omega_c(\boldsymbol{X}) + \frac{1}{2}mU^2$. Here $\boldsymbol{K}$ is the so-called gyrogauge vector; $\widehat{\boldsymbol{e}}_1$ and $\widehat{\boldsymbol{e}}_2$ are arbitrary unit vectors perpendicular to $\boldsymbol{B}$ that locate the position of the gyrating particle in space. Littlejohn (1983, 1984, 1988) has clearly interpreted the appearance of $\boldsymbol{K}$ in terms of the invariance of the formalism with respect to a redefinition of ζ.

The second-order $\overline{H}^{(2)}$ contains ponderomotive terms such as $|\boldsymbol{\nabla}\varphi|^2$ [the complete expression including the effects of nonzero pressure (Dubin et al. 1983) is more complicated]. Those terms are related to Reynolds stresses, which are essentially involved in the generation of zonal flows (see Secs. 3.4 and 5.5). Further complications arise when magnetic inhomogeneities are included. Brizard (1989) described the transformation to gyrocenter variables as two successive steps: first, derive a guiding-center theory that incorporates the effects of magnetic inhomogeneities (ordered as ϵ_B); second, incorporate small fluctuation effects (ordered as ϵ_φ) that destroy the invariance of the guiding-center μ but preserve a modified invariant $\overline{\mu}$ [cf. Taylor (1967)]. When ϵ_B and ϵ_φ are ordered with a common expansion parameter ϵ, one must take care to not miss either cross terms of the order of $\epsilon_B\epsilon_\varphi$ or purely geometrical terms of the order of ϵ_B^2. Explicit expressions for all terms in a consistent choice of $\overline{H}^{(2)}$ were derived only quite recently in a difficult calculation by Parra & Calvo (2010).

Given (an approximate) $\overline{H}$, the gyrocenter drifts follow as $\dot{\overline{z}}^i = \{\overline{z}^i, \overline{H}\}$, where $\{\dots\}$ is a noncanonical Poisson bracket and the gyrokinetic Maxwell equations are obtained from the pull-back $f = \widetilde{T}F$. In practice, the asymptotic expansions must be truncated in two places: (i) the drifts must be truncated in the GKE; (ii) $\widetilde{T}$ must be truncated in the Maxwell equations. Doing this consistently is crucial for the preservation of conservation laws, yet how to do so is possibly unclear. For example, should one truncate at $O(\epsilon^n)$ for some n in both places, can one gain accuracy by working out $\widetilde{T}$ to higher order than the drifts, *etc.*?

2.4 Gyrokinetic field theory

A major advance occurred when it was understood by Sugama (2000) and Brizard (2000) how to derive gyrokinetic-Maxwell systems from field-theoretic variational principles. Brizard's version based on constrained variations is possibly more technically convenient, although it is very subtle. In these methods, an action functional S is constructed from F, $\overline{H}$, and the electromagnetic fields:

$$S = S_{\text{EM}} + S_G[F, \overline{H}(\overline{\boldsymbol{z}}, A_\mu, \overline{\partial}_\nu A_\mu, \dots)], \tag{7}$$

where S_{EM} is the electromagnetic Lagrangian, A_μ is the four-potential, and the forms of S_{EM} and the gyrocenter contribution S_G are not shown here. Variation with respect to F then leads to the (collisionless) GKE in the form $\partial_t F + \{F, \overline{H}\} = 0$, while variation with respect to $\varphi \equiv A_0$ leads to the gyrokinetic Poisson equation. (To obtain the usual quasineutrality condition in which Poisson's $\nabla^2\varphi$ is neglected, the electric-field part of S_{EM} is ignored.) The crucial point is that a single, scalar Hamiltonian generates both the GKE and the gyrokinetic Maxwell equations. $\overline{H}$ may be approximate [e.g., correct only through $O(\epsilon^n)$], but the variational procedure instills that approximation consistently into both the kinetic equation and the Maxwell equations, automatically preserving the conservation laws that follow from Noether methods — a point emphasized by Scott & Smirnov (2010) and Scott et al. (2010). In particular, functional derivation of $\overline{H}$ with respect to φ reduces its order by one, equivalent to a truncation of the $\widetilde{T}$ in the gyrokinetic Poisson equation to one order lower than that of the drifts retained in the GKE. For more discussion, see Sec. 5.5 on momentum conservation.

3 PHYSICAL CONTENT OF GYROKINETICS

The Lagrangian, noncanonical, and field-theoretic derivations of the gyrokinetic-Maxwell system are elegant and lead to self-consistent equations that capture an enormous amount of the physics of the magnetized plasma, including turbulent fluctuations and transport driven by gradients in the background profiles of density n, temperature T, and flow $\boldsymbol{u}$. Most of that physics cannot be described in this short article; see Krommes (2006a) for some tutorial lectures and Sec. 4.2 for some illustrations of the contact between simulations and experiments. However, it is instructive to consider some of the basic content of the gyrokinetic system.

3.1 Polarization and the gyrokinetic vacuum

Although ζ dependence is rigorously removed from the gyrocenter drifts [a process called 'dynamical reduction' by Brizard (2008)], gyration-related effects remain in the pull-back transformation that defines the gyrokinetic Maxwell equations. This redistribution of information is closely related to the treatment of dielectric media in classical electromagnetism. As is well known, it is frequently convenient to separate total charge into free charge and bound charge, the latter describable by a polarization vector $\boldsymbol{P}$ such that Poisson's equation becomes $\boldsymbol{\nabla}\cdot\boldsymbol{D} = 4\pi\rho^{\text{free}}$, where $\boldsymbol{D} \doteq \boldsymbol{E} + 4\pi\boldsymbol{P}$. In gyrokinetics, that decomposition arises inevitably from a representation of the dynamics in terms of the gyrocenter F and $\overline{H}$, with the gyrocenter charge ρ^G playing the role of ρ^{free}. The gyrokinetic quasineutrality condition can be shown to follow from the variational principle as

$$\sum_s \overline{n}_s \int_{\boldsymbol{p}} JF\frac{\delta\overline{H}}{\delta\varphi} = 0. \tag{8}$$

Here $\int_{\boldsymbol{p}}$ denotes integration over momentum variables, J is the Jacobian of the gyrokinetic transformation, and $\delta/\delta\varphi$ denotes the functional derivative with respect to φ, which is nontrivial in the presence of spatial gradients of φ. Upon writing $\overline{H} = \overline{H}^{(0)} + q\varphi + \Delta\overline{H}$ (φ, not $\langle\varphi\rangle_\zeta$, is used here), Eq. (8) becomes $\rho^G - \boldsymbol{\nabla}\cdot\boldsymbol{P} = 0$, where $\rho^G \doteq \sum_s(\overline{n}q)_s \int_{\boldsymbol{p}} JF$ and $\boldsymbol{P} \doteq -\sum_s \overline{n}_s \int JF(\partial\overline{H}/\partial\boldsymbol{E} + \cdots)$, the dots indicating additional terms involving derivatives with respect to second- and higher-order gradients of φ. The most important pedagogical example involves the zero-gyroradius limit of the $\overline{H}^{(2)}$ derived by Dubin et al. (1983): $\Delta\overline{H} = \overline{H}^{(2)} = -\frac{1}{2}mV_E^2$. One finds $\boldsymbol{P} = \mathcal{D}_\perp\boldsymbol{E}_\perp$, where $\mathcal{D}_\perp \doteq \rho_{\text{s}}^2/\lambda_{De}^2 = \omega_{pi}^2/\omega_{ci}^2$, $\rho_{\text{s}} \doteq c_{\text{s}}/\omega_{ci}$ is the so-called sound radius, $c_{\text{s}} \doteq (ZT_e/m_i)^{1/2}$ is the ion sound speed (Z is the atomic number and T_e is the electron temperature), $\lambda_{De} \doteq (4\pi n_e e^2/T_e)^{-1/2}$ is the electron Debye length (e is the electronic charge), and $\omega_{pi} \doteq (4\pi n_i q_i^2/m_i)^{1/2}$ is the ion plasma frequency. Typically $\mathcal{D}_\perp \gg 1$; this inequality defines the gyrokinetic regime (Krommes et al. 1986). This substantial polarization is a consequence of the ion polarization drift $\boldsymbol{V}^{\text{pol}} = \omega_{ci}^{-1}\partial_t(c\boldsymbol{E}_\perp/B)$, as can be seen by integrating the continuity equation for polarization charge $\partial_t\rho^{\text{pol}} + \boldsymbol{\nabla}\cdot(n_i q_i\boldsymbol{V}^{\text{pol}}) = 0$ in the linearized approximation. It is noteworthy that the (effect of the) polarization drift shows up in the gyrokinetic Poisson equation rather than the GKE.

