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Pressure, Chaotic Magnetic Fields and MHD Equilibria

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Analyzes of plasma behavior often begin with a description of the ideal magnetohydrodynamic equilibrium, this being the simplest model capable of approximating macroscopic force balance. Ideal force balance is when the pressure gradient is supported by the Lorentz force, $\nabla p = \mathbf{j} \times \mathbf{B}$. We discuss the implications of allowing for a chaotic magnetic field on the solutions to this equation. We argue that the solutions are pathological and not suitable for numerical calculations. If the pressure and magnetic field are continuous, the only non-trivial solutions have an uncountable infinity of discontinuities in the pressure gradient and current. The problems arise from the arbitrarily small length scales in the structure of the field, and the consequence of ideal force balance that the pressure is constant along the field-lines, $\mathbf{B} \cdot \nabla p = 0$. A simple method to ameliorate the singularities is to include a small but finite perpendicular diffusion. A self-consistent set of equilibrium equations is described and some algorithmic approaches aimed at solving these equations are discussed.

I. INTRODUCTION

A numerical calculation of the equilibrium magnetic field is usually the first step in analyzing plasma behavior. This is a comparatively simple task for a perfectly axisymmetric tokamak (or any system with a continuous symmetry), as the symmetry guarantees that a nested, continuous family of flux surfaces exists, i.e. the magnetic field is integrable. This is a result of the fact that a toroidal magnetic field is analogous to a time-dependent, one degree of freedom Hamiltonian system [1], and by Noether's theorem [2], which states that a Hamiltonian with an ignorable coordinate possess an invariant of the motion. By exploiting axisymmetry, the ideal equilibrium equation, $\nabla p = \mathbf{j} \times \mathbf{B}$, can be reduced to the Grad-Shafranov equation and equilibrium solutions can generally be found numerically.

Perturbations to an axisymmetric system, either from internal plasma motions or coil alignment errors, lead to the formation of magnetic islands, chaotic field-lines and the destruction of flux surfaces. However, from the Kolmogorov-Arnold-Moser (KAM) theorem [3, 4, 5, 6, 7, 8, 9] we know that (under certain conditions) for a Hamiltonian system slightly perturbed from an integrable case, the strongly irrational flux surfaces are likely to survive. We can expect that a realistic tokamak, therefore, will possess a finite-measure of KAM surfaces in addition to the islands and chaotic volumes. This is fortunate, as it is primarily the existence of flux surfaces that results in plasma confinement: if a small perturbation to an integrable system resulted in the immediate destruction of *all* flux surfaces, then one could not expect a realistic tokamak to provide confinement.

Indeed, under some conditions, applied resonant magnetic perturbations (RMP) can *advantageously* be used to suppress edge-localized instabilities, the so-called ELMs [10]. It is plausible to expect that such perturbations will result in the formation of magnetic islands at the rational surfaces, and the overlap of these islands will cause chaotic fields, particularly near the plasma edge. Some understanding of the impact of applied magnetic perturbations may be gleaned, at least in the low pressure case, by superimposing the equilibrium and error fields. The degree of magnetic chaos can then be determined by field line tracing. Such an approach however cannot account for the self-consistent plasma response. To what extent the field becomes chaotic (or whether ideal plasma flows will respond by shielding out the error fields [11, 12]) remains unclear. The importance of computing non-axisymmetric equilibria with chaotic fields is emphasized by noting that it is likely that ITER will employ RMP methods to suppress ELMs.

Stellarators are intrinsically non-axisymmetric, and thus generally possess non-integrable fields. Stellarators are designed to have "good-flux-surfaces" as much as possible [13, 14, 15, 16], but despite one's best efforts, without a continuous symmetry, perfectly integrable fields cannot be achieved. Also, computational evidence suggests [17] that as the plasma pressure increases, stellarator fields become increasingly chaotic. To understand the impact magnetic islands and chaotic fields have on plasma confinement, for both realistic tokamaks and stellarators, a computational algorithm that solves for the plasma equilibrium in the presence of islands and chaotic fields, and a significant volume of robust KAM surfaces, is required.

A given magnetic field may be a continuous, smooth function of space, so that $\mathbf{B}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{B}(\mathbf{x}) + \nabla\mathbf{B} \cdot \delta\mathbf{x} + \mathcal{O}(\delta\mathbf{x}^2)$, but it also may be "chaotic". The term chaotic is really a description of the magnetic field-lines, i.e. the phase space of the magnetic field. The behavior of the field-lines of a chaotic field depends sensitively on position — not just in the sense that nearby trajectories may separate exponentially at a rate given by the Lyapunov exponent, the so-called butterfly-effect, but also in the sense that irregular, 'chaotic' trajectories lie arbitrarily close to regular trajectories and invariant flux surfaces.

A chaotic magnetic field has a *fractal* phase space structure. The fractal structure arises when an integrable field is generally perturbed, as the rational flux surfaces and irrational flux surfaces break apart quite differently. Quoting Grad [18], "What is pathological is the question that is asked, viz., what is the position of a magnetic field-line after *infinitely* many circuits?". Some field-lines trace out structures which are infinitely complex, such as the unstable manifold and the irregular trajectories which seem to come arbitrarily close to every point in a fractal volume. Interspersed between these irregular field-lines are periodic orbits; arbitrarily small, high-order island chains; and irrational field-lines, which may or may not trace out smooth flux surfaces.

Ideal force balance has the consequence that $\mathbf{B}\cdot\nabla p = 0$, so that the pressure is constant along the infinite length of every field-line. The structure of the pressure is exactly tied to the structure of the magnetic field. This paper shall argue that for a chaotic magnetic field, a continuous, non-trivial pressure that satisfies $\mathbf{B}\cdot\nabla p = 0$ must also be fractal.

This paper raises various objections to computational algorithms that seek solutions to ideal force balance, $\nabla p = \mathbf{j}\times\mathbf{B}$, with continuous pressure and chaotic fields. In Sec. II, we review the derivation of ideal force balance from a minimization principle, but discard this as a practical numerical approach for treating chaotic fields as ideal variations do not allow the topology of the field to change. In Sec. III, the solubility conditions on magnetic differential equations are reviewed and applied to chaotic fields. In Sec. IV, the fractal structure of the phase-space of chaotic fields is reviewed. As the structure of the pressure is tied to the structure of the field we conclude, in Sec. V, that a non-trivial, continuous pressure has an uncountable infinity of discontinuities in the pressure gradient, and so therefore must the current. Thus, $\nabla p = \mathbf{j}\times\mathbf{B}$ cannot serve as a coherent mathematical foundation for a computational algorithm. The problems caused by the pathological structure of the solution are not easy to remedy by ad-hoc adjustments to an iterative algorithm and lead to convergence problems, as discussed briefly in Sec. VI, Finally, in Sec. VII, we suggest that it is preferable instead to seek solutions to a well-posed *non-ideal* equilibrium model, and in Sec. VIII we discuss various algorithmic approaches aimed at solving for such an equilibrium.

II. ENERGY MINIMIZATION

Equilibria of conservative dynamical systems are stationary points of an energy functional. The plasma potential energy is [19]

$$W = \int_{\mathcal{V}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dV, \tag{1}$$

where $\mathbf{B}$ is the magnetic field and the pressure, p, is a scalar function of position. Ideal equilibria are states that extremize this functional with respect to ideal variations in the pressure and fields. If we use Faraday's law and the ideal Ohm's law, the plasma displacement, $\delta\boldsymbol{\xi}$, is related to variations in the magnetic field via $\delta\mathbf{B} = \nabla\times(\delta\boldsymbol{\xi}\times\mathbf{B})$. Such variations preserve the topology of the field [20, 21]. The first order variation in W is

$$\delta W = \int_{\mathcal{V}} (\nabla p - \mathbf{j}\times\mathbf{B})\cdot\delta\boldsymbol{\xi}\; dV, \tag{2}$$

where $\mathbf{j} = \nabla\times\mathbf{B}$ is the plasma current and we have used the boundary condition that the magnetic field is tangential to a fixed plasma boundary, $\mathbf{B}\cdot d\mathbf{S}|_{\delta\mathcal{V}} = 0$. Plasma states that extremize the energy functional, subject to arbitrary ideal variations in the plasma position, must satisfy the Euler-Lagrange equation, $\nabla p = \mathbf{j}\times\mathbf{B}$.

An arbitrary magnetic field may be written

$$\mathbf{B} = \nabla\times(\psi\nabla\theta - \chi\nabla\zeta), \tag{3}$$

where (ψ,θ,ζ) is some coordinate system and $\chi(\psi,\theta,\zeta)$ is the field-line Hamiltonian. This representation follows from writing $\mathbf{B} = \nabla\times\mathbf{A}$, using gauge freedom to write $\mathbf{A} = A_\theta\nabla\theta + A_\zeta\nabla\zeta$, and then identifying $\psi = A_\theta$ and $\chi = -A_\zeta$. The equations defining a field line are given by Hamilton's equations: $\dot\theta = \partial_\psi\chi$ and $\dot\psi = -\partial_\theta\chi$, where θ,ψ are analogous to the position and canonical momentum, and the 'dot' denotes derivative with respect to the 'time' coordinate, ζ.

If χ depends only on ψ, $\chi = \chi_0(\psi)$, then ψ is invariant along the field line. The field is called integrable, and θ, ζ are straight field-line coordinates. The flux surfaces coincide with isosurfaces of ψ, and the angles θ and ζ describe how the field-lines wrap around the flux surfaces. The rotational-transform, $\iota\!\!-$, (or transform for short) is generally defined as the average rate of increase of θ with respect to ζ along a field-line, $\iota\!\!- \equiv \lim_{\Delta\zeta\to\infty}\Delta\theta/\Delta\zeta$. This limit may not exist on irregular field-lines, but for the integrable case we simply have $\iota\!\!- = \partial_\psi\chi$.

The magnetic field is completely determined by χ and the coordinate functions ψ, θ and ζ. The latter may be specified inversely: $R = R(\psi,\theta,\zeta)$, $\phi = -\zeta$ and $Z = Z(\psi,\theta,\zeta)$, where (R,ϕ,Z) are the standard cylindrical coordinates (and some arbitrariness has been removed by the restricted choice of toroidal angle). We may vary the shape of the magnetic field, by varying the coordinate transformation, and preserve the topology of the field. In such a manner we may minimize the plasma energy subject to the constraint of fixed topology. This is the approach adopted by the VMEC code [22, 23, 24] (with the exception that VMEC allows the poloidal angle to vary in order to condense the Fourier spectrum of the coordinate transformation [25, 26]).

Chaotic fields may also be written in the form described by Eq. (3), but the field-line Hamiltonian must now be allowed to depend on the angles, $\chi = \chi_0(\psi) + \sum_{mn}\chi_{mn}(\psi)\exp(im\theta - in\zeta)$. The most enigmatic characteristic of a chaotic magnetic field is its topology but, with this general form for the Hamiltonian and arbitrary χ_{mn}, the topology cannot be simply determined. If one were to proceed via a minimization algorithm, then, quoting Kruskal & Kulsrud [19], “we must choose the initial magnetic field to have precisely those topological properties possessed by equilibria of interest”. It seems impossible that we could *a priori* know the *chaotic* structure of the solution that we are searching for, and a computational algorithm must let the topology of the field change. Without a constraint on the topology one cannot derive Eq. (2) from Eq. (1). Except for the trivial solution, $\mathbf{B} = 0$ and $p = 0$ [19], ideal force balance with chaotic fields cannot be solved using a minimization algorithm.

Rather than understanding $\nabla p = \mathbf{j} \times \mathbf{B}$ to be an Euler-Lagrange equation arising from an energy minimization principle, this equation may be understood as statement that an equilibrium is obtained when the pressure gradient is supported by the Lorentz force. Presumably, a numerical solution could be found iteratively. The HINT code [17, 27], and its successor the HINT2 code [28], and the PIES [29, 30, 31] code, seek solutions to $\nabla p = \mathbf{j} \times \mathbf{B}$ without constraints on the topology of the field. The HINT algorithm is based on a relaxation approach [32], whereas the PIES algorithm is based on an iterative scheme [33] using magnetic coordinates [34].

III. MAGNETIC DIFFERENTIAL EQUATIONS

Many of the mathematical problems of ideal MHD can be traced back to equations of the form $\mathbf{B} \cdot \nabla r = s$, which are called magnetic differential equations [19]. If we integrate this equation along a magnetic field-line that, after some distance, returns to the starting point (i.e. a periodic orbit), then for r to be single valued, s must satisfy the solubility condition $\oint s\, dl/|B| = 0$ [35], where $\int dl/|B|$ is the integral along a field-line and is the inverse operator to $\mathbf{B} \cdot \nabla$. Any numerical method for solving this equation will fail unless the solubility conditions on s are satisfied.

A defining property of irregular field-lines is that they come arbitrarily close to any point in a given (fractal) volume, including the starting point. Let us choose a point, $\mathbf{x}_0$, in an irregular region, and measure distance along a field-line with l. Let $l_i, i = 1, 2, \ldots$ label the infinitely many, seemingly random, distances along the field-line at which the field-line returns to within δ of the initial point, $|\mathbf{x}(l_i) - \mathbf{x}_0| < \delta$. For r to be continuous then, for arbitrarily small ϵ, there must exist a δ such that

$$\int^{l_i} s\, dl/B < \epsilon. \tag{4}$$

For a solution to this equation to exist, this infinite set of solubility conditions must be satisfied by each of the infinitely many irregular field-lines present in any chaotic volume.

Ideal force balance has the direct consequence that the pressure is constant along a field-line, $\mathbf{B} \cdot \nabla p = 0$. The integrability condition in this case is trivially satisfied. Another magnetic differential equations arises for the parallel current. By writing the current as $\mathbf{j} = \sigma \mathbf{B} + \mathbf{j}_\perp$ and insisting that $\nabla \cdot \mathbf{j} = 0$, the parallel current must satisfy $\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$. An expression for the perpendicular current may be obtained from force balance, $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$. For an arbitrary pressure gradient, the solubility condition on $\nabla \cdot \mathbf{j}_\perp$ may not be satisfied. The following discussion concentrates primarily on the properties of the solutions to these two magnetic differential equations when the field is chaotic. Unless stated otherwise, we will assume that the pressure is a continuous function of space.

Consider first an integrable field, with continuously nested rational and irrational surfaces that fill space. Each irrational field-line comes arbitrarily close to every point on its flux surface (i.e. an irrational flux surface is the closure of an irrational field-line). To satisfy $\mathbf{B} \cdot \nabla p = 0$, with p being continuous, we must require that the pressure is constant on each irrational surface. A rational flux surface is a family of distinct periodic orbits. As it stands, the equation $\mathbf{B} \cdot \nabla p = 0$ allows for each of these periodic orbits to maintain a distinct value of the pressure; however, each rational surface may be approximated arbitrarily closely by irrational surfaces. For continuity, the pressure must be constant on each rational surface. For integrable fields, the pressure is an arbitrary surface function, $p = p(s)$, were s labels flux surfaces. Indeed, this arbitrary function (and a function describing the current or transform profiles, and the shape of the plasma boundary for example) is required as a boundary condition to the differential equations described by $\nabla p = \mathbf{j} \times \mathbf{B}$, which reduces to the Grad-Shafranov equation for axisymmetric systems.

The solution for the parallel current, even for integrable systems, is not straightforward. In straight-field-line coordinates, the $\mathbf{B} \cdot \nabla$ operator becomes $\sqrt{g}^{-1}(\iota\!\!\!-\partial_\theta + \partial_\zeta)$, and on Fourier decomposing the parallel current, $\sigma = \sum_{mn} \sigma_{mn} \exp(im\theta - in\zeta)$ we obtain

$$\sigma_{mn} = \frac{i\left(\sqrt{g}\nabla \cdot \mathbf{j}_\perp\right)_{mn}}{\kappa_{mn}} + \delta(m\iota\!\!\!- - n), \tag{5}$$

where $\kappa_{mn} = m\iota\!\!\!- - n$. For integrable fields the Dirac delta-function current, $\delta(m\iota\!\!\!- - n)$, is generally required to provide a field that cancels out resonant error fields at the rational surface that would otherwise result in the creation of magnetic islands [36, 37], and has been invoked to explain island-healing phenomena [38]. The Pfirsch-Schlüter current, $i\left(\sqrt{g}\nabla \cdot \mathbf{j}_\perp\right)_{mn}/\kappa_{mn}$, is singular where $\iota\!\!\!- = n/m$. The singularity may be removed by locally flattening the pressure profile at the rational surfaces. Flattening the pressure at the rational surfaces is exactly what the introduction of chaotic fields entails. However, as we describe below, the introduction of chaotic fields introduces more problems than what might naively be anticipated.

One approach [39] for treating the effect of chaotic fields is to exploit an analogy between the magnetic differential equation and the Liouville equation for magnetic field lines [40]. Expanding upon an idea presented by Reiman et al. [41], Krommes & Reiman [39] suggest that by using methods of statistical averaging, that the chaotic field lines (at the macroscopic level) can be described by a diffusion equation, which effectively removes the Pfirsch-Schlüter singularity by introducing a resonance broadening term:

$$\sigma_{mn} = \frac{i\kappa_{mn}\left(\sqrt{g}\nabla\cdot\mathbf{j}_{\perp}\right)_{mn}}{\kappa_{mn}^2+\eta_{mn}^2}, \tag{6}$$

where η_{mn} is a smoothing parameter related to the magnetic diffusion coefficient.

IV. THE FRACTAL STRUCTURE OF CHAOS

Perturbations to an axisymmetric system, or intrinsic three-dimensional effects, destroy flux surfaces and lead to the formation of magnetic islands (and thus chaotic field-lines) if the topology of the field is not constrained. From the Kolmogorov-Arnold-Moser (KAM) theorem [3, 4, 5, 6, 7, 8, 9] we know that for a magnetic field slightly perturbed from an integrable case, only tori with sufficiently irrational transform will survive, where $\iota\!\!\!-$ is sufficiently irrational if it is poorly approximated by rationals and satisfies a Diophantine condition: there exists an $r > 0$ and $k \geq 2$ such that, for all integers p and q, $|\iota\!\!\!- - p/q| > r/q^k$. By the same argument as above, the pressure must be constant on these irrational, *KAM* surfaces.

For a generally chaotic field, there is no region of space foliated with flux surfaces! Quoting Greene [42], "there is a stochastic region in the immediate vicinity of every chain of periodic orbits". For an arbitrarily perturbed field, magnetic islands and irregular field-lines will emerge at the infinitely many rational surfaces that exist between any pair of KAM surfaces. These chaotic volumes are not covered by a single magnetic field-line, but rather are filled by infinitely many (i) unstable manifolds [43] and irregular field-lines that come arbitrarily close to any point within a fractal volume, (ii) stable and unstable periodic orbits [9, 44], and (iii) cantori [45, 46, 47, 48]. Embedded in these ergodic regions, there may exist an infinite "honeycomb" of local regions of stability, namely the elliptic surfaces about the stable periodic orbits [49] (and around these islands there may exist secondary islands with their own elliptic surfaces and resonances *ad infinitum*).

The equation $\mathbf{B}\cdot\nabla p = 0$ allows the pressure on each field-line to be distinct. However, as the irregular field-lines within the ergodic sea may come arbitrarily close to each other, we must conclude that unless the pressure on each of these field-lines is identical, infinite pressure gradients will be created. The only continuous solution is that the pressure is constant between the KAM surfaces. (This is convenient, as the singularity in Eq. (5) is removed by setting pressure gradient to zero across the rational regions.)

The KAM surfaces separate chaotic volumes, but not all KAM surfaces are created equal. Irrational surfaces that are furthest from low order islands are typically the least deformed by low-order resonant perturbations, and consequently are the most robust, in that they survive to comparatively higher levels of chaos [42]. Furthermore, in these particularly irrational regions the phase space density of KAM surfaces is highest.

The KAM surfaces are fragile, in the sense that as the degree of chaos increases the KAM surfaces become increasingly deformed. A surface is called critical when it is continuous but no longer smooth, and an infinitesimally small increase in the chaos will cause the closure of an irrational field-line to disintegrate into invariant, irrational Cantor sets, called cantori [45, 46, 47, 48]. Though the cantori are sets of measure zero, cantori have an important impact as they can form extremely effective partial barriers to field-line transport [50], and field-lines may spend an arbitrarily long time near the cantori (i.e. cantori are 'sticky' [51, 52, 53]). The existence of KAM surfaces, near-critical cantori and magnetic islands violates the assumptions underpinning random walk treatments of field-lines.

As one approaches the irregular region near an unstable periodic orbit, the phase space density of KAM surfaces becomes sparser. The KAM surface that lies adjacent to an irregular region associated with an island chain is called a boundary surface [54], and these surfaces are critical. As a given boundary surface is destroyed by an increase in the degree of chaos, the next closest KAM surface may be a finite distance from the original, so that the location of the closest KAM surface to a given island chain is not a continuous function of perturbation [55].

As the perturbation and chaos increases, the topology of the field breaks up in a rather unpredictable, fractal manner: not-so-irrational KAM surfaces are destroyed leaving behind near-critical cantori; stable periodic orbits bifurcate and become unstable, and additional periodic orbits are born; and field-line transport through gaps in the super-critical cantori increases. Randomly following field-lines in a chaotic region is almost guaranteed to give unreliable results, unless perhaps a large number of field-lines is followed for an *extremely* long distance: for example, in a numerical experiment by Meiss [56] it was shown that about 10^{10} iterates are required for an irregular trajectory to uniformly cover the chaotic region. Following field-lines for this distance is not practical in an iterative scheme.

A common approach is to approximate the effect of chaotic trajectories by assuming a magnetic field-line diffusion [57, 58]. Such an assumption may be justified for *strongly* chaotic fields; however, given that realistic tokamaks and stellarators will likely possess a finite measure of invariant surfaces, one may ask if such an assumption is always reliable. Field-lines that lie on KAM surfaces obviously do not diffuse radially, and neither do the periodic orbits, the cantori, nor the field-lines within the local regions of stability. As the perturbation increases, the stable periodic orbits become unstable and the local regions of stability ultimately vanish, but even after the destruction of the KAM surfaces the existence of cantori has a profound impact on magnetic field line transport. The cantori also restrict anisotropic transport in chaotic fields [59], where cross field transport is modeled by a small perpendicular diffusion; so much so that the temperature profile across a chaotic field is effectively solved by transforming to "chaotic-coordinates", in which the coordinate surfaces are adapted to the cantori and periodic orbits [60].

The approximation that magnetic field-lines diffuse is only reliable in the strongly chaotic case (so called hyperbolic chaos) when all the local regions of stability are destroyed, and all the KAM surfaces are well and truly destroyed. A similar conclusion was reached by Rosenbluth *et al.* [40] who showed that "if resonances do not overlap, then flux surfaces are destroyed in a local region; when resonances overlap strongly, a Brownian motion of field-lines occurs". A robust computational algorithm should be capable of treating completely integrable fields, nearly-integrable fields (in which a few small islands may be present), near-critical fields (in which most irrational surfaces are destroyed but some KAM surfaces survive), as well as strongly chaotic fields, without assuming a Brownian motion of field lines (and preferably without the requirement of inverting singular operators associated with magnetic differential equations).

V. A NON-TRIVIAL PRESSURE

Let us attempt to construct a non-trivial, continuous pressure that is consistent with the fractal structure of the chaotic field. We may imagine a radial coordinate, s, with level surfaces that coincide with the KAM surfaces. For simplicity of discussion, we restrict attention to systems with a monotonic transform profile so that we may label KAM surfaces by their transform. For a given magnetic field, let $\mathcal{S}$ be the subset of the real numbers, $\mathcal{S} \subset \mathbb{R}$, for which $s \in \mathcal{S}$ if and only if a KAM surface with transform $\iota\!\!\!- = s$ exists. On each KAM surface, we may impose a pressure gradient, and across the islands and chaotic volumes we require the pressure gradient to be zero. The pressure gradient, $p'(s)$, may be written $p'(s) = \mathcal{I}_{\mathcal{S}}(s)P(s)$, where $\mathcal{I}_{\mathcal{S}}(s) = 1$ if $s \in \mathcal{S}$ and zero otherwise, and $P(s)$ is some arbitrary function which, for continuity of the pressure, we must assume is bounded. Note that because a small island chain (with generally a chaotic separatrix) will form where the transform is rational, $\mathcal{I}_{\mathcal{S}}(s)$ is zero for a small region about rational $s = p/q$.

Immediately we are in trouble. For an arbitrary chaotic magnetic field, there is no method by which the set $\mathcal{S}$ may be determined. An essential characteristic of $\mathcal{S}$ is that this set has finite measure, by which we may understand that this set has an uncountable infinity of elements. Numerical techniques, such as Greene's residue criterion [42, 61], may be applied to determine if a single KAM surface with given irrational transform exists, and one could imagine an algorithm that successively searched for, and constructed [62], irrational surfaces *ad infinitum*.

To identify and construct a KAM surface that is arbitrarily close to destruction requires an arbitrarily large computational effort, as near-critical KAM surfaces and slightly super-critical cantori are difficult to distinguish. Furthermore, one could determine at most only a countable infinity of KAM surfaces, which remains a set of measure zero. If $\mathcal{S}$ is approximated by a set of measure zero, which presumably it must be if one employs a discrete numerical grid, then integrating $p'(s)$ to obtain $p(s)$ can only result in the trivial function, $p(s) = const.$ To our knowledge, determination of the set $\mathcal{S}$ for a given chaotic system remains an outstanding problem (though there is some interesting work comparing the critical function to the Brjuno function [63]).

There is no bounded function $P(s)$ that gives a non-trivial, continuous pressure gradient. For any rational p/q, we have $\mathcal{I}_{\mathcal{S}}(p/q) = 0$, and so $p'(p/q) = 0$. To have a non-trivial pressure, the function $p'(s)$ must be non-zero on a set with finite measure. Consider some irrational $\iota\!\!\!- \in \mathcal{S}$, where $P(\iota\!\!\!-)$ and therefore $p'(\iota\!\!\!-)$ are non-zero. To show that p' is not continuous at $\iota\!\!\!-$, we may take a sequence of rationals, p_n/q_n, that converges to the irrational: $p_n/q_n \rightarrow \iota\!\!\!-$ as $n \rightarrow \infty$. A suitable sequence is provided by the convergents derived from the continued fraction representation [64]. We have $p'(p_n/q_n) = 0$ for all n, and so $p'(p_n/q_n) \rightarrow 0$ as $n \rightarrow \infty$, but we have assumed that $p'(\iota\!\!\!-)$ is non-zero. Thus, to give a non-trivial pressure, the pressure gradient, ∇p, must have an uncountable infinity of discontinuities. So, therefore, does the perpendicular current, $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$.

In irregular regions we have concluded that $p = const.$ is the only continuous solution to $\mathbf{B} \cdot \nabla p = 0$. We thus have $\mathbf{j}_\perp = 0$, which in turn gives $\mathbf{B} \cdot \nabla \sigma = 0$ and so the parallel current must also be constant in the irregular regions. In the irregular volumes, the field must be a linear, force-free field, i.e. a Beltrami field $\mathbf{j} = \sigma \mathbf{B}$. To obtain a non-trivial, continuous pressure, we must enforce a non-zero, finite pressure gradient on an infinite collection of KAM surfaces. However, ∇p is not continuous at the irrational surfaces, so neither is $\mathbf{j}_\perp$, and $\nabla \cdot \mathbf{j}_\perp$ is not defined. The operator $\mathbf{B} \cdot \nabla$ is not singular on the irrational surfaces, but the solution to $\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$ is not defined.

Let us assume that a non-trivial, non-continuous pressure gradient has been presented and that this function has been integrated to provide the pressure, $p(s)$. Having a gradient with an uncountable infinity of discontinuities, we may conclude that the scale length, L, of the pressure is zero (except where the pressure is constant). As any diligent student of numerical methods is aware, a discrete approximation to a system of equations is only reliable when the scale length, h, of the numerical resolution is smaller than the scale length of the solution; that is, to adequately approximate a function we must have $h \leq L$. A standard finite-difference or finite-element approximation to $p(s)$, even with a countably infinite grid, cannot resolve the structure of the solution to the extent that its gradient ∇p will be accurate. Thus, as it stands, nontrivial solutions to $\nabla p = \mathbf{j} \times \mathbf{B}$ are pathological when the fields are chaotic.

Let us imagine that a chaotic equilibrium has been constructed by an iterative algorithm and consider the properties of the solution. The solution to a system of differential equations is defined by the boundary conditions imposed, which are typically provided at the outset. For determining ideal MHD equilibria, the pressure (and either the current or transform) is required to supplement $\nabla p = \mathbf{j} \times \mathbf{B}$. In the integrable case, as flux surfaces foliate space and thus may serve as coordinate surfaces, an arbitrary function $p(s)$ may be specified.

In the case of chaotic fields, however, the structure of the pressure (a boundary condition) is intimately tied to the structure of the magnetic field (the solution to the differential equation), but the structure of the solution magnetic field is not initially known. Just as the topology of the equilibrium magnetic field cannot be known *a priori*, neither can the pressure. We are in the curious position of having to construct a valid boundary condition simultaneously with constructing the solution.

How do we understand the stability of such an equilibrium? There is no energy functional as there is no topological constraint, and there is no well defined boundary condition. The best statement that one could make regarding stability would merely be a property of one's algorithm rather than any physically motivated concept of stability. (Recall that Newton's method will just as easily converge to an unstable solution to system of equations as it will to a stable solution.) Perhaps we can understand stability by turning on the time evolution under the equations of resistive MHD, which does not constrain the topology of the field. For this it would seem that we would need to construct the equilibrium as a resistive steady state, but a resistive steady state does not satisfy $\nabla p = \mathbf{j} \times \mathbf{B}$ for finite resistivity.

VI. A CHAOS FILTER

To eliminate the discontinuities in the pressure gradient we have two choices. We may approximate the set $\mathcal{S}$ with something non-fractal, or we must allow $\mathbf{B} \cdot \nabla p \neq 0$. Approximating $\mathcal{S}$ with a non-fractal set is equivalent to replacing the chaotic magnetic field by a field with finite volumes foliated by flux surfaces, on which a smooth pressure profile may be imposed, which are perhaps separated by a few "non-ignorable" islands and associated regions of chaos, across which the pressure must be constant. We may write, for example, $\bar{\mathbf{B}} = \mathcal{F}[\mathbf{B}]$ where $\mathcal{F}$ is some filtering operation that takes the true chaotic field, $\mathbf{B}$, and returns an approximation, $\bar{\mathbf{B}}$, with the small scale structures removed. Such a field could then be used to define a piecewise smooth pressure profile consistent with $\bar{\mathbf{B}} \cdot \nabla p = 0$, which in turn could be used to solve for the parallel current from $\bar{\mathbf{B}} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$, as the uncountably infinite set of discontinuities caused by the fractal structure of the chaotic field is removed.

We raise the following objections. By filtering out the chaos, we are to some extent abandoning our original goal: to construct an MHD equilibrium allowing for chaotic fields. Recall that it was the introduction of magnetic islands (and the associated pressure flattening) at the rational surfaces that eliminated the singularity in the Pfirsch-Schlüter current. If the islands at the high-order rational surfaces are removed and replaced by nested flux surfaces, then the Pfirsch-Schlüter singularities at these rational surfaces will re-emerge. The equilibrium becomes schizophrenic: we have one magnetic field that satisfies $\nabla p = \mathbf{j} \times \mathbf{B}$ and is generally chaotic, and we have another magnetic field that satisfies $\bar{\mathbf{B}} \cdot \nabla p = 0$. To filter out the small scale structure, one must introduce a length scale below which the fine scale structure of the field can be ignored; however, there is no length scale in $\nabla p = \mathbf{j} \times \mathbf{B}$, so the introduction of a length scale must be justified by some other means.

In order to have a complete mathematical model of MHD equilibria, an equation describing the filtering operation, $\mathcal{F}$, must be provided: to our knowledge, this has not yet been presented, and so the small-scale filter must instead be constructed using an algorithmic approach. Algorithmic approaches are usually best avoided, as different researchers may devise different algorithms that could lead to different results. This is particularly true in this case considering that the fractal structure of the field is difficult to characterize.

