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Comments on "Guiding Center Plasma Models in Three Dimensions"

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Comments on "Guiding center plasma models in three dimensions" [Phys. Plasmas 15, 092112 (2008)]

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Recent assertions that guiding-center theory breaks down at second order for 3D magnetic fields with nonzero torsion are argued to be incorrect.

Recently Sugiyama (LS) has argued[1] that the asymptotic expansion for guiding-center (GC) motion, which underlies all of modern nonlinear gyrokinetic (GK) theory,[2] "may be undefined even when good magnetic flux surfaces exist," the problem supposedly arising for 3D magnetic fields $\boldsymbol{B}$ with nonzero torsion $\widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\times\widehat{\boldsymbol{b}}$ ($\widehat{\boldsymbol{b}}\doteq\boldsymbol{B}/|\boldsymbol{B}|$). One implication is that for such fields the magnetic moment $\overline{\mu}$ cannot be shown to be adiabatically conserved beyond first order in the expansion in magnetic inhomogeneity ϵ. [The overline distinguishes the conserved quantity from its lowest-order approximation $\mu\doteq\frac{1}{2}v_\perp^2/\omega_c(\boldsymbol{x})$.] Her conclusions present a serious challenge to long-held beliefs about one of the most fundamental analytical formalisms of plasma physics. I argue here that, in a restricted sense, GC theory is well-defined and correct. However, LS's paper remains a valuable reference for the interpretation of some of the unusual terms that arise. [See also the earlier work of Littlejohn (RL).[3]]

In GC theory, the state of a gyrating particle is described with the aid of unit vectors $\widehat{\boldsymbol{e}}_1$ and $\widehat{\boldsymbol{e}}_2$, arbitrary except that the frame field $\mathcal{F}\doteq(\widehat{\boldsymbol{e}}_1,\widehat{\boldsymbol{e}}_2,\widehat{\boldsymbol{b}})$ must form an orthonormal triad. LS asserts that the $\boldsymbol{e}$'s cannot be defined consistently for general 3D $\boldsymbol{B}$ (e.g., torsional or stochastic). In the language of differential geometry,[4,5] she is concerned that $\mathcal{F}$ is in general anholonomic (the unit vectors do not commute). Locally, that implies that certain "rectangular" circuits with side ΔX do not close at $O(\Delta X^2)$. Globally, $\mathcal{F}$ may rotate as it is transported along a field line, not necessarily returning to its original orientation after one circuit around a torus.

That rotation is well known. It manifests itself *via* the appearance at second (relative) order in ϵ of RL's gyrogauge vector $\boldsymbol{R}\doteq(\boldsymbol{\nabla}\widehat{\boldsymbol{e}}_1)\cdot\widehat{\boldsymbol{e}}_2=(\boldsymbol{\nabla}\widehat{\boldsymbol{c}})\cdot\widehat{\boldsymbol{a}}$ ($\widehat{\boldsymbol{a}}$ is in the direction of the gyroradius $\boldsymbol{\rho}$ and $\widehat{\boldsymbol{c}}\doteq\widehat{\boldsymbol{a}}\times\widehat{\boldsymbol{b}}$). LS asserts that for anholonomic $\mathcal{F}$ (in 3D) $\boldsymbol{R}$ cannot be defined because the $\mathcal{F}$'s on neighboring field lines do not fit together consistently. To ensure that they do, she requires that $\mathcal{F}$ be integrable or that the affine connection be flat; that implies that the torsion must vanish, a severe constraint.

RL[3,6] has discussed in detail the interpretation of $\boldsymbol{R}$ and its role in guaranteeing the gyrogauge invariance of the theory. (LS did not cite those papers.) He showed that physical quantities (e.g., $\overline{\mu}$) are independent of the choice of the $\boldsymbol{e}$'s, and that $\boldsymbol{R}$ is the only gyrogauge-dependent quantity that appears in the formalism.

