Princeton Plasma Physics Laboratory

PPPL- 4473

PPPL- 4473

Response to Comment on "On higher-order Corrections to Gyrokinetic Vlasov-Poisson Equations in the Long Wavelength Limit [Phys. Plasmas 16,044506 (2009)]

W. W. Lee, and R. A. Kolesnikov

November, 2009

Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

Response to Comment on "On higher-order corrections to gyrokinetic Vlasov-Poisson equations in the long wavelength limit [Phys. Plasmas 16, 044506 (2009)]"

W. W. Lee, and R. A. Kolesnikov†

Princeton Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543

† *Present Address: Los Alamos National Laboratory, Las Alamos, NM 87544*

We show in this Response that the nonlinear Poisson's equation in our original paper derived from the drift kinetic approach can be verified by using the nonlinear gyrokinetic Poisson's equation of Dubin et al. [Phys. Fluids **26**, 3524 (1983)]. This nonlinear contribution in ϕ^2 is indeed of the order of $k_\perp^4$ in the long wavelength limit and remains finite for zero ion temperature, in contrast to the nonlinear term by Parra and Catto [Plasma Phys. Control. Fusion **50**, 065014 (2008)], which is of the order of $k_\perp^2$ and diverges for $T_i \rightarrow 0$. For comparison, the leading term for the gyrokinetic Poisson's equation in this limit is of the order of $k_\perp^2 \phi$,

This is in response to the Comment on our recent paper [1], in which we questioned the correctness of the nonlinear term in the gyrokinetic Poisson's equation by Parra and Catto [2]. Since these are long wavelength modes with $k_\perp \rho_i \sim o(\epsilon)$ and $k_\perp L \sim o(1)$, we have adopted the drift kinetic approach in the paper, where, ρ_i is the ion gyroradius, L is the scale length of the background inhomogeneity and ϵ is a smallness parameter. Our conclusion was that the higher order perturbation terms, related to the electrostatic potential ϕ^2 in the gyrokinetic Poisson's equation, are of the order of $k_\perp^4 \rho_i^4$ rather than the order of $k_\perp^2 \rho_i^2$, as claimed by Parra and Catto [2]. Since the leading linear ϕ term is of the order of $k_\perp^2 \rho_i^2$, these nonlinear terms should have higher order effects on turbulent and neoclassical transport in tokamaks with $k_\perp L \sim o(1)$. Our original derivation was carried out without requiring the use of a Maxwellian background nor the amplitude ordering of the perturbation. Most of all, this conclusion was obtained without utilizing any of the gyrokinetic equations [3, 4], since the problem at hand was drift kinetic in nature, i.e., $k_\perp \rho_i \ll 1$.

We believe that the focus of the present exchange should be on the correctness of the nonlinear term in Eq. (10) in Ref. [1] rather than on the origin of the nonlinear term of Eq. (55) in Ref. [2], although it is puzzling that the nonlinear term in question remains finite in the limit of $T_i \rightarrow 0$

in our formulation, whereas theirs, in Eq. (55) of Ref. [2] as well as in Eq. (7) of the Comment, become singular in that limit. We will show in this Response that our nonlinear equation, Eq. (10) in Ref. [1], can indeed be obtained from the nonlinear Poisson's equation of Dubin et al. [4], i.e.,

$$\frac{\nabla^2\phi}{4\pi e} = -\int \left\langle \left[F_i + \frac{e}{m_i}(\phi-\bar{\phi})\frac{\partial F_i}{\partial \mu} - \frac{e^2}{m_i^2}\frac{1}{2}\langle(\phi-\bar{\phi})^2\rangle_\varphi \frac{\partial^2 F_i}{\partial \mu^2}\right]\delta(\mathbf{R}-\mathbf{x}+\boldsymbol{\rho})d\mathbf{R}dv_\parallel d\mu \right\rangle_\varphi + n_e, \tag{1}$$

where $F_i(\mathbf{R},\mu,v_\parallel,t)$ is the gyrocenter distribution, $\mathbf{x}=\mathbf{R}+\boldsymbol{\rho}$, $\mu\equiv v_\perp^2/2$, $\mathbf{x}$ represents the particle coordinates, $\mathbf{R}$ denotes the gyrocenter coordinates, $\rho\equiv v_\perp/\Omega_i$ is the ion gyroradius, $v^2=v_\perp^2+v_\parallel^2$, $\mu\equiv v_\perp^2/2$, $\langle\cdots\rangle_\varphi$ is the gyrophase averaging,