Consider a filtering approach based on constructing an arbitrary discrete collection of KAM surfaces (perhaps selected by their transform) and assuming a smooth interpolation. An infinitesimal increase in the chaos may lead to the destruction of one or more of the selected KAM surfaces, but unless the filtering operation is smooth, so that small changes in $\mathbf{B}$ lead to small changes in $\bar{\mathbf{B}}$, we may expect that an iterative algorithm will encounter convergence problems. As flux surfaces become increasingly deformed they presumably can support less pressure, and as flux surfaces are destroyed they can support no pressure, since irregular field-lines may pass through gaps between the remnant irrational set (the cantorus). Again, for stability of an iterative algorithm, one must accommodate the fact that near critical KAM surfaces (e.g. boundary surfaces) cannot be allowed to support a finite pressure, so the filtering algorithm may need to diagnose the structural stability of KAM surfaces (a difficult task in itself) in addition to merely determining the existence of KAM surfaces.

One cannot expect reliable results if one "samples" the structure of phase space on a fixed regular grid: If, for example, we use a piecewise-linear approximation to represent a *smooth* function on a fixed regular grid, with grid size h, then we can expect the associated error to be second order, $\mathcal{O}(h^2)$; but in the case of chaotic fields, any function tied to the topology of the field must be *fractal.*

The above objections are intended to illustrate the problems that will be encountered if one chooses to pursue an *algorithmic* approach to filtering out the small scale structures of the field. Within ideal MHD, there is no justification for ignoring the small scale structure of the field. Certainly, the infinitesimal structures of the field are not important if one considers finite-Larmor radius effects, or if some model of transport across the magnetic field is to be included. However, if one wishes to remove the fractal nature of the solutions by appealing to additional physics, then to provide a complete mathematical model of chaotic equilibrium (rather than presenting an ad-hoc algorithm) then mathematical equations that describe the additional physics must be included.

VII. ALTERNATIVES

The problems for MHD equilibria, as discussed above, arise from the combination of (i) chaotic fields, (ii) continuous pressure and fields; and (iii) ideal force balance. We may avoid these problems if we (i) enforce the constraint that the fields be integrable, even for non-axisymmetric systems, such as in the perturbed equilibrium model [65]; (ii) allow for discontinuous pressure, e.g. the stepped-pressure model [66]; or (iii) approximate an ideal MHD equilibrium by introducing small, non-ideal terms.

In the perturbed equilibrium model one may eliminate the chaotic magnetic field in favor of an integrable field, by exploiting the possibility in ideal MHD that delta-function singular currents may exist at the rational surfaces [65]. The singular currents may be computed [11, 67, 68] so as to exactly cancel the perturbing "error fields" that drive islands and the associated chaos. Expanding the variation in the plasma energy to second order,

$$\delta W = \int_{\mathcal{V}} (\nabla p - \mathbf{j} \times \mathbf{B}) \cdot \delta \boldsymbol{\xi} \, dV + \frac{1}{2} \int_{\mathcal{V}} \mathbf{F} \cdot \delta \boldsymbol{\xi} \, dV, \quad (7)$$

where $\mathbf{F} = \nabla \delta p - \delta \mathbf{j} \times \mathbf{B} - \mathbf{j} \times \delta \mathbf{B}$, we may construct the first-order correction required to bring an initial field (with nested surfaces) that approximates an ideal equilibrium closer to the true equilibrium field (with nested surfaces). The Newton correction is given by $\mathbf{F} \cdot \delta \boldsymbol{\xi} = -(\nabla p - \mathbf{j} \times \mathbf{B})$. The matrix operator, $\mathbf{F}$, is singular at the rationals, and the solution, $\delta \boldsymbol{\xi}$, is generally discontinuous. However, the discontinuities now appear at the rationals, which is at least a countable set. The stability of the equilibrium is given by ideal stability theory.

The stepped-pressure model allows for pressure gradients across chaotic fields by realizing that the KAM surfaces can support a pressure discontinuity, provided that the divergence of the stress tensor vanishes [69]. The plasma volume is partitioned into a set of nested annular regions which are separated by a finite number of KAM surfaces. In each annulus, the plasma energy is given by Eq. (1), and an equilibrium is obtained when the plasma energy is minimized. The plasma energy is minimized subject to the constraint of conserved helicity, $K \equiv \int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B} \, dV$, which is the "most-conserved" invariant for plasmas in which reconnection is allowed [70]. The multi-volume, constrained energy functional, F, for the stepped-pressure model [66] is given

$$F = \sum_i \left(W_i + \frac{\sigma_i}{2} K_i \right), \quad (8)$$

where the σ_i are Lagrange multipliers, and the index i labels the volumes separated by the interfaces. The stepped-pressure model is essentially a multi-volume, Taylor-relaxed equilibrium [71]. (For full details of this approach see Ref. [66]).

The Euler-Lagrange equation derived by minimizing the energy functional, Eq. (8) by allowing variations in the magnetic field is $\nabla \times \mathbf{B}_i = \sigma_i \mathbf{B}_i$, so that between the KAM-interfaces the magnetic field is a force-free, Beltrami field. The Euler-Lagrange equation derived by allowing variations in the geometry of the interfaces is $[[p + B^2/2]] = 0$, where $[[\dots]]$ denotes the jump across the KAM-interfaces, so that the total plasma pressure is continuous. To give a non-trivial pressure profile, only a finite set of interfaces are required. Some steps have been taken toward implementing these ideas in an equilibrium code [72], and the stability of such equilibria has been studied in cylindrical geometry by Hole *et al.* [73, 74, 75]. These equilibria are discontinuous on a finite set, but the number of interfaces, the pressure profile, and the transform profile are selected *a priori*.

The third approach, on which we now concentrate, is to allow small deviations from ideal MHD. The problems discussed in Sec. IV arise because there is no scale length in MHD. The equation $\mathbf{B} \cdot \nabla p = 0$ requires the pressure to be constant along the "infinite-length" of the field-line, despite the fact that the field-line may trace out structures which are vanishingly small. Equivalently, $\mathbf{B} \cdot \nabla p = 0$ implies that the relaxation of the pressure along the magnetic field is infinitely fast.

Clearly, ideal MHD is an oversimplification of plasma dynamics. Collisions and finite-Larmor-radius effects for example, will affect a local smoothing of the pressure, and this is exactly what is required to eliminate the singularities and discontinuities. Any perpendicular transport will naturally introduce a perpendicular scale length, below which the magnetic islands and the fine scale structure of the chaotic magnetic field will be irrelevant. To derive a complete and coherent mathematical model, we must present an equation that approximates this effect.

Motivated by the study of anisotropic heat transport in chaotic fields [59], we consider the case where the parallel transport is characterized by a large but *finite* parallel diffusion coefficient, $\kappa_\parallel$, and is balanced by a small but *non-zero* perpendicular relaxation, characterized by $\kappa_\perp$, so that $\kappa_\parallel >> \kappa_\perp$. We must modify the force balance equation to allow for pressure gradients along the field, and so we include inertial and viscous forces arising from a plasma velocity. To complete the system and provide a complete mathematical model, we must include an equation that constrains the velocity, and so we combine Faraday's law and Ohm's law. The equilibrium is concisely defined by the steady state solution to the following system of equations:

$$\frac{\partial p}{\partial t} = \nabla \cdot \left(\kappa_\parallel \nabla_\parallel p + \kappa_\perp \nabla_\perp p \right) + S, \tag{9}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p - \rho \mathbf{v} \cdot \nabla \mathbf{v} + \mu \nabla \cdot \nabla \mathbf{v}, \tag{10}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B} - \eta \mathbf{j} \right), \tag{11}$$

The derivative along the field-line is $\nabla_\parallel p = \mathbf{b}\, \mathbf{b} \cdot \nabla p$, where $\mathbf{b} = \mathbf{B}/B$, and the perpendicular derivative is $\nabla_\perp p = \nabla p - \nabla_\parallel p$. An inhomogeneous source term for the pressure, S, drives non-trivial solutions. We may set the homogeneous boundary condition that $p = 0$ on some computational boundary and adjust the source so that the computed pressure matches experimental observations. The viscosity, μ, the density, ρ, and the resistivity, η, are (at least from a mathematical perspective) arbitrary constants.

VIII. ALGORITHMS

There is of course nothing radical about approximating an ideal MHD equilibrium by a resistive steady state. We have come to this conclusion by considering the impact of chaotic fields on MHD equilibria, and requiring that the pressure be continuous. The existence of solutions of a similar model of dissipative plasma equilibria has been investigated by Spada & Wobig [76]. A similar system can be derived by taking appropriate limits from the NIMROD [77, 78] and M3D [79] equations. A similar model was suggested by Park *et al.* [32] and forms the basis of the HINT code; however, Park *et al.* replaced Eq. (9) with an artificial sound wave approach to drive the solution towards $\mathbf{B} \cdot \nabla p = 0$.

In the case of the HINT code, the equation $\mathbf{B} \cdot \nabla p = 0$ is enforced iteratively according to

$$p_{new} = \frac{\displaystyle\int_{-L}^{+L} \frac{p_{old}}{B} \mathcal{F} dl}{\displaystyle\int_{-L}^{+L} \frac{1}{B} dl} \tag{12}$$

where L is the integration length along the magnetic field line from an Eulerian grid point, and $\mathcal{F}$ is the weight function:

$$\mathcal{F} = \begin{cases} 1 & \text{for } \; L \le L_c \\ 0 & \text{for } \; L > L_c \end{cases} \tag{13}$$

where L_c is the connection length of the magnetic field line from the starting Eulerian grid point to the boundary. In this method, as is shown in Fig.6 of Ref. [80], the pressure profile develops spikes where the magnetic field is chaotic.

An earlier attempt [80, 81] to resolve this type of numerical result employed a model of parallel transport model, where the form of the weight function was taken as

$$\mathcal{F} = \begin{cases} 1 & \text{for } \; l_{mfp} \le L \le L_c \\ 0 & \text{for } \; \text{others} \end{cases} \tag{14}$$

where l_{mfp} is the mean free path along a magnetic field line. As is shown in Fig.3 of Ref.[81], the pressure profile in the chaotic magnetic field can dramatically change, depending on the length of the field line tracing. This numerical result indicates that the ideal relation, $\mathbf{B} \cdot \nabla p = 0$, is inconsistent with chaotic fields, because Eq. (12) just acts as a nonlinear re-distribution of the initial pressure profile *parallel* to the field, without any regard of the variation of the pressure in the *perpendicular* direction.

An anisotropic diffusion equation for the pressure is preferable. The effect of perpendicular pressure diffusion is a smoothing operation, and magnetic islands (and other structures of the chaotic field) that are smaller than a critical island width, $\Delta w \sim \mathcal{O}(\kappa_\perp/\kappa_\parallel)^{1/4}$, do not affect the structure of the pressure [82]. A major motivation for choosing Eq. (9), Eq. (10) and Eq. (11) as our equilibrium model is that much of the computational architecture has already been implemented in the HINT code. (Work on extending the HINT code to use Eq. (9) has begun, and we hope to present numerical results in a future article.)

The simplest approach to solve for the steady state is to just follow the time-evolution, perhaps using kinetic-energy quenching [32]. However, it may be possible to accelerate convergence by solving Eq. (9) and Eq. (10) directly by setting $\partial_t p = 0$ and $\partial_t \mathbf{v} = 0$. Eq. (9) is a linear equation for the pressure, p, given the magnetic field, $\mathbf{B}$. The large anisotropy $\kappa_\parallel >> \kappa_\perp$ demands that accurate numerical techniques must be applied to ensure that the strong parallel diffusion does not overwhelm the weak perpendicular diffusion, but numerical methods for solving this anisotropic diffusion equation have studied by many authors (see for example Ref.[83] and references therein).

Having solved the pressure for an arbitrary $\mathbf{B}$, a residual force $\mathbf{j} \times \mathbf{B} - \nabla p$ will drive a plasma flow, as described by Eq. (10), which may be interpreted as an equation for the plasma velocity. Writing $\mathbf{v}_{n+1} = \mathbf{v}_n + \delta \mathbf{v}$, a linear equation for the first order correction, $\delta \mathbf{v}$, is obtained which, when embedded in an iterative scheme, will accommodate the non-linear terms. (The steady state solution of Eq. (11) is $\mathbf{v} \times \mathbf{B} - \eta \mathbf{j} = \nabla \Phi$, and so to invert this equation either for $\mathbf{v}$ or $\mathbf{B}$, it seems that the quantity $\nabla \Phi$ must be provided.) To demonstrate convergence with respect to iterations, n, one must confirm that $||\mathbf{B}_{n+1} - \mathbf{B}_n|| / ||\mathbf{B}_n|| < \epsilon$, where ϵ is some desired numerical tolerance and $||\mathbf{f}||$ is some measure of the 'size' of $\mathbf{f}$, e.g. $||\mathbf{f}|| = \int_{\mathcal{V}} |f| dV$.

We have avoided the numerical problem of inverting the operator $\mathbf{B} \cdot \nabla$. Have we satisfied the solubility conditions for the magnetic differential equations? The magnetic differential equation for the pressure has been replaced by Eq. (9). Embedded in this equation is the $\mathbf{B}\cdot\nabla$ operator, but now the right hand side is not independent of the solution; that is, we now have $\mathbf{B}\cdot\nabla p = \kappa_\parallel^{-1} B^2 \int (S - \kappa_\perp \nabla\cdot\nabla p)\, dl/|B|$. Rather than treating the right hand side as an independent, prescribed source, which may or may not solve the solubility conditions, the $\kappa_\parallel^{-1} B^2 \int \kappa_\perp \nabla\cdot\nabla p\, dl/|B|$ term may be thought of as a "source-correction" term that allows the solubility conditions to be satisfied.

The current must still satisfy $\mathbf{B}\cdot\nabla\sigma = -\nabla\cdot\mathbf{j}_\perp$. Rather than first insisting that the perpendicular current satisfy force balance, and then struggle to solve the magnetic differential for the parallel current, we may instead guarantee that the current be divergence free by simply writing $\mathbf{j} = \nabla\times\mathbf{B}$. The parallel current is given by $\sigma = \mathbf{j}\cdot\mathbf{B}/B^2$, and the perpendicular current by $\mathbf{j}_\perp = \mathbf{j} - \sigma\mathbf{B}$. Any error in the force balance Eq. (10) is accommodated by calculating the change in the velocity, which in turn is used to update the magnetic field.

An alternative approach for solving a similar equilibrium model that allows for pressure gradients in chaotic fields was suggested by Reiman et al. [41]. Rather than employing Eq. (9) as the defining equation for the pressure, the pressure is taken as a given, fixed input quantity, on the understanding that this information will be provided by experimental observations. The velocity terms in the perpendicular force balance are assumed to be small compared to the pressure gradient force, so that from Eq. (10) the perpendicular current may be approximated by $\mathbf{j}_\perp = \mathbf{B}\times\nabla p/B^2$. The parallel current is then determined by requiring that $\nabla\cdot\mathbf{j} = 0$ to give a magnetic differential equation for the parallel current, $\mathbf{B}\cdot\nabla\sigma = -\nabla\cdot\mathbf{j}_\perp$. Modeling the effect of chaotic fields by a weak field line diffusion, this magnetic differential equation is statistically averaged, so that the solution is given by Eq. (6). The magnetic field is then given by $\nabla\times\mathbf{B} = \mathbf{j}$.

We raise some concerns about this approach. First, if $\mathbf{j}_\perp = \mathbf{B}\times\nabla p/B^2$, but the pressure is not flattened across the resonances and chaotic regions, there is no guarantee that the solubility conditions for the equation $\mathbf{B}\cdot\nabla\sigma = -\nabla\cdot\mathbf{j}_\perp$ will be satisfied: an arbitrary $\mathbf{j}_\perp = \mathbf{B}\times\nabla p/B^2$ is not consistent with $\nabla\cdot\mathbf{j} = 0$. For the magnetic differential equation for the parallel current to be solved, the extra "source-correction" terms in the perpendicular current arising from the plasma velocity, $\mathbf{B}\times(\rho\mathbf{v}\cdot\nabla\mathbf{v} + \mu\nabla\cdot\nabla\mathbf{v})/B^2$, must be included and be determined so that the solubility conditions are satisfied. Additionally, it is the solution, σ, of this magnetic differential equation that guarantees that $\nabla\cdot(\sigma\mathbf{B} + \mathbf{j}_\perp) = 0$. The statistically-averaged, coarse-grained solution, to the extent that it deviates from the exact solution, would presumably violate this condition. If the divergence of the current is non-zero, it is not clear how one can invert the curl operator to solve for the magnetic field given the current, $\nabla\times\mathbf{B} = \mathbf{j}$

Our second concern with this algorithm is that there is no mechanism by which the structure of the magnetic field influences the structure of the pressure. In strongly magnetized plasmas, the transport parallel to the field overwhelmingly dominates the perpendicular transport: a strongly chaotic field presumably must affect the pressure. If the anisotropic diffusion equation for the pressure is valid, then the pressure gradient must be reduced across the "rational regions" (i.e. islands and chaotic volumes) with width greater than the critical island width (and to the extent that the pressure gradient is reduced, then the additional terms in the perpendicular force balance may become important). The KAM surfaces and cantori have an important effect on both field-line and anisotropic transport [59]. The pressure will deform and adapt to the surviving KAM surfaces and cantori, and the pressure gradient will be comparatively enhanced in these "irrational" regions [60]. We may expect that the enhanced pressure gradients will result in enhanced perpendicular currents, which in turn alter the structure of the field. (Experimental observations of the pressure profile will inherently contain experimental uncertainties, and the effect of these uncertainties is magnified as it is the pressure *gradient* that is required to compute force balance.)

IX. FINAL COMMENTS

There is, of course, additional physics that could be included in the equilibrium model described by Eq. (9), Eq. (10) and Eq. (11). The pressure diffusion coefficients, $\kappa_\parallel$ and $\kappa_\perp$, the viscosity, μ, the density, ρ and the resistivity, η, have been described above as arbitrary constants; these terms should preferably be decided by physical considerations, see for example Ref.[84], or could be chosen to accelerate convergence [32]. (The effect of including these additional terms is to smooth the singularities in the pressure gradient and current and to regularize the linear operators that need to be inverted.)

Nevertheless, the equilibrium equations represent a complete mathematical model that is amenable to numerical computation. In the previous section we described a possible iterative algorithm for computing solutions. However, the equilibrium model itself is independent of the numerical algorithm one may use to obtain a solution, and is similarly independent of any particular numerical discretization: one is free to use finite-differences, finite-elements, or Fourier methods as one wishes; the only discriminating factor being computational speed and accuracy. For any numerical approach, a solution should be obtained that is independent of numerical resolution, which in turn should be achieved when the numerical resolution is sufficient to resolve all structures of the solution.

We have argued that $\nabla p = \mathbf{j}\times\mathbf{B}$, with a continuous pressure, only has solutions with an uncountable infinity of singularities in both the pressure gradient and the current when the field is chaotic. Such solutions are not suited to numerical approximation. By including non-ideal terms we have eliminated the pathological singularities and we have argued that the addition of these terms is required for computational tractability. By including an anisotropic diffusion equation for the pressure, we no longer need to specify the pressure *a priori* as a boundary condition, and the pressure adapts self-consistently to the chaotic structure of the magnetic field. We have a complete mathematical model that can consistently treat pressure gradients in chaotic fields, and in future work we hope to investigate the so-called *soft-beta limit*, where transport is linked to the breaking of magnetic surfaces.

X. ACKNOWLEDGMENTS

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