With regard to the invariance of $\overline{\mu}$, a physical rebuttal to the concern about anholonomic $\mathcal{F}$ is as follows. Suppose that spatially local calculations show that $\overline{\mu}$ is adiabatically conserved through some order in a gyrogauge-invariant (GGI) way (as they do). Then it cannot matter that $\mathcal{F}$ rotates because a spiraling particle does not know where it is going; it makes many nearly closed gyrations before transiting the torus once. It does not explore the magnetic geometry globally (conserving $\overline{\mu}$ all the while), then retroactively decide that $\overline{\mu}$ should not have been conserved during the early part of the motion. Of course, one must ultimately face up to all of the difficulties of Hamiltonian chaos[7]; resonances between the gyration and slower degrees of freedom[8] can break the invariance of $\overline{\mu}$, which is why it is only an "adiabatic" invariant and its construction is only asymptotic. These serious and challenging issues are not addressed here.

LS asserts that certain effects are cumulative, thus may break the asymptotic ordering. But the modern perturbation theories are explicitly designed to prevent that by removing secular terms order by order. In the Lagrangian approach,[9] one works with a set of generating fields g_n^σ and gauge functions $S^{(n)}$ that are chosen to remove gyrophase (ζ) dependence from all components of the one-form $\boldsymbol{\gamma}$ (not to eliminate terms containing $\widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\times\widehat{\boldsymbol{b}}$ as LS asserted); then Noether's theorem[10] shows that $\overline{\mu}$ is conserved to the highest order calculated. (Here σ indexes the GC variables and n indicates the order in ϵ.) One might be concerned that this construction would fail if unsolvable equations were encountered (attempts to transform $\boldsymbol{R}$ entirely away fail for precisely that reason) or if the formalism would constrain $\mathcal{F}$ to be holonomic. Suppose, for example, that one had to solve an equation of the form $\widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}S^{(n)}=f^{(n)}$, where $f^{(n)}$ is known, in order to remove ζ dependence from some component of $\boldsymbol{\gamma}^{(n)}$. It could well be that this magnetic differential equation would be unsolvable due to global constraints. But no such equation or constraint arises. I recall the procedure outlined by RL[9] and implemented by him[11] and Brizard.[12,13] A preparatory transformation is made from $\boldsymbol{z}\doteq\{\boldsymbol{x},\boldsymbol{v}\}$ to lowest-order GC variables $\boldsymbol{Z}\doteq\{\boldsymbol{X},\mu,U,\zeta\}$, where $\boldsymbol{X}\doteq\boldsymbol{x}$, $U\doteq\boldsymbol{v}\cdot\widehat{\boldsymbol{b}}$, and $\zeta\doteq\tan^{-1}(-\boldsymbol{v}\cdot\widehat{\boldsymbol{e}}_1/-\boldsymbol{v}\cdot\widehat{\boldsymbol{e}}_2)$. (In Brizard's version of the method, described here, the lowest-order GC position is the particle position; $\boldsymbol{\rho}$ is not subtracted.) Then the transformation[9,10] $\overline{\boldsymbol{\gamma}}=T^{-1}\boldsymbol{\gamma}+\boldsymbol{d}S$ is implemented perturbatively. With $L_n\equiv L_{g_n}$ [cf. Eq. (2)], one has

$$\overline{\boldsymbol{\gamma}}^{(-1)}=\boldsymbol{\gamma}^{(-1)}+\boldsymbol{d}S^{(-1)}, \tag{1a}$$

$$\overline{\boldsymbol{\gamma}}^{(0)}=\boldsymbol{\gamma}^{(0)}+\boldsymbol{d}S^{(0)}-L_1\boldsymbol{\gamma}^{(-1)}, \tag{1b}$$

$$\overline{\boldsymbol{\gamma}}^{(1)} = \boldsymbol{\gamma}^{(1)} + \boldsymbol{d}S^{(1)} - L_1\boldsymbol{\gamma}^{(0)} + (\tfrac{1}{2}L_1^2 - L_2)\boldsymbol{\gamma}^{(-1)}, \quad (1c)$$

$$\overline{\boldsymbol{\gamma}}^{(2)} = \boldsymbol{\gamma}^{(2)} + \boldsymbol{d}S^{(2)} - L_1\boldsymbol{\gamma}^{(1)} + (\tfrac{1}{2}L_1^2 - L_2)\boldsymbol{\gamma}^{(0)} + (-L_3 + L_2L_1 - \tfrac{1}{6}L_1^3)\boldsymbol{\gamma}^{(-1)}, \quad (1d)$$

etc., where $\boldsymbol{\gamma}^{(-1)} \doteq \boldsymbol{A}\cdot d\boldsymbol{x}$ and $\boldsymbol{\gamma}^{(0)} \doteq (U\widehat{\boldsymbol{b}} + v_\perp\widehat{\boldsymbol{c}})\cdot d\boldsymbol{x} - [\frac{1}{2}U^2 + \mu\omega_c(\boldsymbol{x})]dt$. At any order, gyrogauge invariance is ensured[3,6] by choosing $g^\zeta = \boldsymbol{g}^{\boldsymbol{X}}\cdot\boldsymbol{R} + \Delta g^\zeta$, where Δg^ζ is GGI. At zeroth order [Eq. (1b)], $\boldsymbol{g}^{\boldsymbol{X}}_{1,\perp}$ is chosen to eliminate phase dependence from $\overline{\boldsymbol{\gamma}}^{(0)}_{\boldsymbol{x}}$; one finds $\boldsymbol{g}^{\boldsymbol{X}}_{1,\perp} = -\boldsymbol{\rho}$ [this is the familiar result $\overline{\boldsymbol{X}} = \boldsymbol{x} - \boldsymbol{\rho} + O(\epsilon^2)$]. At first order, the choice $S^{(1)} = 0$ eliminates $\overline{\gamma}^{(1)}_\mu$ and $\overline{\gamma}^{(1)}_\zeta$. δG^U_1 is chosen to remove the fluctuating (δ) part of $\overline{\boldsymbol{\gamma}}^{(1)}_{\boldsymbol{x}}\cdot\widehat{\boldsymbol{b}}$, and $\boldsymbol{g}^{\boldsymbol{X}}_{2,\perp}$ is chosen to remove the GGI part of $\overline{\boldsymbol{\gamma}}^{(1)}_{\boldsymbol{x},\perp}$. Then δg^μ_1 is chosen to remove the δ part of $\overline{\gamma}^{(1)}_t$, and $\boldsymbol{g}^{\boldsymbol{X}}_1\cdot\widehat{\boldsymbol{b}}$ is chosen to set $\overline{\gamma}^{(1)}_U = 0$. At second order, $\delta S^{(2)}$ is chosen to remove the δ part of $\overline{\gamma}^{(2)}_\zeta$ and $\langle g^\mu_1\rangle$ is chosen to set $\overline{\gamma}^{(2)}_\zeta = 0$. Then Δg^ζ_1 is chosen to remove the δ part of $\overline{\gamma}^{(2)}_\mu$, and $\langle S^{(2)}\rangle$ is chosen to set $\overline{\gamma}^{(2)}_\mu = 0$. Finally, $\boldsymbol{g}^{\boldsymbol{X}}_2\cdot\widehat{\boldsymbol{b}}$ is chosen to set $\overline{\gamma}^{(2)}_U = 0$. There is some arbitrariness in the choice of $\langle g^U_1\rangle$; the most physical one sets $\gamma^{(1)}_t = 0$. One can proceed systematically to higher order. (The first-order results are recorded in Appendix A of Ref. 2.)

There are two key features of this strategy. (i) The expansion is GGI; that is, $\boldsymbol{g}^{\boldsymbol{X}}_n$, g^μ_n, g^U_n, and Δg^ζ_n are constructed to be $\boldsymbol{R}$-independent. (ii) Determination of the g's and S's involves only local integrations in ζ and μ; no nonlocal spatial integrations are required. [In particular, $S^{(n)}$ is not determined from $\overline{\boldsymbol{\gamma}}^{(n)}_{\boldsymbol{X}}$, although $\boldsymbol{\nabla}S^{(n)}$ appears in the $\boldsymbol{X}$ component. This is possible because $\boldsymbol{g}^{\boldsymbol{X}}_{n,\perp}$ is determined at $O(\epsilon^{n-1})$.] Thus no difficulty arises in determining higher-order corrections to $\overline{\mu}$:

$$\overline{\mu}(\boldsymbol{X},\mu,U,\zeta) = \left(1 + L_1 + L_2 + \frac{1}{2}L_1^2\right)\mu = \mu + g^\mu_1 + g^\mu_2 + \frac{1}{2}\left(\boldsymbol{g}^{\boldsymbol{X}}_1\cdot\boldsymbol{\nabla} + g^\mu_1\frac{\partial}{\partial\mu} + g^U_1\frac{\partial}{\partial U} + g^\zeta_1\frac{\partial}{\partial\zeta}\right)g^\mu_1 + \cdots. \quad (2)$$

This $\overline{\mu}$ is GGI (contains no $\boldsymbol{R}$) because the operator $\boldsymbol{\nabla}^* \doteq \boldsymbol{\nabla} + \boldsymbol{R}\,\partial_\zeta$ is GGI. The $\boldsymbol{g}^{\boldsymbol{X}}_1\cdot\boldsymbol{\nabla}$ term taken alone generates $\boldsymbol{R}$ dependence, but that is canceled by the $\boldsymbol{R}$-dependent part of the $g^\zeta_1\partial_\zeta$ term. This can be easily proven in general[3,6] and can also be shown explicitly by applying $\boldsymbol{\nabla}^*$ to an arbitrary quantity $Q(\widehat{\boldsymbol{a}}(\boldsymbol{X},\zeta),\widehat{\boldsymbol{b}}(\boldsymbol{X}),\widehat{\boldsymbol{c}}(\boldsymbol{X},\zeta))$ and using the chain rule. By construction, the explicit $\overline{\mu}$ given by Eq. (2) is conserved through second order.

LS asserts that "the equations of motion ... require the curl and gradients of [the effective vector potential] $\boldsymbol{A}^*$, in the phase space spatial variables." That is incorrect; they require just the curl, since $\boldsymbol{A}^* \equiv \overline{\boldsymbol{\gamma}}_{\boldsymbol{X}}$. But one may worry that gradients of $\boldsymbol{R}$ other than its curl ($\boldsymbol{\nabla}\times\boldsymbol{R}$ is GGI) will appear in $\overline{\boldsymbol{\gamma}}^{(n)}_{\boldsymbol{X}}$ or $\overline{\boldsymbol{\gamma}}^{(n)}_t$ for $n \geq 2$. That does not happen, however, as a consequence of GGI. It is not hard to show that by explicit calculation for $n = 2$ [Eq. (1d)]. Consequently the drift equations are GGI.

One can now address LS's concern about integrability as follows. The formalism works with just one field line (and its immediate neighborhood) at a time. Locally, $\boldsymbol{\nabla}\widehat{\boldsymbol{e}}_i$ requires just first-order information about $\widehat{\boldsymbol{e}}_i(\overline{\boldsymbol{X}})$ at a point $\overline{\boldsymbol{X}}$; thus second-order issues related to anholonomic $\mathcal{F}$ do not matter and $\boldsymbol{R}$ is well-defined. Globally, even if a drift trajectory returns close to its spatial starting point after a toroidal circuit, it has not returned to a nearby point in phase space. Given all GC coordinates (including ζ) at time t, one could in principle trace back along the motion and undo the rotation to find the initial state. Thus integrability is not required. That is, the $\widehat{\boldsymbol{e}}_i$'s need not be a coordinate basis: $\widehat{\boldsymbol{e}}_i \neq \partial_{u_i}$.

When electromagnetic potentials are included[2,12,14–17] (the essence of GK, as opposed to GC, theory), the perturbation procedure outlined above must be modified to deal with fluctuations with $k_\perp\rho = O(1)$. But the theory remains local and encounters no difficulty of construction at the second order. For example, the polarization terms in the GK Poisson equation can be shown to be GGI.

Finally, here I have said nothing that is conceptually original. It is impossible to overstate the importance of Littlejohn's pioneering research and the subsequent extensions and applications reviewed in Ref. 2, which have placed the asymptotics of charged-particle motion in a strong, weakly inhomogeneous magnetic field on a very firm and systematic footing.

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