Polarization leads to a useful interpretation of the gyrokinetic–Maxwell system as describing motion of gyrocenters in a 'gyrokinetic vacuum.' The vacuum state, devoid of gyrocenters, is defined (for the example above) to possess a large dielectric permittivity $\mathcal{D}_\perp$ analogous to the permittivity ϵ_0 of free space. Into that vacuum one places gyrocenters, which move with the $\boldsymbol{E}\times\boldsymbol{B}$ and magnetic drifts. This interpretation was first given by Krommes (1993a) and has been discussed in various pedagogical papers by Krommes (2006a, 2009b).

3.2 Equilibrium gyrokinetic statistical mechanics

Although most interest is in nonequilibrium states (see Sec. 3.3), it is instructive to consider what physics emerges from thermal-equilibrium gyrokinetics. Thermal equilibrium arises from intrinsically nonlinear interactions, so predictions derived therefrom can be used to test the nonlinear routines in simulation codes, a rare opportunity. That was already recognized in pre-gyrokinetic simulation theory (Birdsall & Langdon 1985), but gyrokinetics is richer and more subtle.

A gas of discrete gyrocenters in thermal equilibrium exhibits fluctuations whose properties can be calculated from a gyrokinetic fluctuation–dissipation theorem (FDT), which can be formulated in terms of the wave-number- and frequency-dependent gyrokinetic dielectric function $\mathcal{D}(\boldsymbol{k}, \omega)$. That was done first by Krommes et al. (1986) for the electrostatic limit, later by Krommes (1993a,b) for weakly electromagnetic fluctuations. One finds that gyrokinetic fluctuations are strongly suppressed (by the tendency for ion polarization to neutralize charge imbalances) relative to those of the full many-body plasma.

Even in the absence of discreteness effects, gyrokinetic systems appropriately truncated in wave-number and velocity space possess absolute statistical equilibria, as discussed by Zhu & Hammett (2010). Such equilibria are well known in neutral fluids. For example, in 2D the conservation of both energy and enstrophy admits two-parameter Gibbsian equilibrium with possible negative temperature states (Kraichnan 1975). [For a review of 2D turbulence with earlier references, see Kraichnan & Montgomery (1980). Some aspects of the absolute equilibrium problem were also reviewed by Krommes (2002), and a Monte Carlo method for constructing states of N gyrocenters with negative temperature was described by Krommes & Rath (2003).] In a 2D gyrokinetic system truncated in wave number and discretized with N velocity points, there are N entropy-related invariants in addition to an energy invariant, and in a nontrivial calculation Zhu & Hammett were able to find the form of the corresponding Gibbsian equilibria analytically. The plethora of invariants leads to modifications of the equilibrium spectrum and implies nontrivial behavior of forced, dissipative gyrokinetic cascades.

3.3 Drift waves

Confined plasmas cannot be in thermal equilibrium because they possess profile gradients. From the point of view of basic physics, the most important mode in a nonequilibrium gyrokinetic plasma is the drift wave (DW), supported by a gradient in the background density profile. (Modes involving gradients in the background temperature are considered to be more important in practice.) Although the kinetic effects embodied in the full $\mathcal{D}(\boldsymbol{k}, \omega)$ are crucial to quantitative calculations of linear instability, the basic DW can be obtained from a simple fluid description of gyrocenters. Consider the cold-ion limit $T_i \to 0$ [which eliminates finite-Larmor-radius (FLR) effects] and ignore magnetic inhomogeneities. The density of ion gyrocenters then obeys the continuity equation (obtained from the zeroth velocity moment of the collisionless GKE) $\partial_t n_i + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} n_i + \nabla_\parallel (u_{\parallel i} n_i) = 0$. Write $n = \langle n \rangle + \delta n$, where $\langle n \rangle$ denotes the background profile. After linearization, the definition $\boldsymbol{\nabla} \ln\langle n_i \rangle \doteq -L_n^{-1} \widehat{\boldsymbol{x}}$, and the neglect of $u_{\parallel i}$ because ion inertia is large, this becomes $\partial_t (\delta n_i / \langle n_i \rangle) + V_* \partial_y \delta\Phi = 0$, where $\widehat{\boldsymbol{y}} \doteq \widehat{\boldsymbol{b}} \times \widehat{\boldsymbol{x}}$, $V_* \doteq cT_e/eBL_n$ (assumed to be constant) is called the 'diamagnetic velocity,' and $\delta\Phi \doteq e\delta\varphi/T_e$. Electrons are assumed to travel rapidly along field lines and to adjust instantaneously to any ambient potential, so they are assigned the 'adiabatic' or Boltzmann response $\delta n_e/\langle n_e\rangle = \delta\Phi$. Finally, the gyrokinetic Poisson equation relates the ion polarization to the net gyrocenter charge: $-\rho_\mathrm{s}^2\nabla_\perp^2\delta\Phi = \delta n_i/\langle n_i\rangle - \delta n_e/\langle n_e\rangle$. These equations can be combined to a single equation for the potential: $(1-\rho_\mathrm{s}^2\nabla_\perp^2)\partial_t\delta\Phi + V_*\partial_y\delta\Phi = 0$, which yields the DW dispersion relation $\omega = \Omega_{\boldsymbol{k}}$, where $\Omega_{\boldsymbol{k}} \doteq \omega_*/(1+k_\perp^2\rho_\mathrm{s}^2)$ and $\omega_*(k_y) \doteq k_y V_*$. The physics of the wave involves $\boldsymbol{E}\times\boldsymbol{B}$ advection of the background density gradient (V_*) in the presence of adiabatic electron response (the 1 in the denominator) and ion polarization (the $k_\perp^2\rho_\mathrm{s}^2$ term, which leads to wave dispersion; the appearance of ρ_s rather than ρ_i means that polarization is fundamentally not an FLR effect).

When such analysis is repeated without linearization, one is led (Dubin et al. 1983) to the Hasegawa–Mima equation (HME), originally derived more tediously by Hasegawa & Mima (1978) from moment equations for the actual particles:

$$(1-\rho_\mathrm{s}^2\nabla_\perp^2)\partial_t\Phi + V_*\partial_y\Phi + \boldsymbol{V}_E\cdot\boldsymbol{\nabla}(-\rho_\mathrm{s}^2\nabla_\perp^2\Phi) = 0. \tag{9}$$

This equation is conservative; forcing and dissipation can be inserted by hand. Note that the vorticity in the 2D $\boldsymbol{E}\times\boldsymbol{B}$ motion is $\varpi \doteq \widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\times\boldsymbol{V}_E = \omega_{ci}\rho_\mathrm{s}^2\nabla_\perp^2\Phi$ [this point is illustrated by Fig. 2 of Krommes (2006b)], so the nonlinearity in the HME involves the $\boldsymbol{E}\times\boldsymbol{B}$ advection of vorticity. That effect is sometimes called the 'polarization-drift nonlinearity.' The HME is more properly called the Charney–Hasegawa–Mima equation because it has the same form as the equation for Rossby waves well known to geophysicists. This observation provides an entrée to a large literature on 2D geostrophic turbulence (Holloway 1986, Rhines 1979), many basic results from which carry over to some of the plasma paradigms.

When the assumption of adiabatic electron response is relaxed, one finds in a slab model with constant $\boldsymbol{B}$ a 'universal' DW instability, destabilized by inverse electron Landau damping. Drift-wave stability is a voluminous and subtle subject. For example, Antonsen (1978) proved that in slab geometry the DW is absolutely stable in the presence of arbitrarily small amounts of magnetic shear. (Turbulence can exist even in the face of linear stability; see Sec. 5.4.) However, toroidal effects related to the curvature of the magnetic field lines restore instability (Chen & Cheng 1980). Modern literature refers to toroidal versions of ion- and electron-temperature-gradient-driven (ITG and ETG) modes, trapped electron modes (TEM), *etc.*, and elaborate supercomputer codes have been developed to study their linear physics in realistic confinement geometries (Kotschenreuther et al. 1995, Rewoldt et al. 1982). The nonlinear gyrokinetic codes described in Sec. 4 extend those results to the turbulent regime and calculate the 'anomalous' transport and spectral characteristics of nonlinearly saturated steady states, quantities that can be directly compared with experiments. A turbulent transport coefficient D is typically compared to the basic 'gyro-Bohm' scaling $D \sim (\rho_\mathrm{s}/L)\rho_\mathrm{s}c_\mathrm{s} \propto B^{-2}$, where L is a profile scale length. That scaling is a rigorous consequence of dimensional analysis applied to the HME. More generally, it follows heuristically from random-walk considerations that assume that the gyrokinetic turbulence is local, i.e., possesses a correlation length that scales with ρ_s and a correlation time that scales with $L/c_\mathrm{s} = [\omega_*(k_y{=}\rho_\mathrm{s}^{-1})]^{-1}$.

3.4 Zonal flows

Another constituent of magnetized plasma turbulence is the non-wavelike 'zonal flow' (ZF). Zonal flows (frequently called 'zonal jets' in the geophysics literature) are $\boldsymbol{E} \times \boldsymbol{B}$ flows that in toroidal geometry stem from potential fluctuations that are independent of toroidal angle (and are mostly poloidally symmetric as well); they are primarily poloidal and have no radial component. Although ZFs do not directly produce transport in the radial direction, they may regulate DW turbulence levels *via* their shearing effect on DW eddies; thus it is generally believed that a larger level of ZFs (frequently self-generated by the DWs) is associated with a smaller level of turbulent transport. An extreme example is afforded by the Dimits-shift regime discussed in Sec. 5.1, in which turbulence is suppressed completely by the excitation of ZFs. Unfortunately, a thorough discussion of ZFs is impossible here. For many details, one can refer to the reviews by Diamond et al. (2005), Itoh et al. (2006), and Fujisawa (2009). More introductory material can be found in the lectures by Krommes (2006a).

Because $k_\parallel = 0$ for ZFs, the electron response cannot be even approximately adiabatic, as was assumed in the derivation of the HME. [That the proper response strongly enhances ZFs was first emphasized by Hammett et al. (1993).] In the simplest model the electron density fluctuation can be taken to entirely vanish for zonal wave numbers. That leads to a 'generalized' or 'modified' HME (Krommes & Kim 2000) that correctly describes the evolution of zonal vorticity and the coupling of the zonal modes to the drift waves *via* Reynolds stresses.

Frequently ZFs are assumed to be of long wavelength relative to the scales responsible for turbulent transport [see the related work on nonlocal Rossby wave turbulence by Connaughton et al. (2010)]. When that is done, an detailed connection to prior work in neutral-fluid turbulence theory can be demonstrated. Krommes & Kim (2000) applied the disparate-scale expansion to a Markovian closure of an equation that can be simply reduced to either the 2D Navier–Stokes equation (NSE) or the generalized HME by appropriate choice of a single adiabaticity parameter. Note that the positive growth rate $\gamma_{\boldsymbol{q}}^{\text{nl}}$ of the mean long-wavelength fluctuation energy due to nonlocal interaction with the short scales can be interpreted in terms of a negative eddy viscosity: $\gamma_{\boldsymbol{q}}^{\text{nl}} \equiv -\mu_{\text{eddy}} q^2$. For the 2D NSE, the (generally anistropic) result for μ_{eddy} reduces in the isotropic limit exactly to the calculation of Kraichnan (1976). In the generalized HM limit, the analogous result can be interpreted in terms of (and actually defines) an algorithm based on a wave kinetic equation for the DWs. Seminal work on such an algorithm was by Diamond et al. (1998). The more systematic calculation by Krommes & Kim (2000) elucidated the proper form and statistical basis of the wave-kinetic algorithm and corrected some issues related to random Galilean invariance and the role of linear wave dispersion; it incidently identified some technical problems with the important work of Carnevale & Martin (1982) on field-theoretical methods applied to nonlinear wave dynamics in weakly inhomogeneous media. The difficulties are related to the choice of the 'plasmon density' $N_{\boldsymbol{k}}$ to be used in the wave kinetic equation. $N_{\boldsymbol{k}}(\boldsymbol{X})$ is not the formula familiar from linear wave theory; rather, it must be a quantity that when summed over $\boldsymbol{k}$ and integrated over space is conserved under the DW–ZF interaction. The correct form was found for some special cases by Smolyakov & Diamond (1999). More generally, Krommes & Kolesnikov (2004) proved that $N_{\boldsymbol{k}}$ is a Casimir invariant in a field-theoretic Hamiltonian representation (Morrison 1998) of the advective nonlinearity. These results generalize and provide additional perspective on Kraichnan's deep insights about the nature of negative eddy viscosity in 2D flow.

Krommes & Kim assumed that the ZFs had zero ensemble mean. An important method that does not require that assumption is the Stochastic Structural Stability Theory of Farrell & Ioannou (2003). In that technique, a form of stochastic modeling, one first derives the equation for an individual realization of a ZF by applying a zonal average to the dynamical equations. Next, the effects of the short-scale turbulence are modeled by a stochastic forcing. Finally, the zonal average is replaced by an ensemble average under an ergodic assumption. The resulting closure can predict nontrivial spatial structure as well as temporal fixed points, limit cycles, or chaotic regimes for the ZFs. It has been strikingly successful in various comparisons with atmospheric data, for example for the emergence of eddy-driven baroclinic jets (Farrell & Ioannou 2009b), and was applied by Farrell & Ioannou (2009a) to the Hasegawa–Wakatani (HW) system of equations (a collisional generalization of the HME that is a paradigm for some edge-related physics). Further remarks on that calculation are made in Sec. 5.4.

4 GYROKINETIC SIMULATIONS

Numerical solution of the GKE is important both for the exploration of physical processes and for quantitative prediction. I will briefly mention various numerical implementations, which have become highly developed since their inceptions in the early 1980's, then provide a few examples of the physics results and comparisons with experiment that been obtained to date. Extensive additional details and references can be found in the reviews by Garbet et al. (2010) and G.L. Hammett (in preparation).

4.1 Numerical methodology

The two basic approaches to the numerical solution of the GKE are (i) the 'continuum' or 'Vlasov' method, which treats the GKE as a standard Eulerian PDE that evolves in 5D phase space; and (ii) the Lagrangian PIC method. Hybrid Lagrangian–Eulerian techniques have also been developed (Grandgirard et al. 2006). These methods can be applied to either a 'full-F' simulation, which solves Eq. (3), or a 'δF' simulation (Kotschenreuther 1991), which writes $F = F_0 + \delta F$, analytically inserts a known equilibrium F_0, and numerically integrates just the equation for δF. (This ζ-independent δF differs from the $\delta\widetilde{F}$ used previously.)

Because the integration of a PDE that involves 5D plus time is computationally intensive, an alternate 'gyrofluid' approach was developed by Hammett and his coworkers. In that technique, fluid equations are derived from velocity moments of the GKE. The inevitable closure problem (Chapman–Enskog theory is inappropriate for nearly collisionless plasmas) is dealt with by a 'Landau-fluid closure' (Hammett & Perkins 1990), in which unknown moments (e.g., the stress tensor or heat-flow vector) are modeled in such a way that linear response is well reproduced. The method has been successful and can be numerically efficient. Key references include Brizard (1992), Dorland & Hammett (1993), Hammett et al. (1993), and Beer & Hammett (1996), and further authoritative discussion is given in the review by G.L. Hammett (in preparation). However, because the modeling is tricky and can fail to capture certain kinds of nonlinear wave–particle interactions, modern focus has been mostly on simulations of the GKE itself although gyrofluid equations are still actively used in areas such as edge turbulence (Scott 2007) and reduced transport models (Staebler et al. 2007).

4.1.1 The particle-in-cell approach The PIC approach, pioneered for gyrokinetics by Lee (1983), was the first to be implemented. It is based on the fact that the full-F GKE can be written in characteristic form. The characteristic trajectories of many gyrocenters are integrated forward one time step, then gathered onto a spatial grid for the purpose of computing the charge and current. [Bessel functions are evaluated by an n-point averaging technique (Lee 1987); various important smoothing techniques are not described here.] This method avoids the need for a velocity-space grid, is straightforward to program, and is well suited to massively parallel processing. A noncomprehensive list of significant PIC codes includes `GTC` (Lin et al. 1998), `GEM` (Chen & Parker 2003), `GTS` (Wang et al. 2006), `ORB5` (Jolliet et al. 2007), and `XGC1` (Chang et al. 2009).

In the δF method, the PIC approach is implemented by assigning a weight w_i to the ith gyrocenter (called a 'marker' or 'tracer'); $w \doteq \delta F/F$, where F is the tracer PDF. The equation for w is derived analytically, then integrated along with the tracer characteristics. The procedure is more completely expressed in terms of a two-weight scheme (Hu & Krommes 1994). In collisionless theory, it can be shown that the mean-square weight evolves secularly, leading to the so-called 'entropy paradox' in which a statistical observable is changing in time even though by conventional spectral measures the turbulence appears to be saturated. Introduction of collisions resolves this paradox (Krommes & Hu 1994), which is related to the generation of fine scales in velocity space (see Sec. 5.3). For controlling the collisionless limit, Krommes (1999b) suggested an alternate approach involving the use of a generalized thermostat (Evans & Morris 1984) or 'w-stat'; that idea was pursued and extended by McMillan et al. (2008).

The PIC method is essentially a Monte Carlo sampling technique (Aydemir 1994, Hu & Krommes 1994), although it was not originally discussed as such. Thus one must contend with sampling noise. Following Krommes's basic work on gyrokinetic fluctuations (cited in Sec. 3.2) and calculations of δF noise by Hu & Krommes, further theoretical and numerical analyses of gyrokinetic noise were made by Nevins et al. (2005), who showed that some PIC simulations may be noise-dominated. Rigorously, calculation of noise in nonequilibrium situations is nontrivial; the FDT no longer applies, and discreteness effects may be amplified by instabilities and mix nonlinearly with collective effects in complicated ways. Krommes (2007) reviewed the noise issue and discussed the formal problem of calculating nonequilibrium sampling noise, following a general methodology given by Rose (1979). Although the complete formalism is complicated, the structure of the theory supports the general conclusions drawn by Nevins *et al.* Noise can be reduced by increasing the number N of markers (but only by $\sqrt{N}$), or by phase-space smoothing to reduce the weights (Chen & Parker 2007).

4.1.2 The continuum approach Although the PIC approach is very intuitive, it is surprisingly subtle. An alternate approach is to attack the GKE directly with the aid of advanced numerical techniques for PDEs. Specific methods are documented in the publications and web pages for the major continuum codes, a noncomprehensive list of which includes `GS2` (GS2 2000) and its daughter `AstroGK` (Numata et al. 2010), which removes geometry effects from `GS2` and is used for studies of astrophysical gyrokinetics; `GENE` (Jenko et al. 2000); `GYRO` (Candy & Waltz 2003a,b); and `GT5D` (Idomura et al. 2008). Continuum codes have their own algorithmic challenges. However, they have been very effective in attacking practical problems and have made major progress in the push toward successful comparisons with experiment.

4.2 Illustrative simulation results

Figures 1–4 illustrate the kinds of results that can be obtained from the modern codes. They include global simulations of existing machines (Figure 1), impressive agreement with experimental data (Figure 2), and studies of turbulence in regions where magnetic topology changes due to the presence of a divertor (Figure 3). Although workstation-sized runs have been useful for many qualitative physics studies, comparisons against experiment that include realistic geometry and other practical effects require massively parallelized processing on the world's largest supercomputers. Such simulations are not restricted to magnetically confined fusion plasmas. Figure 4 shows an application of a gyrokinetic simulation to the solar wind, and there are other applications to space physics, e.g., the simulations by Kobayashi et al. (2010) of dipolar systems such as planetary magnetospheres.

5 SOME ADVANCED TOPICS IN GYROKINETICS

All of the following topics are still the focus of contemporary research. They demonstrate a nice interplay between numerical simulation and analytical theory, and they illustrate the richness of gyrokinetic physics.

5.1 Transition to turbulence

As simulation codes proliferated, it became important to verify them [here 'verify' is used in a standardized technical sense — see, for example, Greenwald (2010) — and should be contrasted with 'validate' (against experiment)], and a standard case (the 'Cyclone base case') was proposed for detailed study. Dimits et al. (2000) published the results of comparisons between various codes. Part of that work involved the discovery of what is now called the 'Dimits shift,' which provides a window into an important area of gyrokinetic physics.

The Dimits shift arises (at least) in studies of the turbulent ion heat flux Q driven by a gradient in the background ion temperature gradient, parametrized here by $\kappa \doteq R/L_T$, R being the major radius of the torus and L_T being the temperature-gradient scale length. Linearized gyrokinetics makes a definite prediction for the threshold κ_{lin} of linear instability. Conventional arguments suggested that turbulence and Q should turn on for $\kappa > \kappa_{\text{lin}}$; however, the (collisionless) simulations revealed that Q remained essentially zero for a nonzero range $\kappa_{\text{lin}} \le \kappa \le \kappa_*$ for some κ_*; the difference of κ_* from κ_{lin} defines the Dimits shift. Dimits *et al.* gave the correct qualitative interpretation, which is that in the Dimits-shift regime ITG fluctuations are suppressed by the self-generation of zonal flows. Rogers et al. (2000) provided a further compelling analysis (including a discussion of 'tertiary instability') and simple model that argued in favor of the ZF interpretation. That work was important, as the significance of ZFs (already well known to geophysicists) was just beginning to penetrate the consciousness of the fusion-physics community (Diamond et al. 1998, Rosenbluth & Hinton 1998).

Kolesnikov & Krommes (2005, KK) attempted to study the Dimits shift in a simple model by using the systematic machinery of bifurcation analysis (Guckenheimer & Holmes 1983, Kuznetsov 1998), including use of the center manifold theorem to eliminate rapidly damped modes. Zonal flows are only very weakly damped by collisions; KK assumed that they were strictly undamped. That greatly complicates the analysis. In the absence of ZFs, ITG modes are destabilized as κ is increased due to a conventional Hopf bifurcation. In their presence, however, additional undamped ZFs reside at the origin in complex λ space [variations like $\exp(\lambda t)$ are assumed] while the complex-conjugate pair of ITG eigenvalues crosses the imaginary axis. The bifurcation is no longer simple Hopf and exhibits peculiar properties that are difficult to analyze. KK did succeed in analytically calculating a Dimits shift for their model, which was a significant proof of principle. However, they employed a Galerkin truncation in order to reduce the ITG PDEs to a small number of coupled amplitudes, and not surprisingly the predicted shift was sensitive to the truncation. It is probable that a different sort of analysis altogether must be done in order to properly obtain the Dimits shift analytically, especially in realistic problems with magnetic shear. Such a calculation remains one of the outstanding problems in plasma theory.

5.2 The nature of plasma turbulence; damped eigenmodes

In general, sufficiently large drive leads to turbulence. In neutral fluids at high Reynolds number, that is frequently described by the standard paradigm involving long-wavelength energy-containing scales excited by macroscopic instability, an intermediate-scale inertial interval, and very short dissipation scales. For discussion of a similar situation in plasmas, see Schekochihin et al. (2009) and Figure 4. However, plasmas can also behave quite differently. In some cases, especially in magnetic fusion, the range of excited scales may be small, possibly no more than a decade; a well-defined inertial range may not exist. The way in which plasma turbulence saturates may be entirely different from the standard Navier–Stokes scenario.

Although these differences have been long appreciated in general terms, only recently have they been quantified. Terry and coworkers (Hatch et al. 2011, Terry et al. 2006) have demonstrated that a new paradigm, involving coupling to damped eigenmodes and not requiring a cascade in wave number, can sometimes be superior. First consider an n-field fluid description. (For the Hasegawa–Wakatani system, $n = 2$.) Such a system has n linear eigenmodes, most of which are typically damped. Nonlinearity can then couple energy in an unstable mode to stable modes at comparable wave numbers, providing a saturation mechanism. Gyrokinetics is richer, as there are an infinite number of linear eigenmodes. Figure 5 shows a typical eigenmode spectrum for the gyrokinetic description of ITG turbulence; a single unstable ITG mode can couple to a sea of damped eigenmodes. For detailed discussion of Figure 5, see Hatch et al. (2011).

5.3 Entropy, phase-space cascades, and dissipation

It is well known that the behavior of the dissipative Navier–Stokes equation differs profoundly from that of the conservative Euler equation. Analogous discussion of the contrast between the collisional and collisionless gyrokinetic equations was given by Krommes & Hu (1994), who argued that a numerically observed secular growth in the entropy-like quantity $\int dz\, \delta F^2/F$ could be tamed only by the inclusion of collisional dissipation. That insight was verified by careful simulation measurements by Watanabe & Sugama (2004) and Candy & Waltz (2006). For almost-collisionless physics, effective dissipation requires the generation of fine scales in velocity space. That can be accomplished by the parallel streaming term in the linearized GKE, as can be seen by an expansion of the $v_\parallel$ dependence of the GKE into Hermite polynomials (Hammett et al. 1993). But nonlinear phase mixing arising from the $J_0(k_\perp v_\perp/\omega_{ci})$ in the effective potential can be even more efficient. The role of that term in the simultaneous generation of fine perpendicular scales in both position and velocity has been discussed by Schekochihin et al. (2009) in conjunction with the gyrokinetic theory of kinetic Alfvén cascades, believed to be important in solar-wind physics (Figure 4). That mechanism provides "a nonlinear route to dissipation through phase space" (Schekochihin et al. 2008). The analytics lead to concrete predictions for the exponents of self-similar, power-law, phase-space entropy cascades (Plunk et al. 2010). Those have been observed numerically (Navarro et al. 2011, Tatsuno et al. 2009) and provide a basis for interpreting some solar-wind spectra (Howes et al. 2008).

5.4 Submarginal and non-normal turbulence

'Submarginal' or 'subcritical' turbulence exists, by definition, in regimes where the linear eigenmode spectrum is entirely stable. This phenomenon, intensively studied in various neutral-fluid situations such as Poiseuille or planar Couette flow, requires that the linear operator L be non-normal, i.e., $[L, L^\dagger] \neq 0$ (Henningson & Reddy 1994). Non-normality can be important even when some modes are linearly unstable. Submarginal or more generally non-normal turbulence can exist in gyrokinetic plasmas for relevant parameters; it may be particularly important to tokamak edges, the physics of which is considered to be of critical importance for the understanding of magnetic confinement. Scott (1992a,b) demonstrated numerically that drift-wave turbulence can be submarginal, and he described a plausible scenario for the nonlinear self-sustainment. Itoh et al. (1996) described simple closure approximations that suggested that submarginal turbulence can exist quite generally. Krommes (1999a) discussed some of the ideas that Waleffe (1995) had developed for the description of shear flows close to transition, providing an interpretation and generalization of some earlier plasma work by Drake et al. (1995). Thorough study of the HW equations, which are non-normal, was done by Farrell & Ioannou (2009a), who used the Stochastic Structural Stability Theory introduced in Sec. 3.4. Their analytical closure predicted states of both high (H) and low (L) transport. [Qualitatively similar regimes have been observed experimentally, as described in the review by Wagner (2007), and are a subject of longstanding interest.] They argued that their low-transport states could be accessed through appropriate manipulations of external parameters. Some details of the results are not definitive because the calculation was not fully energetically self-consistent and the small-scale turbulence was modeled crudely. However, Farrell & Ioannou (1996) have advanced powerful arguments that the predictions should be relatively insensitive to the details of the stochastic modeling when the linear operator is non-normal. Further application of these ideas to models of gyrokinetic turbulence should be a fruitful line of research.

Recently, gyrokinetic simulations were used to obtain a description of ITG heat flow Q parametrized by temperature gradient κ_T and flow shear κ_u (Highcock et al. 2010). Nonzero flow shear allows the possibility of submarginal turbulence. Parra et al. (2011) used the unusual shape of $Q(\kappa_T, \kappa_u)$ to predict the optimal amount of momentum input that minimizes transport, and showed that it admits the possibility of bifurcations between regimes of high and low transport. Such results provide a fresh look at the mechanisms for the transition between the L and the H mode as well as the formation of internal transport barriers, understanding of which is important for the operation of future devices such as ITER.

5.5 Momentum conservation and toroidal rotation

For numerical studies of microturbulence in toroidal devices, a simulation should be run for at least a few turbulence autocorrelation times τ_{ac} in order that good statistics can be obtained by time averaging, and that is already very challenging. Nevertheless, as computational resources continue to improve, attention is beginning to shift toward simulations on the transport time scale on which the macroscopic profiles evolve; that can be orders of magnitude larger than τ_{ac}. One appealing approach invokes a multiple-time-scale strategy in which a δF turbulence code, run on the τ_{ac} scale to obtain local fluxes, is coupled to a coarse-grained transport code that advances mean profiles on macroscopic time scales [see Figure 2, Barnes et al. (2010), and the work of Sugama & Horton (1997) on the derivation of gyrokinetic transport equations]. But it is at least conceptually interesting to inquire whether a full-F gyrokinetic code could be integrated directly to transport times. In a provocative PhD dissertation, Parra (2009) argued to the contrary in the context of the calculation of the radial electric field and toroidal rotation. [The latter is of interest for magnetohydrodynamic stability and flow-shear stabilization of microturbulence (Terry 2000).] Two fundamental assumptions were made: (i) the 'low-flow ordering' $u/c_{\mathrm{s}} = O(\epsilon)$; (ii) gyro-Bohm transport scaling. Then the basic assertion was that the truncations of the gyrokinetic system that are usually implemented in the codes drop higher-order terms that are essential for a correct evaluation of the rotation that develops at long times. That has been highly controversial and has generated a considerable amount of discussion and publications. Many of those are cited in the overview by Parra & Catto (2010b). (The collected papers of those authors serve as an authoritative primer on many facets of gyrokinetics.)

In the original derivations of nonlinear gyrokinetics, truncations were in principle made independently in the kinetic equations and in the Poisson equation. That can obviously lead to problems with conservation laws. As an illustration, Parra & Catto (2010a) revisited the slab calculation of Dubin et al. (1983) and showed that if one retains terms through second order in both places, then a spurious nonconservative term emerges in a momentum evolution equation. They recognized that the spurious term would be eliminated if third-order drifts were retained in the kinetic equation. However, those are very complicated and possibly impractical to code. Moreover, in the presence of magnetic inhomogeneities it seemed possible that even higher-order effects would be required.

The correct way to truncate is given a definitive answer, and in fact is superceded, by the field-theoretic derivations of gyrokinetics. As was described in Sec. 2.4, truncation should be done directly on the action functional S; use of an nth-order Hamiltonian in S is equivalent to truncations of $O(\epsilon^n)$ in the kinetic equation and $O(\epsilon^{n-1})$ in the gyrokinetic Poisson equation. From Noether arguments in the presence of toroidal symmetry, that ensures a conservative form of the equation for toroidal momentum. Left open is the possibility that the momentum fluxes are calculated inaccurately if n is too small [errors of $O(1)$ in the predicted rotation profile may be possible even if $n = 2$]. It was thus significant that Scott & Smirnov (2010) derived the exact form of the gyrokinetic conservation law for toroidal angular momentum, expressing the fluxes in terms of an arbitrarily accurate Hamiltonian $\overline{H}$. [A concise and elegant rederivation of that law by A.J. Brizard (2010, private communication) has been very helpful to the author.] Knowledge of that conservation law is technically useful, as it can serve to verify the fidelity of gyrokinetic simulation codes; an authoritative discussion of related issues was given by Scott et al. (2010). Moreover, it is conceptually important, as it bears on the ordering issue. Scott & Smirnov concluded that one must use an $\overline{H}$ valid through second order (i.e., second-order drifts but first-order polarization). That supports a basic complaint of Parra & Catto that most extant codes use only first-order drifts. However, naive estimates suggest that under the assumptions (i) and (ii) the momentum fluxes stemming from $\overline{H}^{(3)}$ may be of the same order as those from $\overline{H}^{(2)}$ (the latter, nominally of second order, must in fact be smaller in order to recover gyro-Bohm scaling). At the time of writing the necessity of an $\overline{H}^{(3)}$ remained unresolved; subtle symmetry considerations are involved. The issue would disappear if the actual transport were larger than gyro-Bohm, and that is the case in some important physical situations. In any event, while further research should lead to a consensus, it must be emphasized that the full-F quasineutral method may not be practical for long-time gyrokinetic simulations; alternate hybrid approaches may be superior (Parra & Catto 2009, and references therein).

6 SUMMARY POINTS

1. Nonlinear low-frequency gyrokinetics is the major formalism used to study microturbulence in magnetically confined plasmas.
2. The nonlinear gyrokinetic equation (a 5D PDE in gyrocenter phase space) coupled with the gyrokinetic Maxwell equations is a unique dynamical system describing the drift motion of gyrocenters in a ‘gyrokinetic vacuum’ with large permittivity due to ion polarization.
3. Modern derivations of the gyrokinetic–Maxwell system exploit variational principles, noncanonical Lagrangian methods, and Lie perturbation theory.
4. Gyrokinetics provides a good description of the nonlinear interactions between drift waves and self-consistently generated zonal flows. The former cause turbulent transport; the latter regulate the turbulence level.
5. Numerical simulation of the nonlinear gyrokinetic equation, though computationally very challenging, is becoming a quantitatively predictive tool.
6. A nonlinear phase-mixing mechanism unique to gyrokinetics is responsible for entropy cascade in phase space, which provides a route to collisional dissipation at fine scales.
7. The gyrokinetic conservation law for angular momentum can be used to address subtle issues relating to the development of toroidal rotation on long time scales.
8. Modern nonlinear gyrokinetics is elegant, subtle, and powerful. It should reside in the toolbox of every plasma physicist.

7 FUTURE CHALLENGES

1. Continue to develop simulations of the edge and scrapeoff-layer regions of toroidal devices (an endeavor that is complicated because of the change of

magnetic topology from closed to open field lines).

2. Continue to explore effective ways of performing gyrokinetic simulations on the transport time scale, with particular focus on momentum transport.
3. Pursue an integrated program of theory, simulation, and experiment to deeply understand the submarginal and non-normal nature of plasmas.
4. Develop new gyrokinetic simulation algorithms that exploit modern computer hardware such as graphical processing units.

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ANNOTATIONS TO REFERENCES

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SIDEBAR: BRIEF HISTORY OF MODERN STATISTICAL CLOSURE THEORY FOR PLASMAS

The transition to numerical gyrokinetics was not instantaneous; it substantially overlapped with the development of analytical statistical closures for plasmas over a period of several decades. The history and theory of statistical closures for magnetized plasmas were reviewed by Krommes (2002). 'Modern' statistical plasma turbulence theory dates from the mid-1970's with the development of Kraichnan's direct-interaction approximation (DIA) for plasma physics by DuBois & Espedal (1978) and Krommes (1978). [An overview of Kraichnan's substantial contributions to statistical turbulence theory is given by Eyink & Frisch (2010).] Similon & Sudan (1990) described various early (1980's) plasma applications of the DIA. Some efforts at Markovian statistical closures for plasmas were also made during the 1980's (Waltz 1983). But systematic development of Markovian closures, and proper comparison of their predictions with direct numerical simulations, did not occur until the 1990's with the work of Bowman et al. (1993), Bowman & Krommes (1997), and Hu et al. (1995, 1997). Closure theory continues to be of use in specific contexts such as the theory of zonal flows (Krommes & Kim 2000), it provides the natural framework for qualitative interpretations of numerical and experimental data, inspires diagnostic techniques (Itoh et al. 2005), and may serve to motivate useful sub-grid-scale models for large eddy simulations (Morel et al. 2011, Smith & Hammett 1997). However, it has always been clear that realistic plasma turbulence (being inhomogeneous, anisotropic, and 6D) will not yield in any quantitative way to analytical statistical closures. By the early 1980's, the time was ripe for a new tool.

ACRONYMS

1. DW: drift wave
2. FLR: finite-Larmor-radius
3. GKE: gyrokinetic equation
4. HME: Hasegawa–Mima equation
5. HW: Hasegawa–Wakatani
6. ITER: This is not (any more) an acronym; for discussion, see `www.wikipedia.org/wiki/Iter`. It is now understood to mean 'the way' in Latin.
7. ITG: ion temperature gradient
8. PDF: probability density function
9. PIC: particle-in-cell
10. ZF: zonal flow

KEY TERMS

Adiabatic invariant: A quantity that is conserved through all orders under slow variations.

Canonical variables: $2N$ generalized coordinates z^α whose Poisson brackets are $\{z^\alpha, z^\beta\} = \sigma^{\alpha\beta}$, where $\boldsymbol{\sigma} \doteq \begin{pmatrix} 0 & \mathsf{I} \\ -\mathsf{I} & 0 \end{pmatrix}$, I being the $N \times N$ identity matrix.

Dimits shift: In collisionless ITG turbulence, the amount by which the gradient required for the onset of turbulence exceeds the threshold for linear instability.

Gyrokinetics: The study of fluctuations in magnetized plasmas having frequencies much smaller than the ion gyrofrequency.

Gyrokinetic vacuum: The background state, endowed with large dielectric permittivity due to ion polarization, in which gyrocenters move with the effective $\boldsymbol{E} \times \boldsymbol{B}$ and magnetic drifts.

Magnetic moment (μ): The adiabatic invariant associated with the gyration of a charged particle around a magnetic field line.

Noether Theorem: Loosely, the statement that any symmetry of the Lagrangian is associated with a conservation law.

Polarization-drift nonlinearity: The $\boldsymbol{E} \times \boldsymbol{B}$ advection of the vorticity of $\boldsymbol{E} \times \boldsymbol{B}$ motion; the nonlinearity in the Hasegawa–Mima paradigm for drift waves.

Pull-back (transformation): Operator $\widetilde{T}$ that transforms the gyrocenter PDF $\widetilde{F}$ into the particle PDF f: $f = \widetilde{T}\widetilde{F}$.

Zonal flow: $\boldsymbol{E} \times \boldsymbol{B}$ flow (mostly poloidal) generated from a potential that is toroidally and (mostly) poloidally symmetric.

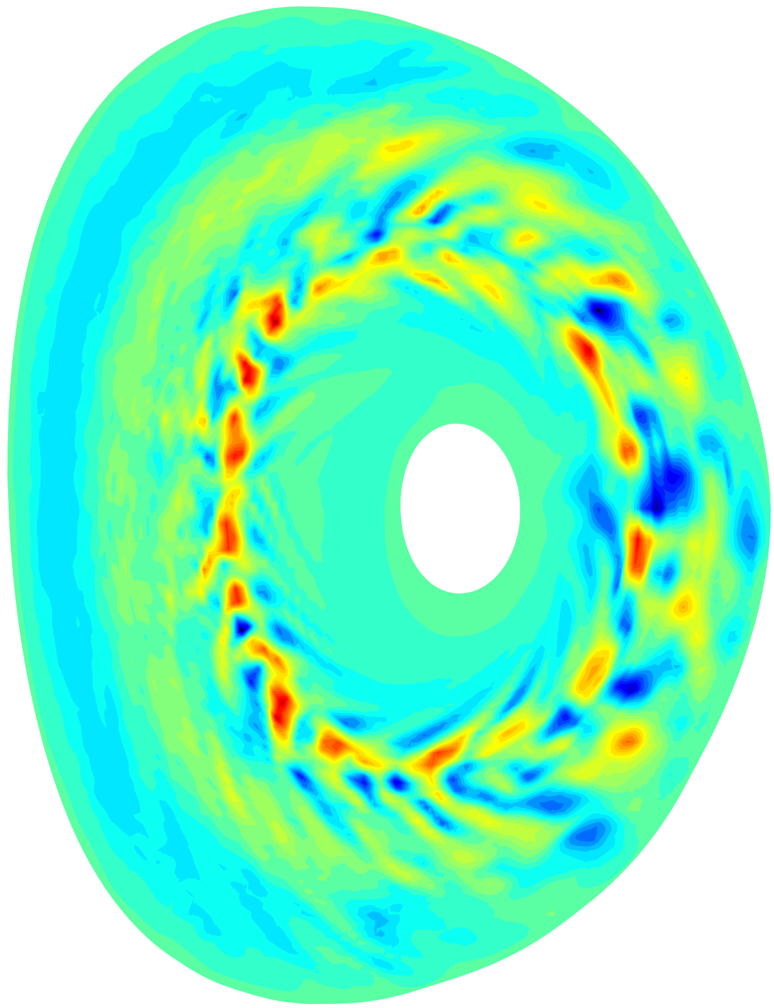

Figure 1: Full-torus ('global') `GENE` simulation of a discharge in the TCV tokamak that exhibits an internal transport barrier (ITB). Realistic input data and comprehensive physics are used. The figure displays contours of electrostatic potential (stream function). In this cross section, the ITB corresponds to a fairly narrow ring near the mid minor radius of the torus. Reprinted from Görler et al. (2011, Figure 8). Copyright 2011 by the American Institute of Physics; used with permission *(obtained from author, but not yet from AIP)*.

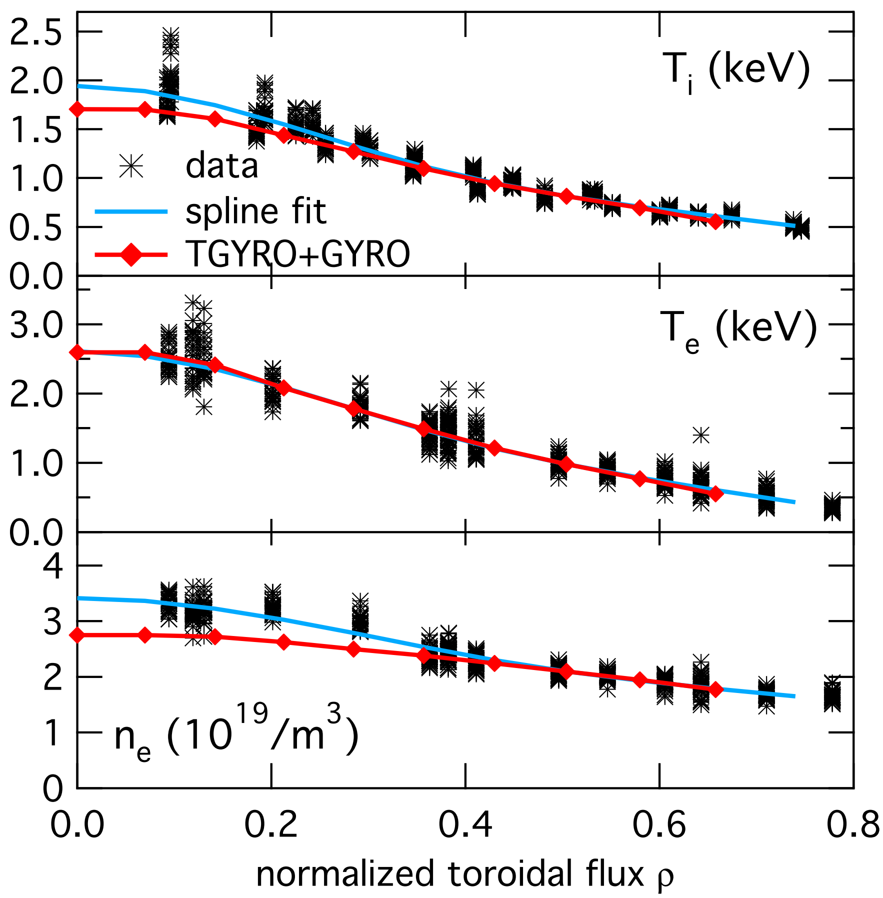

Figure 2: Predictions (red curves) of the `TGYRO` code (`https://fusion.gat.com/theory/Tgyrooverview`) for DIII-D discharge 128913 compared with experimental measurements (discrete data points). The electron temperature is particularly well-reproduced. Fits to the experimental data (blue curves) are also shown. For this case, 10 simulation radii (10 instances of `GYRO`) were used. All three profiles (n_e, T_e, T_i) were evolved by `TGYRO` using nonlinear `GYRO` calculations of the turbulent electron particle flux, electron energy flux, and ion energy flux. Unpublished figure courtesy of J. Candy.

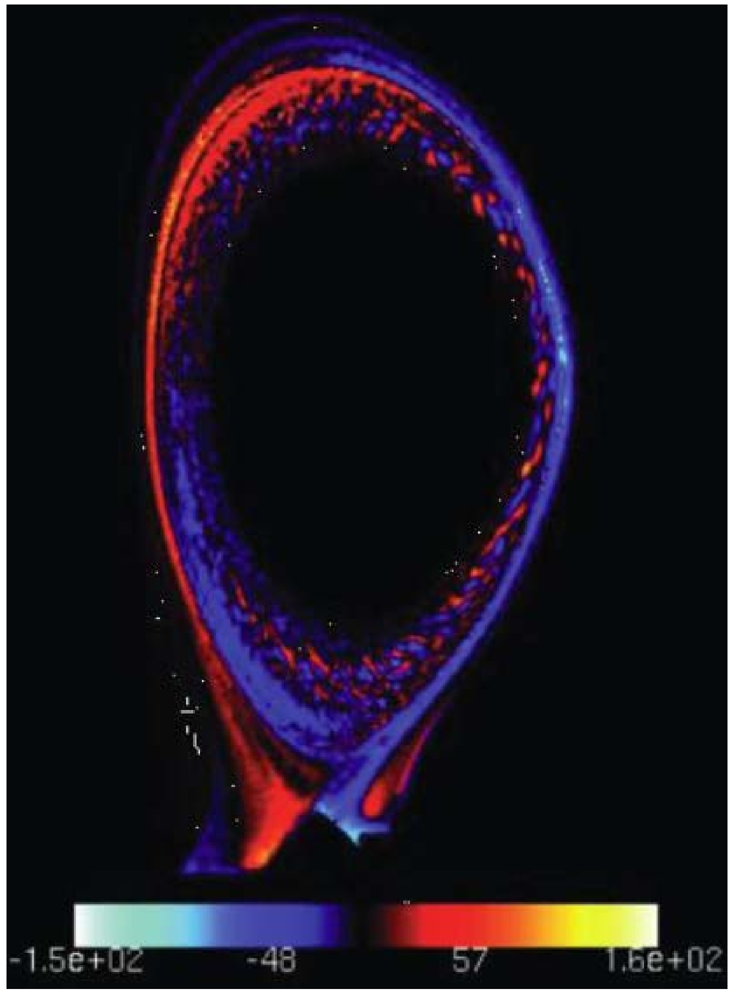

Figure 3: Gyrokinetic simulation of ITG turbulence by the PIC code `XGC1` in an edge pedestal region with the realistic diverted geometry of the DIII-D device. More than 10^9 marker particles were used. Such simulations, still in an early stage of development, are difficult because of the change of magnetic topology associated with the divertor separatrix. Reprinted from Chang et al. (2009, Figure 6). Copyright 2009 by the American Institute of Physics; used with permission.

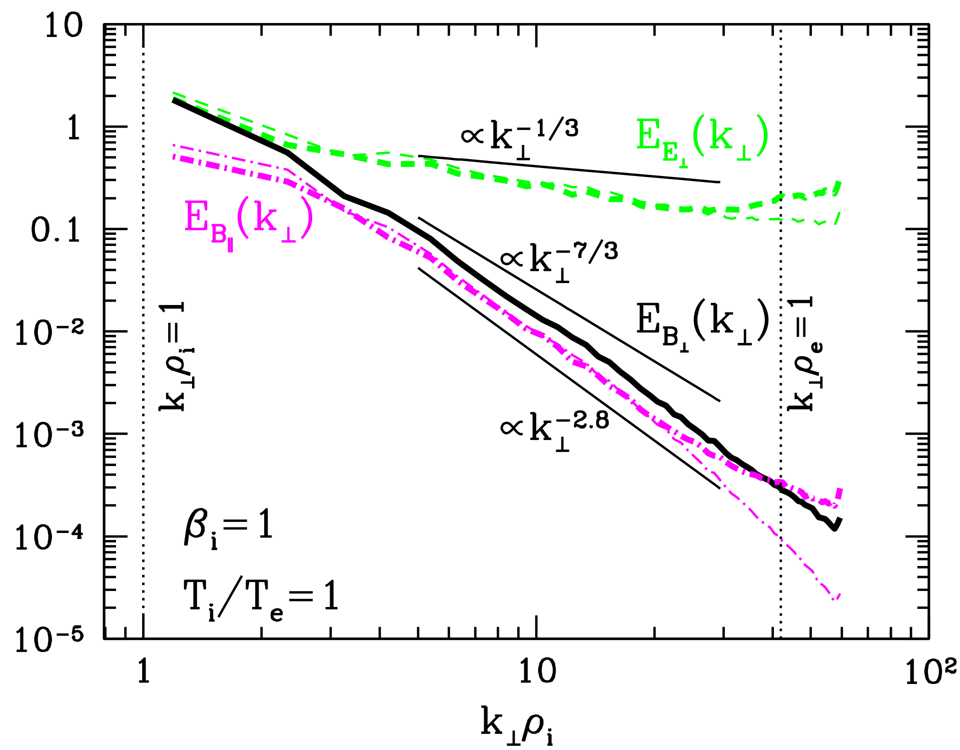

Figure 4: One-dimensional energy spectra of the perpendicular magnetic (solid), electric (dashed), and parallel magnetic (dot-dashed) fields from gyrokinetic simulations (with `AstroGK`) of kinetic Alfvén wave (KAW) turbulence from the ion to electron Larmor radius scales. These spectra demonstrate that KAW turbulence can indeed yield energy spectra reaching the electron scales, as found in recent observations (Sahraoui et al. 2009). Thin lines are the perpendicular electric (dashed) and parallel magnetic energy (dot-dashed) spectra predicted from the perpendicular magnetic energy spectrum using the linear kinetic Alfvén wave eigenfunction, demonstrating that, even in fully developed turbulence, the fluctuations retain the character of linear wave modes. Reprinted from Howes et al. (2011, Figure 1). Copyright 2011 by the American Physical Society; used with permission *(obtained from author, but not yet from APS)*.

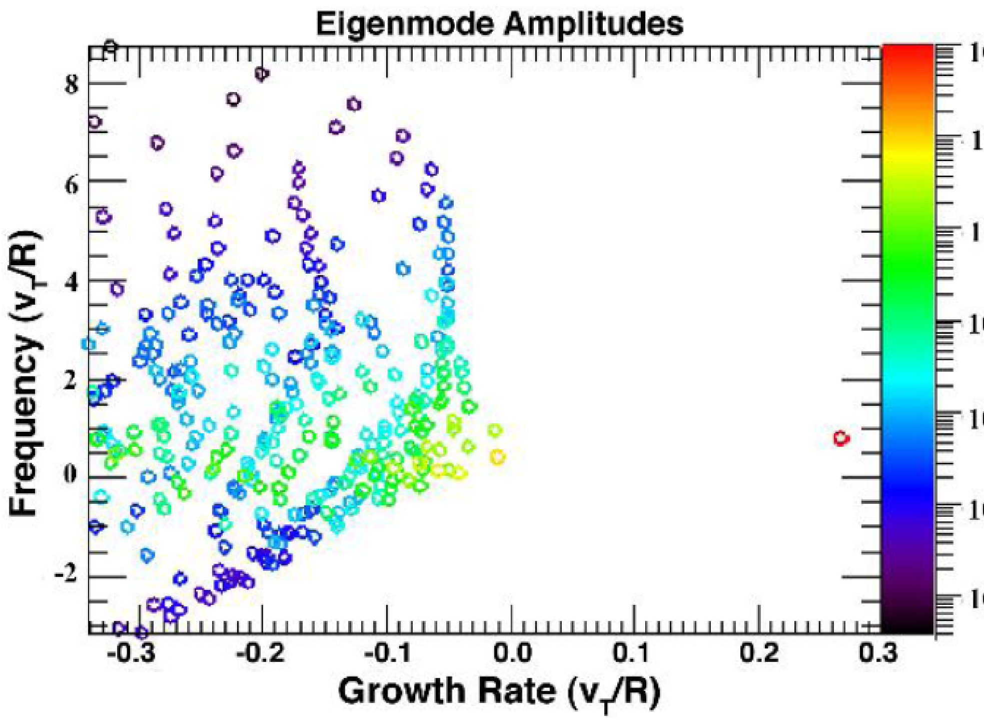

Figure 5: Gyrokinetic eigenmode spectrum for ITG fluctuations, demonstrating the possibility of nonlinear coupling of an unstable ITG mode (red) to damped eigenmodes. Reprinted from Hatch et al. (2011, Figure 2). Copyright 2011 by the American Institute of Physics; used with permission *(obtained from author, but not yet from AIP)*.

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