$$\bar{\phi}(\mathbf{R}) = \langle\phi(\mathbf{x})\rangle_\varphi = \sum_{\mathbf{k}}\phi(\mathbf{k})J_0(k_\perp v_\perp/\Omega_i)exp(i\mathbf{k}\cdot\mathbf{R}),$$

and n_e is the electron guiding center density. To simplify the calculation, we can assume that $\partial F_i/\partial\mu \approx -F_{Mi}/v_{ti}^2$ and $\partial^2 F_i/\partial\mu^2 \approx F_{Mi}/v_{ti}^4$ for a Maxwellian F_i, where $v_{ti}^2=T_i/m_i$. From

$$(\phi-\bar{\phi}) = \sum_{\mathbf{k}}\phi(\mathbf{k})[1-J_0(k_\perp v_\perp/\Omega_i)\exp(i\mathbf{k}\cdot\boldsymbol{\rho})]\exp(i\mathbf{k}\cdot\mathbf{x}), \tag{2}$$

we obtain

$$\langle(\phi-\bar{\phi})\rangle_\varphi = \sum_{\mathbf{k}}\phi(\mathbf{k})[1-J_0^2(k_\perp v_\perp/\Omega_i)]\exp(i\mathbf{k}\cdot\mathbf{x}) \tag{3}$$

and

$$\begin{aligned}\langle(\phi-\bar{\phi})^2\rangle_\varphi = \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''}&\phi(\mathbf{k}')\phi(\mathbf{k}'')[1-J_0^2(k'_\perp v_\perp/\Omega_i)-J_0^2(k''_\perp v_\perp/\Omega_i)\\ &+J_0(k'_\perp v_\perp/\Omega_i)J_0(k''_\perp v_\perp/\Omega_i)J_0(k_\perp v_\perp/\Omega_i)]\exp(i\mathbf{k}\cdot\mathbf{x}).\end{aligned} \tag{4}$$

For the long wavelength modes of interest with $\mathbf{k}\neq 0$, they become

$$\langle(\phi-\bar{\phi})\rangle_\varphi \approx -\frac{1}{2}\frac{v_\perp^2}{\Omega_i^2}\nabla_\perp^2\phi(\mathbf{x}) \tag{5}$$

and

$$\langle(\phi-\bar{\phi})^2\rangle_\varphi \approx \frac{1}{4}\frac{v_\perp^4}{\Omega_i^4}\nabla_\perp^2\phi(\mathbf{x})\nabla_\perp^2\phi(\mathbf{x}), \tag{6}$$

respectively, where $J_0(k_\perp v_\perp/\Omega_i) = J_0(k'_\perp v_\perp/\Omega_i)J_0(k''_\perp v_\perp/\Omega_i)$ is used. The resulting nonlinear gyrokinetic Poisson equation takes the form of

$$\nabla^2\phi + \frac{\omega_{pi}^2}{\Omega_i^2}\nabla_\perp^2\phi\left[1-\rho_i^2\nabla_\perp^2\frac{e\phi}{T_i}\right] = -4\pi e(n_i-n_e), \tag{7}$$

where $n_i=\int F_i dv_\parallel d\mu$ is the ion (guiding center) density, $\rho_i\equiv v_{ti}/\Omega_i$ is the ion thermal gyroradius, $\omega_{pi}(\equiv\sqrt{4\pi n_0 e^2/m_i})$ is the ion plasma frequency, Ω_i is ion cyctron frequency, and n_0 is the average number density. Thus, the nonlinear term is the same as that of Eq. (10) in Ref. [1] and the conclusion remains the same - it is higher order in $k_\perp \rho_i$ and, therefore, is small, but finite for $T_i \to 0$.

On the other hand, the linearized gyrokinetic Poisson equation, i.e, Eq. (1) without the nonlinear term, and the gyrokinetic equation with linearized fields [3, 4],

$$\frac{\partial F}{\partial t} + \left[\mathbf{v}_\parallel + \frac{c}{B^2}\bar{\mathbf{E}}_\perp \times \mathbf{B}\right] \cdot \frac{\partial F}{\partial \mathbf{R}} + \frac{q}{m}\bar{\mathbf{E}}_\parallel \cdot \frac{\partial F}{\partial \mathbf{v}} = 0, \tag{8}$$

where $\bar{\mathbf{E}} = -\partial\bar{\phi}(\mathbf{R})/\partial\mathbf{R}$, give rise to a system that conserves total energy (kinetic energy and field energy) [4], i.e.,

$$\left\langle \frac{m_i}{2}\int v^2 F_i d\mathbf{v} + \frac{m_e}{2}\int v^2 F_e d\mathbf{v} + \frac{1}{2}\frac{e^2}{m_i}\int \frac{\partial}{\partial\mu}\langle(\phi-\bar{\phi})^2\rangle_\varphi F d\mathbf{v}\right\rangle_{\mathbf{x}} = const.. \tag{9}$$

Substituting the relationship of

$$\langle(\phi-\bar{\phi})^2\rangle_\varphi = \sum_{\mathbf{k}'}\phi(\mathbf{k}')\phi^*(\mathbf{k}')[1 - J_0^2(k'_\perp v_\perp/\Omega_i)] \approx \frac{1}{2}\frac{v_\perp^2}{\Omega_i^2}|\nabla_\perp\phi(\mathbf{x})|^2,$$

from Eq. (4), for the $\mathbf{k} = \mathbf{k}' + \mathbf{k}" = 0$ modes, and carrying out the velocity space integration using a Maxwellian background, we then obtain the usual energy conservation, where the field energy, the last term on the LHS of Eq. (9), takes the form of

$$\frac{n_o T_i}{2}\sum_{\mathbf{k}}(1-\Gamma_0)\left|\frac{e\phi(\mathbf{k})}{T_i}\right|^2 \approx \frac{n_0 T_e}{2}\rho_s^2\left|\nabla_\perp\frac{e\phi}{T_e}\right|^2, \tag{10}$$

with $\Gamma_0[\equiv I_o(b)exp(-b)]$, $b = k_\perp^2\rho_i^2$ and $\rho_s \equiv \sqrt{T_e/T_i}\rho_i$. This is in agreement with the conservation property given by Eq. (28) of Ref. [4], and, for $k_\perp\rho_i^2 \ll 1$, we then recover Eq. (8) in Ref. [1] for the energy conservation. It should be noted here that the term $\langle(\phi-\bar{\phi})^2\rangle_\varphi$ appears in both Eqs. (1) and (9), [similarly, in Eqs. (6) and (7) of the present Comment], but results as $(\nabla_\perp^2\phi)^2$ and $|\nabla_\perp\phi|^2$ in Eqs. (7) and (10), respectively.

In conclusion, the statement by Lee and Kolesnikov in Ref. [1], that nonlinear ϕ^2 terms in the gyrokinetic Poisson's equation are of the order of $k_\perp^4$, based on the drift kinetic approach, is valid, as verified by the new derivation given here using the nonlinear gyrokinetic Poisson's equation of Dubin et al. [4]. Hopefully, this Response would resolve the interesting, yet controversial, issue originally brought up by Parra and Catto [2]. Most importantly, our derivations in Ref. [1] as well as those given here have shown that the commonly used gyrokinetic Poisson's equation [3, 4], by keeping only the linear polarization density term, is valid for the global simulations of $k_\perp\rho_i \sim o(\epsilon)$

and $k_\perp L \sim o(1)$. In the event that the nonlinear ϕ^2 term become important, it should be of the form given by Eq. (10) in Ref. [1] or Eq. (7) of this Response, which is of the order of $k_\perp^4\phi^2$, rather than that given by Eq. (55) in Ref. [2] or Eq. (7) of the Comment, which is of the order of $k_\perp^2\phi^2$. Since the lowest order polarization density term is of the order of $k_\perp^2\phi$ in the original gyrokinetic Poisson's equation, the effects of our nonlinear term [1] on the modes with large ϕ but small $k_\perp$ are significantly different from theirs [2].

One of us (WWL) would like to thank Dr. John Krommes and Dr. Roscoe White at PPPL and Ms. Lu Wang, presently visiting PPPL from Peking University, for useful discussions.

---

[1] W. W. Lee and R. A. Kolesnikov, Phys. Plasmas **16**, 044506 (2009).

[2] F. I. Parra and P. J. Catto, Plasma Phys. Control. Fusion **50**, 065014 (2008).

[3] W. W. Lee, Phys. Fluids **26**, 556 (1983).

[4] D. E. Dubin, J. A. Krommes, C. Oberman and W. W. Lee, Phys. Fluids **26**, 3524 (1983).

The Princeton Plasma Physics Laboratory is operated
by Princeton University under contract
with the U.S. Department of Energy.

Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2245
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov