Princeton Plasma Physics Laboratory

PPPL- 4401

PPPL- 4401

Plasma Equilibrium in a Magnetic Field with Stochastic Regions

J.A. Krommes and Allan H. Reiman

April, 2009

Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

Plasma Equilibrium in a Magnetic Field with Stochastic Regions

John A. Krommes* and Allan H. Reiman†
Plasma Physics Laboratory, Princeton University, P.O. Box 451, Princeton, NJ 08543–0451
(Dated: April 20, 2009)

The nature of plasma equilibrium in a magnetic field with stochastic regions is examined. It is shown that the magnetic differential equation that determines the equilibrium Pfirsch-Schlüter currents can be cast in a form similar to various nonlinear equations for a turbulent plasma, allowing application of the mathematical methods of statistical turbulence theory. An analytically tractable model, previously studied in the context of resonance-broadening theory, is applied with particular attention paid to the periodicity constraints required in toroidal configurations. It is shown that even a very weak radial diffusion of the magnetic field lines can have a significant effect on the equilibrium in the neighborhood of the rational surfaces, strongly modifying the near-resonant Pfirsch-Schlüter currents. Implications for the numerical calculation of 3D equilibria are discussed.

I. INTRODUCTION

There is an extensive literature on the theory of plasma transport in the presence of stochastic magnetic field lines (see Refs. 1–3 and references therein). The existence of an underlying plasma equilibrium is implicitly assumed in such studies. In this paper, we examine the nature of plasma equilibria in a field with stochastic field lines, expanding upon the work of Reiman *et al.* in Ref. 4. We take advantage of a similarity in form between (i) the magnetic differential equation that determines the equilibrium Pfirsch-Schlüter currents (and a similar equation for the variation of the pressure along the field), (ii) the Liouville equation for magnetic field lines,[5] and (iii) nonlinear equations for turbulent plasmas, such as the Vlasov or drift-kinetic equations, to apply mathematical methods of turbulence theory to the magnetic differential equations. We shall focus particularly on an analytically tractable model (the limit of short correlation lengths) that has previously been studied in the context of resonance-broadening theory.[3] However, we generalize previous calculations to include the important periodicity constraints that are required in toroidal configurations.

Equilibria in stochastic regions are of current interest for tokamaks with ergodic limiters[6] and in the context of stellarator experiments at high pressure, where there is evidence of the formation of a large region of stochastic field lines at the plasma edge with a nonzero pressure gradient in that region.[4,7–9] For diverted tokamaks, there has been some success in suppressing edge localized modes (ELMs) by the imposition of nonaxisymmetric fields near the plasma edge,[10] and the possible role of the stochastic layer produced near the diverter separatrix is a subject of current research. Additionally, it has recently been shown that the observed electron thermal conductivity in one type of discharge in the National Spherical Torus Experiment is in quantitative agreement with a theory of transport due to magnetic-field-line stochasticity produced by magnetic perturbations from microtearing instabilities.[11]

In this paper we shall assume that the magnetic field $\boldsymbol{B}$ can be expressed as a sum of two pieces, $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}$, where $\boldsymbol{B}_0$ is an underlying field with nested flux surfaces and $\delta\boldsymbol{B}$ is a small perturbation that breaks the flux surfaces and causes the magnetic field lines to weakly diffuse relative to the unperturbed flux surfaces. We will see that if the underlying surfaces are three-dimensional (3D), even a very weak radial diffusion of the lines can have a significant effect on the equilibrium in the neighborhood of the rational surfaces (strongly modifying the near-resonant Pfirsch-Schlüter currents), and we describe a numerical procedure to calculate that equilibrium.

A. Overview of the calculation

The essential idea that is explored in this work is quite simple, although some of the details relating to toroidal geometry are tedious, and if the equations are pursued in all generality, one is led into the difficult and only incompletely understood area of strong-turbulence theory (of magnetic field-line stochasticity and the associated current flows). It is useful to appreciate the structure and challenges of the general theory, and we will discuss some of those. However, in practical applications one can make considerable simplifying approximations. To orient the reader, in this section we sketch the analysis as concisely as possible, without defining all of our terms or providing supporting logic. Then, in the remainder of the manuscript, we develop the topics in detail.

First assume one is given a magnetic field with good, nested flux surfaces. That is compatible with the standard scalar-pressure equilibrium force balance

$$\boldsymbol{j} \times \boldsymbol{B} = \boldsymbol{\nabla} p. \tag{1}$$

Such equilibria may be calculated numerically, for example with the Variational Moments Equilibrium Code[12] (`VMEC`) or with the Princeton Iterative Equilibrium Solver (`PIES`) code.[13]

Now add a small perturbation $\delta\boldsymbol{B}$. In general, such perturbations give rise to stochastic regions. If Eq. (1) is taken literally, it requires that the pressure be flattened in the stochastic regions; otherwise, the pressure force would have an unbalanced projection along the stochastic lines. However, the true force balance contains additional small terms that have been omitted from Eq. (1), and those terms can balance the pressure gradient in the stochastic regions. This argument, based on microscopic parallel force balance, implies after coarse-graining that nonzero macroscopic pressure gradients can be supported across stochastic regions.

Given such pressure (and current) profiles, we focus on the calculation of the equilibrium $\boldsymbol{B}$. (Ultimately, those profiles must be determined self-consistently by solution of long-time-scale transport equations, which we do not address here.) For Ampère's law $\boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{j}=j_{\parallel}\widehat{\boldsymbol{b}}+\boldsymbol{j}_{\perp}$, one requires $j_{\parallel}$ and $\boldsymbol{j}_{\perp}$. The field line stochasticity enters through the calculation of $j_{\parallel}$. Define $\mu\doteq j_{\parallel}/B$. (We use $\doteq$ for definitions.) Then the quasineutrality condition $\boldsymbol{\nabla}\cdot\boldsymbol{j}=0$ leads to the magnetic differential equation $\boldsymbol{B}\cdot\boldsymbol{\nabla}\mu=-\boldsymbol{\nabla}\cdot\boldsymbol{j}_{\perp}$. In a coordinate system for which $\boldsymbol{B}_0$ has straight magnetic field lines, the resonant Fourier components of the operator $(1/B_0^{\phi})\boldsymbol{B}_0\cdot\boldsymbol{\nabla}$ vanish at rational surfaces, implying that the resonant components of μ may be large in thin boundary layers surrounding the rational surfaces. To calculate their size precisely, we coarse-grain the magnetic differential equation *via* a quasilinear analysis that represents the magnetic stochasticity by a diffusion operator $-D\partial^2/\partial\psi^2$. Here $D[\langle\delta B^2\rangle]$ is the magnetic diffusion coefficient, dependent on the magnetic fluctuation level. (Brackets denote functional dependence.) The stochastic diffusion broadens the resonances at $k_{\parallel}(\psi)=0$ and limits the size of the resonant amplitudes.

We can now reformulate the equilibrium problem in the more tractable form $\boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{j}[\boldsymbol{B}]$. To define the functional $\boldsymbol{j}[\boldsymbol{B}]$, given $\boldsymbol{B}$, we first find a nearby field with good surfaces, $\boldsymbol{B}_0$. For numerical solution of the equilibrium equation, $\boldsymbol{B}_0$ can be calculated by using the `VMEC` equilibrium code, which assumes good flux surfaces, or by interpolating between regions of good surfaces of $\boldsymbol{B}$. (In the latter case, it may be helpful to construct the flux surfaces of $\boldsymbol{B}_0$ to approximately coincide with cantori in any large stochastic regions.[14]) Define $\delta\boldsymbol{B}\doteq\boldsymbol{B}-\boldsymbol{B}_0$. Calculating $D[\langle\delta\boldsymbol{B}^2\rangle]$, we may solve the coarse-grained magnetic differential equation for μ. The equation $\boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{j}[\boldsymbol{B}]$ may be solved numerically by standard methods for solving nonlinear partial differential equations, such as Picard iteration[13,15,16] or a Newton-Krylov scheme.[17]

The most important technical details missing from this abbreviated discussion are the mathematics of representing the stochastic diffusion, given that the physics should look diffusive whether one traverses the magnetic lines forwards or backwards. That can be handled by a Green's-function technique, as we will show. An additional complication is that the Green's functions must be periodic in the magnetic coordinates θ (generalized poloidal angle) and ϕ (toroidal angle). We will show how that periodicity can be enforced by working in the covering space of the underlying magnetic surface, then using a shifted-sum representation (techniques familiar from ballooning theory).

B. Organization

The organization of this paper is as follows. In Sec. II we describe the basic logic in more detail, and we derive the coupled system of Ampère's law and the magnetic differential equation for parallel current that must be solved. In Sec. III we discuss the interpretation of that magnetic differential equation as a stochastic differential equation,[18] including remarks on the statistical closure problem but ignoring periodicity considerations. In Sec. IV we obtain the Green's-function solution of the statistically coarse-grained magnetic differential equation with periodicity constraints enforced. We describe numerical procedures in Sec. V, and briefly discuss outstanding issues in Sec. VI. In Appendix A we illustrate statistical closure theory for passive advection with the aid of a tractable model; that discussion serves as background for the quasilinear analysis performed in the body of the paper. Finally, in Appendix B we review the theory of Green's functions, paying particular attention to the difficulties engendered by periodicity constraints and the equivalence of alternate formulations.

II. FORCE BALANCE IN THE PRESENCE OF STOCHASTIC MAGNETIC FIELDS

The usual MHD equilibrium equation (1) implies that the pressure gradient vanishes in stochastic regions, since it predicts that $\boldsymbol{B}\cdot\boldsymbol{\nabla}p=0$. However, for sufficiently small perturbations we intuitively expect the stochasticity to affect the radial transport, but not necessarily to completely flatten the pressure; this is the scenario studied in many papers that discuss the effect of field-line stochasticity on radial transport. There is indeed experimental evidence that nonzero pressure gradients are supported in regions of stochastic field lines in both tokamaks and stellarators.[4,19]

To handle a pressure gradient in the stochastic region, we follow Ref. 4 and allow for the presence of weak anisotropic terms in the pressure tensor ($\mathsf{P}=p\mathsf{I}+\boldsymbol{\pi}$, with $|\boldsymbol{\nabla}\cdot\boldsymbol{\pi}|\ll|\boldsymbol{\nabla}p|$) as well as a weak flow ($|\rho_m\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}|\ll|\boldsymbol{\nabla}p|$, where ρ_m is the mass density). Ultimately, these quantities and the global pressure and current profiles evolve and are determined by solution of the long-time transport problem, which we do not address in this paper. However, we will see that for calculation of the instantaneous equilibrium it is adequate to consider merely the steady-state force-balance

$$\boldsymbol{j}\times\boldsymbol{B}-\rho_m\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}-\boldsymbol{\nabla}\cdot\boldsymbol{\pi}=\boldsymbol{\nabla}p. \tag{2}$$

In the remainder of this section, we shall qualitatively discuss both the parallel and perpendicular components of this equation in the presence of stochastic magnetic fields. Our goal here is merely to outline the strategy; detailed treatments of the stochastic differential equations that arise are given in Sec. III and subsequently.

A. Parallel force balance

If the field lines diffuse weakly relative to the unperturbed flux surfaces, the pressure gradient across the unperturbed flux surfaces gives rise to a small pressure gradient along the weakly diffusing field lines that can be balanced by small terms in the force balance (2). Specifically, let

$$\boldsymbol{g} \doteq -\rho_m \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\pi} \tag{3}$$

and assume that $\boldsymbol{g}$ is known. Then, upon taking the scalar product of $\boldsymbol{B}$ with Eq. (2), one gets the *magnetic differential equation*[20]

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}p = \boldsymbol{B}\cdot\boldsymbol{g}. \tag{4}$$

In Sec. II B we will see that a similar equation arises for the quantity $\mu \doteq j_{\parallel}/B$. We will treat the μ equation thoroughly because its solution is central to our reformulation of the equilibrium problem. The analysis of Eq. (4), which we just sketch here, is closely parallel to that for μ.

As discussed in the Introduction, we assume that $\boldsymbol{B}$ can be expressed as a sum of two pieces, $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}$, where $\boldsymbol{B}_0$ is a field with nested flux surfaces and $\delta B \ll B_0$. We shall write $\boldsymbol{B}_0 \equiv \langle\boldsymbol{B}\rangle$, where the angular brackets denote a statistical average (discussed in detail in Sec. III C 1), and treat $\delta\boldsymbol{B} \doteq \boldsymbol{B} - \langle\boldsymbol{B}\rangle$ as a random variable. Then the statistical average of Eq. (4) is

$$\langle\boldsymbol{B}\rangle\cdot\boldsymbol{\nabla}\langle p\rangle + \boldsymbol{\nabla}\cdot(\delta\boldsymbol{B}\,\delta p) = \langle\boldsymbol{B}\cdot\boldsymbol{g}\rangle. \tag{5}$$

For stochastic regions, a quasilinear estimate of the second term (discussed in detail in Sec. III C 2) is $-D\nabla^2\langle p\rangle$, where $D\nabla^2$ stands for an operator that is possibly complicated due to the details of general geometry. A flux-surface average of Eq. (5) over the flux surfaces of $\boldsymbol{B}_0$ (denoted by an overline and defined precisely in Sec. III C 1) then annihilates the first term and leads to the equation

$$-D\frac{\partial^2\overline{\langle p\rangle}}{\partial\psi^2} = \overline{\langle\boldsymbol{B}\cdot\boldsymbol{g}\rangle}, \tag{6}$$

where ψ is the toroidal flux. This shows explicitly how the pressure gradient across a stochastic region can be supported by the small correction terms to the MHD equations. We will see in Sec. III C 3 that D is $O\big((\delta B/B)^2\big)$. It follows that a $\boldsymbol{g}$ with a magnitude of the order of $(\delta B/B)^2$ is sufficient for parallel force balance.

This analysis shows that the parallel force-balance equation couples to the transport equations (which ultimately determine $\boldsymbol{g}$). Fortunately, we will see in the next section that it decouples from the perpendicular force balance, so the solution of the parallel force balance can be regarded as being part of the transport problem rather than the equilibrium problem; we shall not deal with it further.

B. Perpendicular Force Balance

Upon taking the cross product of $\boldsymbol{B}$ with Eq. (2), one obtains an expression for $\boldsymbol{j}_\perp$, the component of the current density perpendicular to the magnetic field:

$$\boldsymbol{j}_\perp = \boldsymbol{B}\times\boldsymbol{\nabla}p/B^2 - \boldsymbol{B}\times\boldsymbol{g}/B^2. \tag{7}$$

We will be interested in retaining terms through $\mathrm{O}(\delta B/B)$ in Eq. (7). Since $\boldsymbol{g} = O\big((\delta B/B)^2\big)$, the last term can be neglected and one obtains the familiar expression

$$\boldsymbol{j}_\perp \approx \boldsymbol{B}\times\boldsymbol{\nabla}p/B^2. \tag{8}$$

We will be particularly interested in the resonant components of the current. We will see that D is actually of second order in the resonant components of $\delta\mathbf{B}/B$, so that the approximation remains valid for calculating the resonant components of $\mathbf{j}$.

The component of $\boldsymbol{j}$ parallel to the magnetic field, $j_\parallel$, is determined by the quasineutrality condition $\boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$, which gives the magnetic differential equation

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\mu = -\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp. \tag{9}$$

Given the pressure, Eqs. (8) and (9) specify $\boldsymbol{j}$ as a functional of $\boldsymbol{B}$: $\boldsymbol{j} = \boldsymbol{j}[\boldsymbol{B}]$. With Ampère's Law $\boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{j}$, one has a closed set of equations that can be solved for the equilibrium magnetic field:

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{j}[\boldsymbol{B}] = \mu\boldsymbol{B} + \boldsymbol{j}_\perp. \tag{10}$$

For evaluating the right-hand side of Eq. (8), we may to lowest order approximate $\boldsymbol{\nabla}p \approx \boldsymbol{\nabla}\langle p\rangle$ and take the latter quantity from experiment or transport modeling. (We will argue in Sec. III C 4 that pressure fluctuations are negligible.) Thus, because $\boldsymbol{g}$ has been neglected, it is unnecessary to simultaneously solve the parallel force balance in order to determine $\boldsymbol{B}$. (As we noted in the previous section, solution of the parallel force balance can be treated as part of the transport problem. We do not address slow evolution on the transport time scale in this paper.)

The PIES 3D equilibrium code[13] was developed to solve Eq. (10) numerically. In practice, Eq. (8) gives an explicit expression for $\mathbf{j}_\perp$; Eq. (9) then determines μ. The remainder of this paper deals largely with the solution of Eq. (9); we focus particularly on the role of magnetic stochasticity in that equation.

In calculating 3D equilibria with the PIES code, it has generally been the practice to flatten the pressure profile in stochastic regions, as dictated by the MHD equilibrium equation (1). In that case, the issue of solving the magnetic differential equation along stochastic magnetic-field-line trajectories does not arise. However, the problem becomes more complex when there are stochastic regions with nonzero pressure gradients. Accurate integrations along field lines fail because of the well-known extreme sensitivity of the trajectories to small changes in initial conditions. Shadowing theorems imply that statistics calculated from long-time integrations may still be valid. However, instead of proceeding entirely numerically, we will use statistical methods to replace the effects of the stochasticity by diffusion operators. This greatly simplifies numerical solution of (the statistically averaged version of) Eq. (9). Recently the PIES code has been modified to allow nonzero pressure gradients in stochastic regions, using a model for $j_\parallel$ along the lines discussed in this paper. Further details of the numerical procedure are given in Sec. V.

III. EQUATION FOR THE PFIRSCH-SCHLÜTER CURRENTS ALONG STOCHASTIC FIELD LINES

In this section we discuss Eq. (9), which determines the Pfirsch-Schlüter current (the pressure-driven part of $j_\parallel$). We first describe the solution on good flux surfaces, then we turn to a discussion of the impact of the stochastic regions.

A. Solution for $\boldsymbol{j}_\parallel$ on good flux surfaces

On a good flux (ψ) surface, Eq. (9) can be solved by transforming to magnetic coordinates (ψ, θ, ϕ). Those are flux coordinates with straight field lines: $\boldsymbol{B}\cdot\boldsymbol{\nabla}\psi = 0$, while $\iota(\psi) \doteq \boldsymbol{B}\cdot\boldsymbol{\nabla}\theta/\boldsymbol{B}\cdot\boldsymbol{\nabla}\phi$ is constant on the flux surface. Upon taking ψ to be the toroidal flux, one can write $\boldsymbol{B}$ in the form

$$\mathbf{B} = \nabla\psi \times \nabla\theta + \iota\nabla\phi \times \nabla\psi. \tag{11}$$

In magnetic coordinates, Eq. (9) can be rewritten as

$$\frac{\partial\mu}{\partial\phi} + \iota\frac{\partial\mu}{\partial\theta} = f \doteq -\frac{\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp}{B^\phi}. \tag{12}$$

We use superscripts to denote contravariant components, e.g., $B^\phi \equiv \boldsymbol{B}\cdot\boldsymbol{\nabla}\phi$. We shall sometimes write $f \equiv f^{\text{ext}}$, where "ext" stands for external; f^{ext} is to be viewed as an externally imposed source for the purposes of solving for μ. (Later we will also discuss an internal source f^{int} arising from nonlinearity.) Fourier transformation in θ and ϕ according to

$$\mu(\psi,\theta,\phi) = \sum_{mn} \widehat{\mu}_{mn}(\psi)\mathrm{e}^{i(m\theta - n\phi)} \tag{13}$$

(cf. space-time variations $\sim \exp[i(kx-\omega t)]$) gives

$$i\kappa_{\parallel mn}(\psi)\widehat{\mu}_{mn}(\psi) = \widehat{f}_{mn}(\psi), \tag{14}$$

where $\kappa_{\parallel mn}(\psi) \doteq \iota(\psi)m - n$ is essentially the wave number parallel to the unperturbed field lines.

The left-hand side of Eq. (14) vanishes for $m = 0$ and $n = 0$. To see that $\widehat{f}_{00}(\psi)$ also vanishes for all ψ, recall that the Jacobian of the transformation between the Cartesian components x^i and generalized coordinates u^i is $J \doteq (\boldsymbol{\nabla}u^1\cdot\boldsymbol{\nabla}u^2\times\boldsymbol{\nabla}u^3)^{-1}$. From Eq. (11), one sees that the Jacobian of the $(\psi,\theta,\phi) \equiv u^i$ magnetic coordinate system is $J = 1/B^\phi$. Therefore the $(0,0)$ Fourier component of $\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp/B^\phi$ is proportional to the ψ derivative of the volume integral of $\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp$ over the region bounded by ψ, which vanishes by Gauss's law if $p = p(\psi)$. An alternate proof uses the formula, for arbitrary vector $\boldsymbol{A}$,

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{J}\frac{\partial}{\partial u^i}(JA^i). \tag{15}$$

Integration of $J\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp$ over θ and ϕ eliminates the θ and ϕ components; the result then vanishes with $j_\perp^\psi$.

Equation (12) does not determine the Fourier coefficient $\widehat{\mu}_{00}(\psi)$, which serves as a constant of integration. Specification of the $\widehat{\mu}_{00}$ profile is equivalent to the specification of the net poloidal or toroidal current profile.[21] Thus, determination of a general nonaxisymmetric equilibrium requires specification of two profiles, a pressure profile and a current profile (or a q profile, where $q \doteq \iota^{-1}$), just as is the case for axisymmetric equilibria.

B. Effect of stochastic magnetic fields on $j_\parallel$

The apparent singularities of $\widehat{\mu}_{mn}$ at the rational surfaces $[\kappa_{\parallel mn}(\psi) = 0]$ can be resolved if one allows for the presence of magnetic islands[22] or magnetic stochasticity. A heuristic generalization of the solution of Eq. (14) that resolves the singularity is

$$\widehat{\mu}_{mn} = \left(\frac{-i\kappa_{\parallel mn}}{\kappa_{\parallel mn}^2 + \eta_{mn}^2}\right)\widehat{f}_{mn}, \tag{16}$$

where η is an effective spread in parallel wave number and a measure of the stochasticity in the vicinity of the rational surface. In Sec. IV we will show how this formula arises from systematic statistical formalism. First, however, we motivate the use of statistical methods, discuss the role of small terms in the determination of $j_\parallel$, and introduce various issues related to the calculation of the magnetic diffusion coefficient, the size of which ultimately determines η.

1. *Motivation for statistical methods*

The component of the current density parallel to the magnetic field, $\boldsymbol{j}_\parallel = \mu\boldsymbol{B}$, is in principle determined by Eq. (9), which is an ordinary differential equation along the magnetic field lines. If one uses ζ as a coordinate along a given line, the equation for μ along the field line takes the form

$$\frac{d\mu}{d\zeta} = -\frac{\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp}{B^\zeta} \doteq f. \tag{17}$$

This equation can in principle be solved numerically to any desired accuracy. In practice, however, if one integrates the equation sufficiently far along a given chaotic field-line trajectory, small numerical errors amplify and swamp the solution.

To get some additional insight into this situation, Fourier decompose f as a function of θ and ϕ, then write $\mu = \sum_{mn} \mu^{(mn)}$, where

$$d\mu^{(mn)}/d\zeta = \widehat{f}_{mn}(\psi(\zeta))\mathrm{e}^{i[m\theta(\zeta)-n\phi(\zeta)]}. \tag{18}$$

Let ζ_{l} represent a scale length such that for $\zeta \ll \zeta_{\mathrm{l}}$ the trajectory may be regarded as unperturbed by $\delta\mathbf{B}$, and let ζ_{nl} represent a nonlinear mixing length (to be discussed in Sec. IV C), with $\zeta_{\mathrm{nl}} > \zeta_{\mathrm{l}}$. For (m,n) satisfying $\kappa_{\|mn}\zeta_{\mathrm{l}} \gg 1$, the wavelength along the field line is short compared to ζ_{l}, and $\mu^{(mn)}$ is well approximated in terms of the local value of ψ by $\mu^{(mn)}(\psi,\theta,\phi) \approx -i\widehat{f}_{mn}(\psi)\mathrm{e}^{i(m\theta-n\phi)}/\kappa_{\|mn}(\psi)$. On the other hand, for (m,n) satisfying $\kappa_{\|mn}\zeta_{\mathrm{nl}} \ll 1$, the wavelength along the field line is long compared to the mixing length and accurate calculation of $\mu^{(mn)}$ becomes difficult. Furthermore, field-line trajectories that are initially close will deviate significantly for $\zeta \gg \zeta_{\mathrm{nl}}$, so the physically relevant quantity is a statistical average (over either the trajectory or initial coordinates). It is thus natural to adopt statistical methods and study a suitably averaged $\langle\mu\rangle_{mn}$.

2. *Effect of small terms on* $\boldsymbol{j}_\|$

The $m = 0$, $n = 0$ component of Eq. (18) requires special consideration. Consider first the case where $\boldsymbol{\nabla} p = \mathbf{0}$. The approximate Eqs. (1) and (9) imply $\boldsymbol{B}\cdot\boldsymbol{\nabla}\mu = 0$; however, the considerations discussed in Sec. II A for the equation $\boldsymbol{B}\cdot\boldsymbol{\nabla} p = 0$ apply here as well. Specifically, a nonzero $\boldsymbol{\nabla}\mu$ across the unperturbed surfaces corresponds to a very small $\boldsymbol{B}\cdot\boldsymbol{\nabla}\mu$ along the weakly diffusing field lines. That small $\boldsymbol{B}\cdot\boldsymbol{\nabla}\mu$ can be balanced by contributions from the small additional terms retained in the expression for $\boldsymbol{j}_\perp$ in Eq. (7).

Consider a case where a magnetic field that initially has good flux surfaces is stochasticized by a small magnetic-field perturbation. Initially there is a $\widehat{\mu}_{00}(\psi)$ profile, which can be determined by specifying either $q(\psi)$ or a net poloidal or toroidal current as a function of ψ. Turning on a small magnetic-field perturbation that breaks the flux surfaces, so that the field becomes stochastic in the region of interest, produces a slow radial diffusion of the field lines. The electrons move along the field lines much more rapidly than the ions, and they carry electron momentum with them as they move along the radially diffusing lines, producing an electron viscosity. This effect was first studied by Stix.[23] The resulting term in Ohm's law is called *hyper-resistivity*; it has been discussed extensively in the context of tearing-mode turbulence (see, e.g., Ref. 24 and references therein). As with the weak variation of p along the radially diffusing field lines, the weak variation of μ along the radially diffusing field lines in this case is properly dealt with in the context of transport theory rather than equilibrium theory.

In the more general case where $\boldsymbol{\nabla} p$ is nonzero, the same considerations lead to the conclusion that $\widehat{\mu}_{00}$ varies very slowly along the magnetic field lines; that variation is properly dealt with in the context of transport theory.

3. *Stochastic magnetic fields and resonance broadening: Heuristic considerations*

We shall assume that $B_0^\phi \gg B_0^\theta$ and that the three components of $\delta\boldsymbol{B}$ are of the same order, so the term containing δB^ϕ in the magnetic differential equation can be neglected. Thus, in the presence of $\delta\boldsymbol{B}$ Eq. (12) acquires two additional terms:

$$\frac{\partial\mu}{\partial\phi} + \iota(\psi)\frac{\partial\mu}{\partial\theta} + \frac{\delta B^\psi}{B_0^\phi}\frac{\partial\mu}{\partial\psi} + \frac{\delta B^\theta}{B_0^\phi}\frac{\partial\mu}{\partial\theta} = f^{\mathrm{ext}}. \tag{19}$$

The homogeneous part of Eq. (19) is identical in form to the Liouville equation for magnetic field lines discussed in the fundamental paper by Rosenbluth, Sagdeev, Taylor, and Zaslavskii (RSTZ).[5] Those authors showed that when the perturbing fields are stochastic their effect can be described at the macroscopic level by a diffusion equation, and they found the quasilinear expression for the magnetic diffusion coefficient D. A related equation for test-particle transport in stochastic magnetic fields was discussed by Krommes[25] and Krommes *et al.*[2] (There has been a resurgence of interest in such equations; see, for example, Ref. 26 and references therein.)

One can also draw some parallels between Eq. (19) and other stochastic differential equations commonly encountered in plasma physics. If ι is a monotonic function of ψ in the region of interest, one can adopt it as the radial variable. Equation (19) then becomes

$$\frac{\partial\mu(\iota,\theta,\phi)}{\partial\phi} + \iota\frac{\partial\mu}{\partial\theta} + \frac{\delta B^\iota}{B_0^\phi}\frac{\partial\mu}{\partial\iota} + \frac{\delta B^\theta}{B_0^\phi}\frac{\partial\mu}{\partial\theta} = f^{\mathrm{ext}}. \tag{20}$$

We first compare this with the collisonless Vlasov equation in one spatial dimension for an unmagnetized plasma with a turbulent electrostatic field δE:

$$\frac{\partial f(x,v,t)}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}\delta E\frac{\partial f}{\partial v} = 0. \tag{21}$$

The homogeneous part of Eq. (20) has the same form as Eq. (21) if one ignores δB^θ. (We identify $\mu \to f$, $\theta \to x$, $\iota \to v$, and $\phi \to t$.) For a random δE that is given (i.e., not a functional of f), Eq. (21) defines the so-called *stochastic acceleration problem*, which has been studied by Sturrock,[27] Dupree,[28] Orszag and Kraichnan,[29] and Orszag,[30] for example.

An even closer parallel is to the drift-kinetic equation for a plasma in a strong magnetic field parallel to the

z direction and possessing a fluctuating $\boldsymbol{E}\times\boldsymbol{B}$ velocity $\delta\boldsymbol{V}_E$:

$$\frac{\partial f(\boldsymbol{x},v_\parallel,t)}{\partial t}+v_\parallel\frac{\partial f}{\partial z}+\delta\boldsymbol{V}_E\cdot\boldsymbol{\nabla} f+\frac{q}{m}E_\parallel\frac{\partial f}{\partial v_\parallel}=0. \quad (22)$$

The parallel nonlinearity is frequently neglected. Then the nonlinear advection term $\delta\boldsymbol{V}_E\cdot\boldsymbol{\nabla} f=\delta V_{E,x}\partial_x f+\delta V_{E,y}\partial_y f$ is seen to be analogous to the δB terms in Eq. (20); note that $\boldsymbol{\nabla}\cdot\boldsymbol{V}_E=0$ for constant B, analogous to $\boldsymbol{\nabla}\cdot(\delta\boldsymbol{B}/B_0^\phi)=0$ for constant B_0^ϕ.

In Refs. 25 and 2, Krommes pointed out that the similarity in form of equations such as Eq. (20) to the Vlasov, drift-kinetic, and other nonlinear equations allows one to apply to magnetic-field problems the mathematical methods that have been developed extensively in the context of stochastic differential equations[18] and turbulence.[3] To do so, one begins by considering δB^θ and δB^ψ (or δB^ι) to be random functions. Then all of the methodology of statistical closure theory becomes available. A review article on that topic is Ref. 3.

In this section, we introduce the closure issues heuristically. In the next section and in Appendix A, we consider more formal aspects of the statistical problem.

In the simplest (resonance-broadening) approximation, the effects of the fluctuating terms are replaced by diffusion operators. For example, if in Eq. (21) one considers δE to be random and further assumes that it is Gaussian white noise (delta-correlated in time), it follows from Fokker-Planck theory that the probability density function (PDF) obeys

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\frac{\partial}{\partial v}D_v(v)\frac{\partial f}{\partial v}=0, \quad (23)$$

where the velocity-space diffusion coefficient is

$$D_v\doteq\left(\frac{q}{m}\right)^2\int_0^\infty d\tau\,\langle\delta E(\widetilde{x}(\tau),\tau)\delta E(\widetilde{x}(0),0)\rangle \quad (24)$$

and the integral is taken over the turbulent particle trajectories $\widetilde{x}(\tau)$ (the tilde denotes a random quantity). This is a principle result of Dupree's 1966 resonance-broadening formalism,[28] which has been discussed extensively.[3]

Note that the assumption of Gaussian white noise is not essential; it merely allows one to rigorously transform an equation of the form of Eq. (21) to the more tractable form of Eq. (23). When the turbulence has a nonzero correlation time or length, the transformation still holds approximately provided that the effective *Kubo number* for the problem (see Appendix A) is small.[3,31]

At virtually the same time as Dupree's work leading to Eq. (23), RSTZ[5] derived the analogous equation for the PDF f_m of magnetic field lines. Note that the operator

$$\frac{\partial}{\partial\phi}+\iota\frac{\partial}{\partial\theta}\equiv\frac{\partial}{\partial\zeta} \quad (25)$$

is the directional derivative along the unperturbed field lines. RSTZ treated ζ as literally time-like in that they advanced the Liouville distribution forward along the lines, analogous to the causal way in which the initial-value problem is usually studied in turbulence theory. Upon performing an appropriate average, they found a diffusion equation that in the present notation would have the form

$$\frac{\partial f_m}{\partial\zeta}-\frac{\partial}{\partial\psi}D\frac{\partial f_m}{\partial\psi}=0, \quad (26)$$

where D is the diffusion coefficient of the magnetic lines.

One might expect to be able to follow an analogous procedure for the magnetic differential equation (19). However, while Eq. (19) is similar to the equation studied by RSTZ, there is an important nuance. In an application to a real device, one must solve for $\langle\mu\rangle(\theta,\phi)$ on a domain that is 2π-periodic in both θ and ϕ; however, periodic systems are not spatially causal. To get to any particular physical point, one can move either forward or backward along a line; the physics should look diffusive in either direction. Thus, *in the presence of periodicity $\langle\mu\rangle$ itself does not obey a diffusion equation.* Instead, one must represent $\langle\mu\rangle$ as a superposition of solutions that are constructed from both causal (forward integration) and anti-causal (backward integration) Green's functions. It will be the causal Green's function that obeys a standard diffusion equation (with ζ increasing). Thus, if one defines $\delta b^i\doteq\delta B^i/B_0^\phi$, takes the δb^i in Eq. (19) to be Gaussian white noise (as seen along the unperturbed field lines), and assumes that $\boldsymbol{j}_\perp$ is statistically independent of $\delta\boldsymbol{B}$ [strictly speaking, this is not true; see the discussion of term (c) below Eq. (41)], one may argue (ignoring details of general geometry) that the causal Green's function G^+ should obey

$$\begin{aligned}\frac{\partial G^+}{\partial\zeta}-\epsilon\left(\frac{\partial}{\partial\psi}D^\psi\frac{\partial G^+}{\partial\psi}+\frac{\partial}{\partial\theta}D^\theta\frac{\partial G^+}{\partial\theta}\right)&\\ =\delta(\psi-\psi')\delta(\theta-\theta')\delta(\zeta-\zeta'),&\quad (27)\end{aligned}$$

where

$$\epsilon D^\psi\doteq\int_0^\infty d\zeta\,\langle\delta b^\psi(\widetilde{\psi}(\zeta),\widetilde{\theta}(\zeta),\zeta)\delta b^\psi(\widetilde{\psi}(0),\widetilde{\theta}(0),0)\rangle, \quad (28a)$$

$$\epsilon D^\theta\doteq\int_0^\infty d\zeta\,\langle\delta b^\theta(\widetilde{\psi}(\zeta),\widetilde{\theta}(\zeta),\zeta)\delta b^\theta(\widetilde{\psi}(0),\widetilde{\theta}(0),0)\rangle, \quad (28b)$$

and the integrals are taken along the field-line trajectories. The ϵ coefficients have been included to emphasize that the diffusion coefficients are assumed to be small. In the analogous equation for the anti-causal (adjoint) Green's function G^-, the sign of the diffusion terms is reversed.

An implicit assumption of the resonance-broadening theory is that the fluctuating field is *passive.* That is, it is statistically specified, not self-consistently related to the dependent variable (e.g., the particle PDF). In our application, this is actually not true because the solution for $\langle\mu\rangle$ will be fed into Ampère's law, from which $\delta\boldsymbol{B}$ ultimately follows. Nevertheless, for the purpose of simplifying the equations for the Green's functions, we shall use a passive approximation, as a complete theory of self-consistent statistics is quite involved.[3]

The introduction of ζ has the unfortunate side effect of complicating the periodicity constraints. We will solve the periodic generalization of Eq. (27) in Sec. IV, where we also show how to construct $\langle\mu\rangle$ from G^{+} and its adjoint G^{-}. Before doing so, however, it is necessary to resolve several other issues:

1. One must clearly understand how to work with the ensemble average $\langle\ldots\rangle$.

2. It is well known that standard resonance-broadening theory violates certain conservation properties. One must understand the implications of those in the present context and argue that they are unimportant.

3. Because $\boldsymbol{j}_{\perp}$ depends on $\boldsymbol{B}$ according to Eq. (8), it depends on the fluctuating $\delta\boldsymbol{B}$. Therefore, $\delta\boldsymbol{j}_{\perp}$ is statistically correlated to $\delta\boldsymbol{B}$; this violates the assumption of statistical independence between the random coefficient and the inhomogeneous source term that is used in the derivation of the equation for $\langle\mu\rangle$.

4. Fundamentally, we find an equation for $\langle\mu\rangle$. However, the current $\widetilde{\boldsymbol{j}} = \widetilde{\mu}\widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{j}}_{\perp}$ on the right-hand side of Ampère's law involves the total $\widetilde{\mu} \doteq \langle\mu\rangle + \delta\mu$. One must either discuss how to treat the fluctuating $\delta\mu$ and $\delta\boldsymbol{j}_{\perp}$ or argue that they are negligible.

The resolutions of these issues are nontrivial. Some discussion is given in the next section. We have also found it instructive to use a simple stochastic model to illustrate various points, especially items 2 and 3 above. In order to not interrupt the flow of the logic, we have relegated discussion of that model to Appendix A.

C. Statistical methodology and the diffusion equation

We now consider a more formal approach to the derivation of the diffusion equation. We begin with some general discussion of statistical averaging techniques. Then in Sec. III C 2 we introduce a stochastic Langevin equation and use it to derive the diffusion equation.

1. *Statistical averaging techniques*

The instant one introduces a diffusion coefficient, as we did heuristically in the last section, one commits to the use of some sort of statistical averaging. Ensemble averages are the most flexible, as they permit systematic discussion of temporally nonstationary and spatially inhomogeneous problems without the need for somewhat ill-defined multiple-scale methods. A clear statement was made by Balescu (Ref. 32, Appendix A.7). In self-consistent problems, one generally envisions an ensemble of initial conditions (and then argues that the final, nonlinearly and self-consistently determined, statistics are sensibly independent of the initial state[33]). For passive problems, one additionally postulates an ensemble of realizations of the random coefficient. The use of ensembles reflects the fact that in stochastic regimes dynamical realizations are extremely sensitive to small changes in initial conditions. Diffusion coefficients describe the consequences of that sensitivity in a coarse-grained, mean-square sense.

Statistical methods can be problematical. In nonstochastic (integrable) regimes, indiscriminate use of statistical approximations can introduce spurious dissipation or decay of correlations.[3] In the present problem, we also face an additional difficulty. Although in traditional analytical turbulence theory one solves equations only for ensemble-averaged quantities (e.g., the mean field and two-point correlation function), we intend to solve Ampère's law (numerically) *in a particular realization.* Stochastic fluctuations will be involved in that procedure, so we must be very careful.

As we have stated, we divide the total magnetic field into (i) a part $\boldsymbol{B}_0$ possessing good flux surfaces everywhere, and (ii) a stochastic part $\delta\boldsymbol{B}$. This decomposition has meaning even for a single realization defined by a microscopic $\boldsymbol{j}_{\perp}$ field. However, to aid us in dealing with the extreme sensitivity of the field lines (and thus $\boldsymbol{j}_{\parallel}$) in stochastic regions to small changes in initial conditions, we treat the magnetic field as a random variable $\widetilde{\boldsymbol{B}} = \langle\boldsymbol{B}\rangle + \delta\boldsymbol{B}$ and envision a formal ensemble averaging such that $\langle\boldsymbol{B}\rangle = \boldsymbol{B}_0$. We will need to say little about the operational definition of $\langle\ldots\rangle$. However, it is instructive to understand that it is neither a simple integration over angles nor a flux-surface average. In simple problems with homogeneous statistics, it is sometimes stated that ensemble averages may be replaced by spatial integration over a single realization. That assertion is incomplete, for one must define multipoint correlation functions as well as the mean field. Furthermore, it holds only for ergodic flows, whereas ensemble averages are not so restricted. In any event, spatial integration is not appropriate in the present case because in general even fields with good flux surfaces have nontrivial dependencies on all coordinates. For example, in straight field-line coordinates $u^i = (\rho, \theta, \phi)$, where ρ labels the flux surface, it can be shown[34] that

$$B_0^{\rho} = 0,\ B_0^{\theta} = \frac{\dot{\Psi}_{\rm pol}(\rho)}{2\pi J(\rho,\theta,\phi)},\ B_0^{\phi} = \frac{\dot{\Psi}_{\rm tor}(\rho)}{2\pi J(\rho,\theta,\phi)}, \quad (29)$$

where $\Psi_{\rm pol}$ and $\Psi_{\rm tor}$ are the poloidal and toroidal fluxes, the dot means the derivative with respect to ρ, and J is the Jacobian. Thus angle dependence enters the contravariant components through J. When $\boldsymbol{B}_0$ is constructed in terms of the covariant basis vectors, i.e., $\boldsymbol{B}_0 = B_0^i \boldsymbol{e}_i$ (summation convention implied), additional coordinate dependence enters through the $\boldsymbol{e}_i$'s. One cannot extract the $\dot{\Psi}$'s merely by integrating $\boldsymbol{B}(\rho, \theta, \phi)$ over θ and ϕ.

For the magnetic field vector only, a coordinate-averaging operation that might appear to work is the following. Assume that one is given the set of covariant basis vectors $\boldsymbol{e}_i$ and the reciprocal, contravariant set $\boldsymbol{e}^i$ that are associated with the good flux surfaces expressed in straight field-line coordinates. (In reality, the $\boldsymbol{e}$'s are not known until the flux surfaces are found.) J is then determined according to $J = \boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)$. Define

$$\langle \boldsymbol{B} \rangle \doteq \boldsymbol{e}_i \frac{V'}{(2\pi)^2 J} \overline{B^i}, \tag{30}$$

where $B^i \doteq \boldsymbol{e}^i \cdot \boldsymbol{B}$, $V' \doteq \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \, J$, and the overline denotes the *flux-surface average*: For any scalar A,

$$\overline{A}(\rho) \doteq \frac{1}{V'} \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \, J \, A. \tag{31}$$

(A thorough discussion of flux-surface averaging can be found in Ref. 34.) Because $\overline{V'/[(2\pi)^2 J]} = 1$ and $\boldsymbol{e}_i \cdot \boldsymbol{e}^j = \delta_i^j$, this procedure defines a proper projection operator. Equation (30) becomes

$$\langle \boldsymbol{B} \rangle = \boldsymbol{e}_i \left(\frac{1}{J} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} J B^i \right). \tag{32}$$

We see that this averaging procedure extracts the angle-independent part of the J*-weighted* contravariant component, but still defines a $\langle \boldsymbol{B} \rangle$ that depends on θ and ϕ.

For this average to be useful, one would hope to apply it to Ampère's law and write something like $\boldsymbol{\nabla} \times \langle \boldsymbol{B} \rangle = \langle \boldsymbol{j} \rangle$. However, *in general geometry the* J*-weighted average does not commute with the curl!* One has

$$(\boldsymbol{\nabla} \times \boldsymbol{B})^i = \frac{1}{J} \epsilon^{ijk} \frac{\partial B_k}{\partial u^j}, \tag{33}$$

naturally expressed in terms of the *covariant* components of $\boldsymbol{B}$. One may introduce the contravariant components with the aid of the metric tensor $g_{ij} \doteq \boldsymbol{e}_i \cdot \boldsymbol{e}_j$, viz., $B_k = g_{kl} B^l$. However, although the J^{-1} in Eq. (33) is canceled by the J in the weighted average (32), the presence of the angle-dependent metric factors introduces a nonlinear coupling of Fourier harmonics and precludes the average from acting directly on the B^l. In fact, in stellarators this coupling may be the principal mechanism for the generation of resonant (stochastic) Fourier components (see Sec. III C 4).

To deal with this difficulty, we shall eschew a statistical treatment of Ampère's law; instead, we will solve the full, unaveraged equation numerically, given $\boldsymbol{j} = \mu \boldsymbol{B} + \boldsymbol{j}_\perp$. Having obtained the total $\boldsymbol{B}$, one can use standard interpolating techniques to construct a field $\boldsymbol{B}_0 \equiv \langle \boldsymbol{B} \rangle$ having good surfaces everywhere. Then $\delta \boldsymbol{B}$ is defined from $\delta \boldsymbol{B} = \boldsymbol{B} - \boldsymbol{B}_0$.

Unfortunately, use of this procedure raises another issue, which is how to define the total μ (treated as a random variable $\widetilde{\mu}$). Since we desire to introduce an analytical approximation and broaden the resonances by means of diffusion terms, we cannot avoid some kind of averaging; we must write $\widetilde{\mu} = \langle \mu \rangle + \delta \mu$ and calculate both $\langle \mu \rangle$ (from a diffusion equation) and $\delta \mu$. In order to ultimately obtain a time-like diffusion equation for $\langle \mu \rangle$ (or its underlying Green's functions), we need to be able to write

$$\langle \boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \widetilde{\mu} \rangle = \boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \langle \mu \rangle = \frac{\dot{\Psi}_{\text{tor}}}{2\pi J} \left(\frac{\partial \langle \mu \rangle}{\partial \phi} + \iota \frac{\partial \langle \mu \rangle}{\partial \theta} \right). \tag{34}$$

However, as is well known,[34] *the flux-surface average annihilates* $\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}$, so use of that operation would predict $\boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \langle \mu \rangle = 0$. But we expect that an appropriately defined $\langle \mu \rangle$ is angle-*dependent* in a general toroidal equilibrium, so contains a spectrum of m's and n's. Only for $\langle \widehat{\mu} \rangle_{00}$ does $\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}$ vanish identically. Therefore a flux-surface average is not appropriate for our needs.

There is a simple and intuitive way of understanding why the flux-surface average seems to work for $\boldsymbol{B}$ but is inadequate for μ. Consider an analogy to the time-dependent problems of standard turbulence theory. Given a temporal evolution equation for a generic random field $\widetilde{\psi}(t)$ [analogous to $\widetilde{\mu}(\zeta)$], one may average that equation to find $\partial_t \langle \psi \rangle (t) + \cdots = 0$. From an arbitrary initial condition, the mean field evolves in time (analogous to $\boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \langle \mu \rangle \neq 0$). Ultimately, the system saturates due to nonlinear terms and achieves a steady state in which $\partial_t \langle \psi \rangle = 0$. The steady-state $\langle \psi \rangle$ is analogous to the $\langle \widehat{\mu} \rangle_{00}$ term, which is annihilated by $\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}$; the temporal Fourier components $\widehat{\langle \psi \rangle}(\omega)$ of the nonsteady $\langle \psi \rangle (t)$ are analogous to the $\langle \widehat{\mu} \rangle_{mn}$. In contrast, in our calculation of the mean magnetic field there is no analog to nonsteady states. $\boldsymbol{B}_0$ is a global configuration, not "evolving" in field-line distance ζ. (Slow temporal evolution occurs on the transport time scale, but we do not address the transport problem in this paper.) It is analogous to the saturated steady state of the time-dependent problems. The flux-surface average of the contravariant magnetic-field component extracts that state directly.

Given such difficulties, we shall eschew attempts to define averages in terms of spatial integration. Instead, by analogy with the more familiar time-dependent problems, we shall apply formal ensemble-averaging procedures to the $\widetilde{\mu}$ equation, assuming only that the average commutes with $\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}$. That will lead us to a diffusion equation that contains coefficients involving the two-point correlation functions of the δb^i's. Only at that stage will we need to say something about the operational definition of the average; we will be able to do that plausibly.

2. A stochastic Langevin equation and derivation of a diffusion equation for $\mu \doteq j_\parallel/B$

The statistical average of Eq. (19) is

$$\left(\frac{\partial}{\partial\phi}+\iota\frac{\partial}{\partial\theta}\right)\langle\mu\rangle+\frac{1}{B_0^\phi J}\frac{\partial}{\partial u^i}(B_0^\phi J\langle\delta b^i\delta\mu\rangle) = -\langle\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp/B_0^\phi\rangle=\langle f^{\rm ext}\rangle. \qquad (35)$$

We used $\boldsymbol{\nabla}\cdot\delta\boldsymbol{B}=0$ to write $\delta\boldsymbol{B}\cdot\boldsymbol{\nabla}\mu=\boldsymbol{\nabla}\cdot(\delta\boldsymbol{B}\,\mu)$, then employed formula (15). Equation (35) is exact. To proceed, one must introduce a statistical closure such as the direct-interaction approximation (DIA).[3] Although such approximations are usually stated in terms of means and correlation functions, we shall for pedagogical reasons instead assert a stochastic Langevin model. (See Appendix A for the simplest version of this procedure as applied to a problem dependent only on time.) In order to focus on the statistical aspects of the problem, *we shall ignore periodicity issues for the remainder of this section.* The exact equation for $\delta\mu$ is

$$\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\delta\mu+\boldsymbol{\nabla}\cdot(\delta\boldsymbol{B}\,\delta\mu-\langle\delta\boldsymbol{B}\,\delta\mu\rangle)=-\delta\boldsymbol{B}\cdot\boldsymbol{\nabla}\langle\mu\rangle-\boldsymbol{\nabla}\cdot\delta\boldsymbol{j}_\perp. \qquad (36)$$

By using straight field-line coordinates based on the mean field $\boldsymbol{B}_0$, one can rearrange and simplify this to

$$\left(\frac{\partial}{\partial\phi}+\iota\frac{\partial}{\partial\theta}\right)\delta\mu+(\text{n.l. terms})=-\delta b^i\frac{\partial\langle\mu\rangle}{\partial u^i}+\delta f^{\rm ext}, \qquad (37)$$

where "(n.l. terms)" refers to the $\delta\boldsymbol{B}\,\delta\mu$ terms in Eq. (36) and $\delta f^{\rm ext}$ refers to the last term of Eq. (36) divided by B_0^ϕ. A statistical closure is intended to provide a workable model of the nonlinear terms. One expects that one effect of those terms will be to broaden the resonances by introducing a diffusion operator acting on $\delta\mu$; we shall write that as $-D'\nabla^2\delta\mu$, where "$D'\nabla^2$" is a symbolic notation for a possibly complicated formula. (That is, D' is really a tensor and should appear inside one of the derivatives, as $\boldsymbol{\nabla}\cdot\mathsf{D}'\cdot\boldsymbol{\nabla}$, and one must worry about metric factors associated with the nonorthogonal coordinate system. We use the prime to distinguish D' from the diffusion coefficient D that will shortly appear in the equation for $\langle\mu\rangle$.) This diffusion term describes so-called *coherent response*, i.e., it acts on the same $\delta\mu$ that appears under the time-like ϕ derivative. In general, there is also a phase-*incoherent* part of the nonlinearity, which we shall call $\delta f^{\rm int}$ ("int" stands for internal). (In the general theory, as discussed in Appendix A, there also arises another piece $\Sigma'\delta b$. However, Σ' vanishes in the white-noise limit, so we do not write it here.) Thus one has

$$\left(\frac{\partial}{\partial\phi}+\iota\frac{\partial}{\partial\theta}\right)\delta\mu-D'\nabla^2\delta\mu=-\delta b^i\frac{\partial\langle\mu\rangle}{\partial u^i}+\delta f^{\rm int}+\delta f^{\rm ext}. \qquad (38)$$

In the passive DIA, the incoherent internal noise $\delta f^{\rm int}$ is modeled by

$$\delta f^{\rm int}=-\delta\boldsymbol{B}\cdot\boldsymbol{\nabla}\delta\xi, \qquad (39)$$

where $\delta\xi$ is a centered Gaussian random variable, statistically independent of $\delta\boldsymbol{B}$, whose variance and cross correlations with $\delta\boldsymbol{B}$ and $\delta f^{\rm ext}$ are pinned to those of the $\delta\mu$ that is ultimately determined from the Langevin equation (38). This construction is subtle and possibly unfamiliar; see further discussion in Appendix A.

For forward motion, one can solve Eq. (38) by introducing the causal response function R that obeys

$$\left(\frac{\partial}{\partial\phi}+\iota\frac{\partial}{\partial\theta}\right)R(1;1')-D'\nabla^2R=\delta(1-1') \qquad (40)$$

subject to $R(\phi;\phi')=0$ for $\phi<\phi'$ on a given field line. (The precise meaning of "forward" will be clarified in Sec. IV with the introduction of field-line-following coordinates.) Thus, with $\star$ denoting convolution,

$$\delta\mu=R\star\left(\underbrace{-\delta b^i\frac{\partial\langle\mu\rangle}{\partial u^i}}_{\text{(a)}}+\underbrace{\delta f^{\rm int}}_{\text{(b)}}+\underbrace{\delta f^{\rm ext}}_{\text{(c)}}\right). \qquad (41)$$

3. Role of $\delta\mu$ in the $\langle\mu\rangle$ equation

The $\delta\mu$ so determined is required for both the $\langle\mu\rangle$ equation and for Ampère's law. Here we concentrate on its role in Eq. (35) for $\langle\mu\rangle$, for which we need $\langle\delta b^i\delta\mu\rangle$. All terms of Eq. (41) contribute to this, in principle; label them (a), (b), and (c). Term (a) evaluates directly to

$$\langle\delta b^i\delta\mu\rangle^{\rm (a)}=-D^{ij}\frac{\partial\langle\mu\rangle}{\partial u^j}, \qquad (42)$$

where the magnetic diffusion tensor is schematically $D^{ij}\doteq\langle\delta b^iR\,\delta b^j\rangle$. In writing these forms, we have implicitly made the Markovian approximation, namely that the fluctuating magnetic field has a short Lagrangian correlation length, so the action of R is only on the second δb, not on $\langle\mu\rangle$.

The diffusion tensor can be simplified by neglecting off-diagonal correlations. That need not be true in general, but it is a legitimate approximation for the boundary-layer problem we will ultimately solve, where the divergence of term (a) is dominated by $D^\psi\partial^2/\partial\psi^2$. As we stated earlier, we shall neglect δb^ϕ because the background toroidal field is large. For any diagonal component, one has

$$D^{ii}=\left\langle\delta b^i(\rho,\theta,\phi)\int d\overline{\rho}\,d\overline{\theta}\,d\overline{\phi}\,R(\rho,\theta,\phi;\overline{\rho},\overline{\theta},\overline{\phi})\delta b^i(\overline{\rho},\overline{\theta},\overline{\phi})\right\rangle \qquad (43)$$

(no sum on i). In the subsequent discussion, we shall drop the i label for brevity. In the absence of resonance broadening and periodicity constraints, the solution of Eq. (40) for R is $R\approx R_0$, where

$$R_0(\rho,\theta,\phi;\rho',\theta',\phi')=H(\phi-\phi')\delta\big(\theta-\theta'-\iota(\phi-\phi')\big)\delta(\rho-\rho') \qquad (44)$$

and $H(\zeta)$ is the unit step function [see Eq. (A9)]. Upon Fourier-decomposing δb and defining $\overline{\zeta} \doteq \phi - \overline{\phi}$, one finds

$$D = \int_0^\infty d\overline{\zeta} \sum_{m'n'} \sum_{\overline{m}\,\overline{n}} \langle \delta\widehat{b}_{m'n'}(\rho)\delta\widehat{b}_{\overline{m}\,\overline{n}}(\rho)\rangle \times e^{i(m'+\overline{m})\theta} e^{-i(n'+\overline{n})\phi} e^{i(\overline{n}-\iota\overline{m})\overline{\zeta}}. \quad (45)$$

This is a complicated angle-dependent expression. However, since the principal role of D is merely to broaden the resonances in the $\langle\mu\rangle$ equation, we argue that its size is more important than its detailed form. One can simplify expression (45) by using one of two closely-related arguments. First, one can assert that distinct Fourier components are uncorrelated. That assumption is legitimate in a statistically homogeneous geometry, and a homogeneity assumption is plausible for a stochastic magnetic field with short correlation length. Alternatively, one can average D over θ and ϕ by arguing that the D should be representative of the (assumed homogeneous) stochastic region as a whole. Either way, one gets the angle-independent formula

$$D = \int_0^\infty d\overline{\zeta} \sum_{\overline{m}\,\overline{n}} \langle |\delta\widehat{b}_{\overline{m}\,\overline{n}}|^2\rangle e^{i\kappa_{\|\overline{m}\,\overline{n}}\overline{\zeta}}. \quad (46)$$

(The general Green's-function formalism to be described in Sec. IV and Appendix B can in principle handle an angle-dependent D, such as might arise from weak poloidal variation of $\langle\delta\widehat{b}^2\rangle$.) Because the spacing between rational surfaces is small, it is appropriate to treat $\kappa_\|$ as a continuous variable (see discussion in the next paragraph) and introduce the spectral density $\widehat{C}_n(\kappa_\|)$, normalized such that

$$\langle\delta b^2\rangle = \sum_{\overline{n}} \int_{-\infty}^{\infty} \frac{d\overline{\kappa}_\|}{2\pi}\, \widehat{C}_{\overline{n}}(\overline{\kappa}_\|) = \sum_{\overline{m}\,\overline{n}} \langle |\delta\widehat{b}_{\overline{m}\,\overline{n}}|^2\rangle. \quad (47)$$

Formula (46) then becomes

$$D = \int_0^\infty d\overline{\zeta}\, C_L^{\rm QL}(\overline{\zeta}), \quad (48)$$

where

$$C_L^{\rm QL}(\zeta) \doteq \int_{-\infty}^{\infty} \frac{d\overline{\kappa}_\|}{2\pi}\, e^{i\overline{\kappa}_\|\zeta} \sum_{\overline{n}} \widehat{C}_{\overline{n}}(\overline{\kappa}_\|) \quad (49)$$

is the quasilinear approximation to the *Lagrangian correlation function.* (Lagrangian implies measurement along the field lines.) [Compare Eq. (48) to Eqs. (28).] It is reasonable to expect $C_L(\zeta)$ to decay as $|\zeta| \to \infty$. The *correlation length* or *integral scale* $\zeta_{\rm ac}$ is defined by

$$\zeta_{\rm ac} \doteq C^{-1} \int_0^\infty d\zeta\, C_L(\zeta), \quad (50)$$

where $C \doteq C(\zeta = 0) = \langle\delta b^2\rangle$. Thus

$$D = C\zeta_{\rm ac}. \quad (51)$$

In the quasilinear approximation (49), formula (50) can be written as $\zeta_{\rm ac}^{\rm QL} = \frac{1}{2}C_L(k_\| = 0)/C$ or, in terms of the spectral width $\Delta\kappa_\|$ defined from $C = (2\pi)^{-1}\Delta\kappa_\|\widehat{C}_L(\kappa_\| = 0)$, $\zeta_{\rm ac}^{\rm QL} = \pi/\Delta\kappa_\|$. Thus

$$D^{\rm QL} = \pi\langle\delta b^2\rangle\Delta\kappa_\|^{-1}. \quad (52)$$

Such formulas are well known in the literature on stochastic magnetic fields.[2]

The key assumption in this argument is that $\kappa_\|$ may be treated as a continuous variable even though it is built from discrete Fourier mode numbers. This issue is well known in the closely analogous problem of the quasilinear velocity-space diffusion coefficient for electrostatic Vlasov turbulence.[3,35] The resolution is understood to be that in the stochastic regime (i.e., when the Chirikov criterion for stochasticity is satisfied) the resonance broadening, contained in Eq. (40) but neglected in the approximate solution (44), is large enough to justify the continuum approximation. Note that the final formula for $D^{\rm QL}$ does not depend on the strength of the resonance broadening. That is, if we think of the stochasticity as providing an extra damping with decay length ζ_D, we are working in the quasilinear limit $\zeta_{\rm ac} \ll \zeta_D$. That is the case when the diffusion is sufficiently weak, i.e., when one is not too far above the threshold for stochasticity.

The appearance of the spectral component at $\kappa_\| = 0$ tells us that the diffusion of the magnetic lines is a resonant phenomenon. Frequently one evaluates D by performing the $\overline{\zeta}$ integration first in Eq. (46); in that procedure, and in the absence of resonance broadening, the quantity $\delta(\overline{\kappa}_\|)$ appears [analogous to the $\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})$ in Vlasov quasilinear theory]. This unfortunately introduces the embarrassment of a Dirac delta function evaluated with a discretely varying (in $\overline{m}$ and $\overline{n}$) argument; the resulting $D(\rho)$ would be a singular function of ρ. The problem is again cured by passing to the continuum limit or by introducing some resonance broadening. However, both physically and mathematically, introducing the Lagrangian correlation function and performing the $\overline{\zeta}$ integral last is the better way to proceed. (These ideas have a long history. In plasma physics, they go back to at least the 1976 PhD dissertation of Tetreault.[36] For fluids, basic notions of the Lagrangian correlation function were already considered by G. I. Taylor[37] in 1921.)

To this point, we have ignored periodicity considerations in the calculation of D. Clearly the toroidal periodicity constraint should affect the value of D when $\zeta_{\rm ac}$ becomes greater than or comparable to 2π. In that case, one must use more sophisticated, periodically constrained, response functions; see the discussion in Appendix B. In practice, it is frequently found that correlation lengths are of the order of the connection length qR, where R is the major radius. That implies $\zeta_{\rm ac} \sim qR/2\pi R = q/2\pi$, so one is marginally in the regime where periodicity constraints on D are unimportant. Of course, that is consistent with our initial assumption of "short" correlation lengths, to which we restrict ourselves here.

Next we consider the contribution to $\langle \delta b\, \delta\mu \rangle$ due to the internal incoherent noise [term (b)]. This term vanishes because

$$\langle \delta b\, \delta\mu \rangle^{(\mathrm{b})} = \langle \delta b\, R\, \delta f^{\mathrm{int}} \rangle \sim \langle \delta b\, R\, \delta b\, \delta\xi \rangle = 0, \tag{53}$$

the last result following because $\delta\xi$ is statistically independent of $\delta\boldsymbol{B}$ and has zero average.

Finally, we must assess the contribution of δf^{ext} [term (c)] to $\langle \delta b\, \delta\mu \rangle$. The answer is not immediate because, as we remarked two paragraphs above, δf^{ext} is a functional of $\delta\boldsymbol{B}$ and is therefore cross-correlated with δb. The theory of passive advection in the presence of such cross correlations is relatively unfamiliar, although some calculations have been done.[38] In Appendix A we work out an explicit example that demonstrates the consequences of the cross correlation X. There one sees that X effectively behaves as an $O(\langle \delta b^2 \rangle)$ correction to $\langle f^{\mathrm{ext}} \rangle$. Since for our problem $\langle f^{\mathrm{ext}} \rangle \sim \boldsymbol{\nabla} \cdot \boldsymbol{j}_\perp$ is of zeroth order in δb, we shall neglect term (c). (This argument fails for $\widehat{\mu}_{00}$; however, we will specify $\widehat{\mu}_{00}$ as part of the definition of the equilibrium, not determine it from the $\langle \mu \rangle$ equation.)

4. Role of fluctuating terms in Ampère's law

Although the incoherent noise makes no direct contribution to the evolution of $\langle \mu \rangle$, it obviously does contribute to $\delta\mu$, which appears on the right-hand side of Ampère's law. If that contribution were retained, the complexity of the problem would increase considerably since one would have to work explicitly with many stochastic realizations and generate appropriate samples of $\delta\xi$ at each stage of the field-line integration that determines $\langle \mu \rangle$. Since the representation (39) describes a closure approximation, it would also be unclear how the random statistics of $\delta\xi$ would affect the detailed structure of the stochastic field. Fortunately, we can argue that when the stochastic component is small δf^{int} is negligible relative to δf^{ext}. The latter involves the fluctuating $\boldsymbol{j}_\perp$, which is the fluctuating part of Eq. (8). That involves either $\delta\boldsymbol{B}$ or pressure fluctuations δp. Crudely speaking, those arise at linear order, whereas the incoherent noise is nonlinear and should therefore be small.

Such an argument is dangerous and incomplete; the sizes of stochastic terms may differ in mean square from nominal orderings. In Appendix A, we work out a tractable model in complete detail and focus in particular on the effects of the incoherent noise. In fact, that noise does turn out to be important at scales long compared to the appropriate correlation scale, as its presence is necessary in order to guarantee conservation of quadratic nonlinear invariants. However, we argue in Appendix A that on the scale at which the stochastic field saturates the effect of the incoherent noise should be small. We shall therefore neglect its contribution to $\delta\mu$.

Whether fluctuations are important at all on the right-hand side of Ampère's law appears to depend on the application. Those terms arise only in stochastic regions, so one must inquire into the dominant mechanism for the generation of resonant harmonics. In stellarators, which are intrinsically nonaxisymmetric, the metric factors involved in the curl operator depend on both θ and ϕ; nonlinear beats between that dependence and nonresonant harmonics of $\langle \boldsymbol{B} \rangle$ can drive substantial resonant components. Therefore, for stellarators we shall ignore all fluctuating terms on the right-hand side of Ampere's law.

This argument does not apply to axisymmetric tokamaks, for which the metric factors are independent of ϕ. The fluctuating terms are therefore crucial and must be retained, at least through first order. Explicitly,

$$\delta\boldsymbol{j} = \delta\mu \langle \boldsymbol{B} \rangle + \langle \mu \rangle \delta\boldsymbol{B} + \delta\mu\, \delta\boldsymbol{B} - \langle \delta\mu\, \delta\boldsymbol{B} \rangle + \delta\boldsymbol{j}_\perp, \tag{54}$$

where from Eq. (8) $\delta\boldsymbol{j}_\perp$ involves $\delta\boldsymbol{B}$ and δp. We shall neglect the small products of fluctuating quantities. Because $\delta\boldsymbol{B}$ is known at any step in the iterative solution of Ampère's law, calculation of any term involving $\delta\boldsymbol{B}$ presents no problem; that includes term (a) in Eq. (41). Term (c) in Eq. (41) involves δp, as does $\delta\boldsymbol{j}_\perp$. One can estimate the size of δp as follows. From Eq. (5), the basic size of the small terms $\boldsymbol{g}$ [Eq. (3)] is $O(\delta B^2)$. From Eq. (4), nonresonant parts of δp are also $O(\delta B^2)$ and therefore negligible. The maximum size of a resonant part is $O(g/\eta_D)$; since we show in Sec. IV C that $\eta_D = O(D^{1/3}) = O(\delta B^{2/3})$, one estimates that the resonant $\delta p = O(\delta B^{4/3})$. This is slightly smaller than the basic $\delta\boldsymbol{B}$; furthermore, it exists only in a narrow stochastic layer [whose width is $O(\delta B^{2/3})$]. Therefore, we will neglect δp altogether.

There are two sources of $\delta\boldsymbol{B}$. First, a seed $\delta\boldsymbol{B}$ can be introduced externally by either field errors or purposely produced nonaxisymmetric fields, such as those used to suppress ELMs. The plasma responds to the seed $\delta\boldsymbol{B}$ as dictated by Ampère's law to produce the true $\delta\boldsymbol{B}$ of the equilibrium. Second, a $\delta\boldsymbol{B}$ can arise spontaneously from a saturated nonaxisymmetric instability, such as a tearing mode or resistive wall mode. In each case, the Fourier decomposition of $\delta\boldsymbol{B}$ in magnetic coordinates reveals a broad Fourier spectrum that generally includes resonant components corresponding to the rational surfaces in the plasma. This effect is of first order in δB and can therefore be expected to dominate the second-order coupling terms in Eq. (54).

IV. PERIODIC SOLUTION IN THE WEAK DIFFUSION LIMIT

In this section, we solve the periodic generalization of Eq. (27) for the Pfirsch-Schlüter current under the assumption that the diffusion terms are relatively weak. For definiteness, we adopt the toroidal flux ψ as the radial coordinate ρ.

A. Introductory remarks on periodic Green's functions

For maximal generality, we shall use the method of Green's functions. The periodicity requirements raise special concerns and interesting subtleties, so the formalism is reviewed and illustrated in Appendix B. That discussion describes several equivalent formulations distinguished by the boundary conditions applied to Green's function. When the stochastic diffusion is weak there may be numerical advantages to the use of homogeneous boundary conditions, as is traditionally done in boundary-value problems; see Sec. B 3 c. However, the simplest formal solution involves a function $G_{2\pi}$, constrained to be 2π-periodic in both θ and ϕ, such that

$$\langle\mu\rangle(\boldsymbol{u}) = \int_0^{\psi_{\rm wall}} d\overline{\psi} \int_0^{2\pi} d\overline{\theta} \int_0^{2\pi} d\overline{\phi}\, G_{2\pi}(\boldsymbol{u};\overline{\boldsymbol{u}})\langle f\rangle(\overline{\boldsymbol{u}}). \quad (55)$$

This formula is to be used for all Fourier harmonics $(m,n) \neq (0,0)$; again, $\widehat{\mu}_{00}$ is to be specified.

It is not immediately obvious what equation $G_{2\pi}$ should obey. For an initial-value problem, G should obey a diffusion equation along the magnetic lines, as discussed in Sec. III C 2. In a doubly periodic torus, however, one can traverse magnetic lines in either of the forward or backward directions, and the physics should look diffusive in either direction. That is, if one chooses $\zeta = 0$ as an arbitrary origin, one intuitively expects an equation of the form

$$\partial_\zeta\langle\mu\rangle - \mathrm{sgn}(\zeta)D\nabla^2\langle\mu\rangle = \langle f\rangle. \quad (56)$$

In such situations, a standard way of proceeding is to introduce $+$ and $-$ parts according to $A^\pm = H(\pm\zeta)A(\zeta)$ ($A = \langle\mu\rangle$ or $\langle f\rangle$) and calculate each part separately. Thus, we will first find the appropriately periodic solution $\langle\mu\rangle^+$ for forward traversal. Then we obtain the analogous one $\langle\mu\rangle^-$ for backward traversal. Finally, we superimpose $\langle\mu\rangle^+$ and $\langle\mu\rangle^-$ to obtain the final solution. Symmetry considerations described in Sec. B 3 b show that $\langle\mu\rangle = \frac{1}{2}(\langle\mu\rangle^+ + \langle\mu\rangle^-)$.

Although Green's formalism is quite general, we will be able to make analytical progress only by simplifying the diffusion operator. Although D^ψ and D^θ can be assumed to be of comparable size, diffusion will be important only in a narrow layer around a rational surface. Accordingly, the second derivative with respect to ψ will dominate, so

$$D\nabla^2 \approx D^\psi \frac{\partial^2}{\partial\psi^2}. \quad (57)$$

This argument justifies both passing D^ψ through the first ψ derivative and ignoring the angle dependence of J in the formula (15) for the divergence in general geometry.

B. Field-line-following coordinates and periodicity

Let $G_{2\pi}^\pm(\psi,\theta,\phi;\psi',\theta',\phi')$ be Green's function for forward ($+$) or backward ($-$) traversal. It is convenient to transform to the field-line-following coordinates

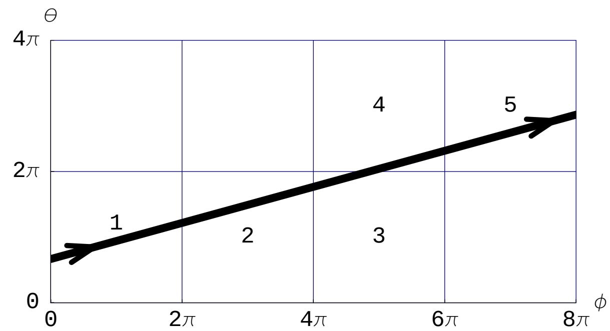

FIG. 1: A portion of the covering space for a magnetic flux surface, showing a sample magnetic field line. Coordinates are chosen such that field lines on good surfaces are straight [with slope $\iota(\psi)$]. The numbers indicate the consecutive domains that are pierced as the line is traversed in the direction of increasing ϕ. Periodicity wraps a line into the fundamental square, as shown in Fig. 2.

$$\chi \doteq \psi, \quad \alpha \doteq \theta - \iota(\psi)\phi, \quad \zeta \doteq \phi. \quad (58)$$

With $\overline{G}^+(\chi,\alpha,\zeta;\chi',\alpha',\zeta') \doteq G_{2\pi}^+(\psi,\theta,\phi;\psi',\theta',\phi')$, one finds the diffusion equation

$$\partial_\zeta\overline{G}^+ - D\overline{\nabla}^2\overline{G}^+ = \delta(\chi-\chi')\delta(\alpha-\alpha')\delta(\zeta-\zeta'), \quad (59)$$

where the effective Laplacian in the new coordinates is

$$D\overline{\nabla}^2 = D^\psi\left(\frac{\partial}{\partial\chi} - \iota'\zeta\frac{\partial}{\partial\alpha}\right)^2, \quad (60)$$

with $\iota' \doteq \partial\iota/\partial\psi$ describing the magnetic shear. (Note the well-known secular ζ dependence that is introduced by the variable transformation.) As is well known, the expression of physical periodicity in the field-line-following coordinates is nontrivial.[39] Specifically, one has

$$\overline{G}(\alpha+2\pi,\zeta) = \overline{G}(\alpha,\zeta), \quad (61a)$$

$$\overline{G}(\alpha,\zeta+2\pi) = \overline{G}(\alpha+2\pi\iota,\zeta). \quad (61b)$$

Thus α is periodic but ζ is not. The α periodicity permits Fourier transformation according to $\overline{G}_m(\zeta) = (2\pi)^{-1}\int_0^{2\pi} d\alpha\, e^{-im\alpha}\overline{G}(\alpha,\zeta)$. The condition (61b) is then stated as

$$\overline{G}_m(\zeta+2\pi) = e^{i\beta_m}\overline{G}_m(\zeta), \quad (62)$$

where

$$\beta_m \doteq 2\pi m\iota. \quad (63)$$

C. The causal field-line propagator

One way of dealing with the aperiodicity in ζ is to work in the infinite covering space[40] depicted in Figs. 1

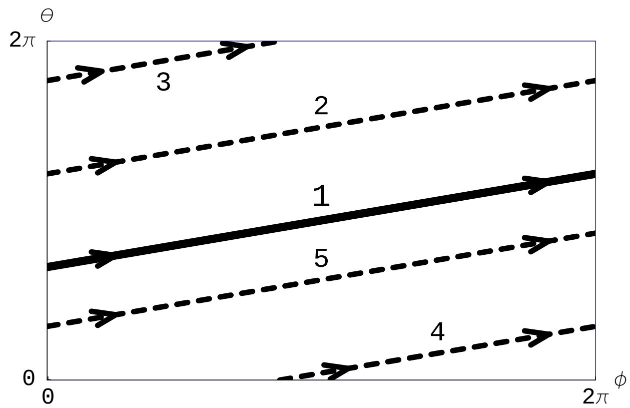

FIG. 2: The fundamental domain of the covering space is pierced repeatedly by the magnetic field line as it winds around the torus. The periodicity constraint on Green's function captures contributions from all segments of the line and permits integration over only the fundamental domain $\phi \in [0, 2\pi)$, i.e., effectively along just segment 1.

and 2, then construct properly periodic solutions by using a shifted-sum representation (i.e., the Poisson sum formula). The technique is illustrated for a closely related example in Appendix B. (The material in Sec. B 2 b is integral to the present discussion and should be surveyed at this time.) As described in Sec. B 2 b, the first task is to determine the causal function R that obeys

$$\begin{aligned}&(\partial_\zeta - D\overline{\nabla}^2)R(\chi,\alpha,\zeta;\chi',\alpha',\zeta')\\ &\quad = \delta(\chi-\chi')\delta(\alpha-\alpha')\delta(\zeta-\zeta') \quad [R(\zeta<\zeta')=0]. \quad (64)\end{aligned}$$

Here ζ is defined on the real line $\mathcal{R}$; the α dependence remains periodic according to Eq. (61a). Upon Fourier transformation with respect to $\alpha - \alpha'$, one has

$$\begin{aligned}&\left[\frac{\partial}{\partial\zeta} - D^\psi\left(\frac{\partial}{\partial\chi} - i\iota' m\zeta\right)^2\right] R_m(\chi,\zeta;\chi',\zeta')\\ &\quad = \delta(\chi-\chi')\delta(\zeta-\zeta'). \qquad (65)\end{aligned}$$

D^ψ and ι' may be evaluated at χ' because the stochasticity is important only near the resonance. Equation (65) has the same form as Eq. (23) for constant D after the streaming term has been transformed away if one identifies $\alpha \to x$, $\chi \to v$, $\zeta \to t$. The equation can be completely solved analytically by Fourier transformation with respect to χ; the result is quoted in Ref. 3, Appendix E.1.2, for the test-particle problem. One finds

$$\begin{aligned}R_m(\chi,\zeta;\chi',\zeta') &= H(\zeta-\zeta')\exp[im\iota'(\chi-\chi')\tfrac{1}{2}(\zeta+\zeta')]\\ &\quad\times \Phi(\chi-\chi' \mid 2D^\psi\zeta)\\ &\quad\times \exp[-\tfrac{1}{12}(m\iota')^2 D^\psi(\zeta^3-\zeta'^3)], \quad (66)\end{aligned}$$

where $\Phi(\psi \mid \sigma^2) \doteq (2\pi\sigma^2)^{-1/2}\exp(-\psi^2/2\sigma^2)$ is the diffusion Green's function. The last factor introduces the characteristic "length" (dimensionless because ζ is the toroidal angle)

$$\zeta_D \doteq (\tfrac{1}{4}m^2\iota'^2 D^\psi)^{-1/3}. \qquad (67)$$

[The factor of $\frac{1}{4}$ is included to bring formulas (68) and (84a) into a standard form.] ζ_D describes the well-known resonance-broadening effect of Dupree.[28] [Had the θ diffusion been included, the length $\zeta_\perp \doteq (m^2D^\theta)^{-1}$ would also have appeared; however, in the regime of interest one has $\zeta_D \ll \zeta_\perp$.] When the radial dispersion $\sigma^2 = 2D^\psi\zeta$ is evaluated at ζ_D, σ^2 is also small, so we shall approximate $\Phi(\chi-\chi') \approx \delta(\chi-\chi')$. That reduces formula (66) to

$$R_m(\chi,\zeta;\chi',\zeta') \approx H(\zeta-\zeta')K_m(\zeta;\zeta')\delta(\chi-\chi'), \quad (68)$$

where

$$K_m(\zeta;\zeta') \doteq K_m(\zeta)/K_m(\zeta'), \qquad (69a)$$

$$K_m(\zeta) \doteq \exp[-\tfrac{1}{3}(\zeta/\zeta_D)^3]. \qquad (69b)$$

Note that K possesses the semigroup property $K(\zeta;\overline{\zeta})K(\overline{\zeta};\zeta') = K(\zeta;\zeta')$ even though it is not translationally invariant.

That ζ_D comprises effects that are $\frac{1}{3}$ diffusive and $\frac{2}{3}$ shear-related was originally noted in the context of magnetic field lines by Krommes[2,25] and has been exploited in various analogous situations by Hirshman and Molvig[41] and Biglari, Terry, and Diamond.[42] In the present context, one estimates the spread η in $\kappa_\parallel$ due to diffusion over a width $\Delta\psi$ [namely $\eta = \Delta(\iota m - n) \approx m\iota'\Delta\psi$] as follows. Let $\zeta_{\rm nl}$ be the distance (to be determined) over which nonlinear mixing effects become important. Over a distance $\zeta_{\rm nl}$ along an unperturbed reference line on a rational surface at ψ_0, the actual line has diffused radially an amount $\Delta\psi^2 = 2D^\psi\zeta_{\rm nl}$. But because of magnetic shear, the line also has moved poloidally according to

$$\frac{d\Delta\theta}{d\zeta} = \iota(\psi) - \iota(\psi_0) \approx \iota'\Delta\psi, \qquad (70)$$

or $\Delta\theta = \iota'\Delta\psi\zeta_{\rm nl}$. Substantial mixing should occur at $m\Delta\theta \sim 1$, which leads to $\zeta_{\rm nl} = (m\iota'\Delta\psi)^{-1}$. That immediately gives $\eta = \zeta_{\rm nl}^{-1}$. Substituting from the diffusion law for $\Delta\psi$ then determines $\zeta_{\rm nl} \sim (m^2\iota'^2D^\psi)^{-1/3} \sim \zeta_D$.

Do not confuse the nonlinearly determined $\zeta_{\rm nl}$ (or the resonance-broadening width $\eta = \zeta_{\rm nl}^{-1}$) with the quasilinear correlation length $\zeta_{\rm ac}^{\rm QL}$ determined from the spectral width $\Delta\kappa_\parallel = (\zeta_{\rm ac}^{\rm QL})^{-1}$. By definition of the quasilinear regime, one has $\zeta_{\rm ac}^{\rm QL} \ll \zeta_{\rm nl}$. $\Delta\kappa_\parallel$ is related to the number of excited m's and n's, whereas the Δ in $\eta = \Delta[\iota(\psi)m - n]$ acts on the ψ coordinate for fixed m and n.

D. Construction of the causal periodic Green's function

We must now construct the appropriate periodic Green's function from R. Upon omitting the χ dependence for conciseness, one has

$$G^{+}_{2\pi}(\theta,\phi;\theta',\phi') = \sum_m e^{im(\theta-\theta')} e^{-im\iota(\phi-\phi')} \overline{G}^{+}_m(\phi;\phi'). \tag{71}$$

The phase factor involving ι arises from the transformation from the field-line coordinates back to the original periodic ones. A function that respects the periodicity constraint (61b) is

$$\overline{G}^{+}_m(\phi;\phi') = A_m R_{2\pi,m}(\phi;\phi'), \tag{72}$$

where

$$R_{2\pi,m}(\phi;\phi') \doteq \sum_{l=-\infty}^{\infty} e^{i\beta_m l} R_m(\phi - 2\pi l;\phi') \tag{73}$$

and A_m is a constant to be determined from the jump condition

$$\lim_{\epsilon\to 0}[\overline{G}^{+}_m(\phi'+\epsilon;\phi') - \overline{G}^{+}_m(\phi'-\epsilon;\phi')] = 1. \tag{74}$$

Now for any $\phi' \in \mathcal{I}_L$, where $\mathcal{I}_L \doteq [2\pi L, 2\pi(L+1))$, the only discontinuity in $\mathcal{I}_L$ arises from $R(\phi - 2\pi L;\phi')$, which contributes a unit jump (see Fig. 3). Thus the jump in $\overline{G}^{+}$ is $A_m e^{i\beta_m L}$, so $A_m = e^{-i\beta_m L}$. We need work only in the fundamental domain $\mathcal{I}_0$, in which case $A_m = 1$ and contributions from $l > 0$ do not enter due to the one-sided (causal) nature of $R \propto H(\phi - \phi')$; for example, $R(\phi - 2\pi;\phi') = 0$. Thus

$$\overline{G}^{+}_m(\phi;\phi') = \sum_{l\le 0} e^{i\beta_m l} R(\phi - 2\pi l;\phi') \quad (\phi,\phi' \in [0,2\pi)). \tag{75}$$

Upon writing $\overline{G}^{+}_m(\phi;\phi') = \overline{G}^{+>}_m(\phi;\phi')H(\phi-\phi') + \overline{G}^{+<}_m(\phi;\phi')H(\phi'-\phi)$ and defining

$$K_{2\pi,m}(\phi) \doteq \sum_{l\le 0} e^{i\beta_m l} K_m(\phi - 2\pi l), \tag{76}$$

one obtains

$$\overline{G}^{+>}_m(\phi;\phi') = K_{2\pi,m}(\phi)/K_m(\phi'), \tag{77a}$$

$$\overline{G}^{+<}_m(\phi;\phi') = e^{-i\beta_m} K_{2\pi,m}(\phi+2\pi)/K_m(\phi'), \tag{77b}$$

where in obtaining the last expression we noted that the $l = 0$ term does not contribute to formula (75) for $\phi < \phi'$, then introduced $\bar{l} \doteq l + 1$ to shift the sum from $l \le -1$ to $\bar{l} \le 0$. Note that

$$\overline{G}^{+<}_m(\phi;\phi') = e^{-i\beta_m} \overline{G}^{+>}_m(\phi+2\pi;\phi'); \tag{78}$$

this is illustrated in Fig. 3.

Although the formulas (77) are usable as they stand, it is instructive to rewrite them in a way that makes the jump condition and the effects of periodicity more explicit. Define

$$\mathcal{D}^{-}_m(\phi') \doteq 1 - \Delta_m(\phi'), \tag{79a}$$

$$\Delta_m(\phi') \doteq e^{-i\beta_m} K_{2\pi,m}(\phi'+2\pi)/K_{2\pi,m}(\phi'). \tag{79b}$$

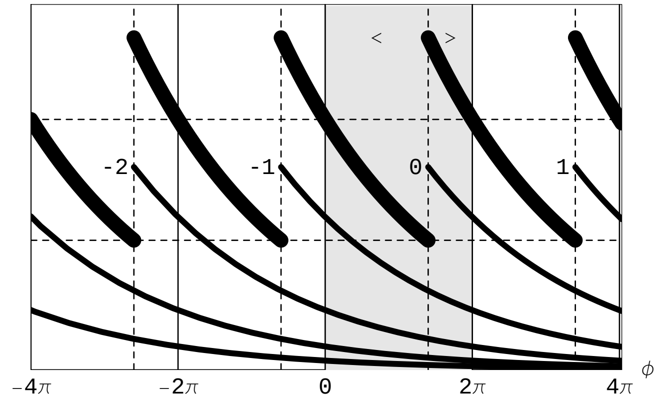

FIG. 3: Illustration of the periodic Green's function $G_{2\pi}(\phi;\phi')$ (thick curve) and its constituents (thin solid curves, labeled by their l values). Each constituent involves the phase factor $e^{i\beta_m l}$. The vertical dotted lines indicated $\phi' + 2\pi l$ for a representative ϕ'. The fundamental domain $\mathcal{I}$ is shown in gray, and the parts corresponding to $G^<$ and $G^>$ are indicated by $<$ and $>$. The horizontal dotted lines indicate the vertical extent of $G^{<}_{2\pi}$ and show that the result $G^{<}_{2\pi}(\phi;\phi') = e^{-i\beta_m} G^{>}_{2\pi}(\phi+2\pi;\phi')$ is valid.

Then one finds

$$\overline{G}^{+>}_m(\phi;\phi') = \frac{1}{\mathcal{D}^{-}_m(\phi')} \frac{K_{2\pi,m}(\phi)}{K_{2\pi,m}(\phi')}, \tag{80a}$$

$$\overline{G}^{+<}_m(\phi;\phi') = \frac{\Delta_m(\phi)}{\mathcal{D}^{-}_m(\phi')} \frac{K_{2\pi,m}(\phi)}{K_{2\pi,m}(\phi')}. \tag{80b}$$

The jump condition $G^{+>}(\phi';\phi') - G^{+<}(\phi';\phi') = 1$ is then obvious. For the simple model $K(\zeta) = e^{-\eta\zeta}$ (no m dependence, and $\iota = 0$) discussed in Appendix B, the sum defining $K_{2\pi}$ can be worked out explicitly by summing a geometric series [see Eqs. (B18)]; one then finds

$$K_{2\pi}(\phi) = e^{-\eta\phi}/\mathcal{D}^{-}, \quad \Delta = e^{-2\pi\eta}, \tag{81}$$

which recovers the result (B13).

The Δ terms arise from the periodicity constraint. The relative importance of $\overline{G}^{>}_m$ and $\overline{G}^{<}_m$ therefore depends on the strength of the diffusion, i.e., on whether ζ_D is greater than or less than 2π. This remark should not be confused with our discussion of the quasilinear magnetic diffusion coefficient $D^{\rm QL}$ in Sec. III C 2, where we argued that periodicity constraints are unimportant for $D^{\rm QL}$ provided that the Lagrangian correlation length $\zeta_{\rm ac}$ was less than 2π. [Although a small amount of resonance broadening $\zeta_D^{-1} \ll (\zeta_{\rm ac}^{\rm QL})^{-1}$ is necessary in order to properly define the calculation, the value of $D^{\rm QL}$ does not sensibly depend on ζ_D.] In the present calculation of $\overline{G}$, the comparison is instead between ζ_D and 2π. The difference is that $\overline{G}_m$ describes the response of a single Fourier mode, whereas $\zeta_{\rm ac}^{\rm QL}$ describes the phase mixing associated with the broadband nature of the mode spectrum. Thus, if $\zeta_{\rm ac} \lesssim 2\pi$ and $\zeta_D \gg \zeta_{\rm ac}$, one is in the regime in which $R(\zeta)$ decays little in the fundamental domain

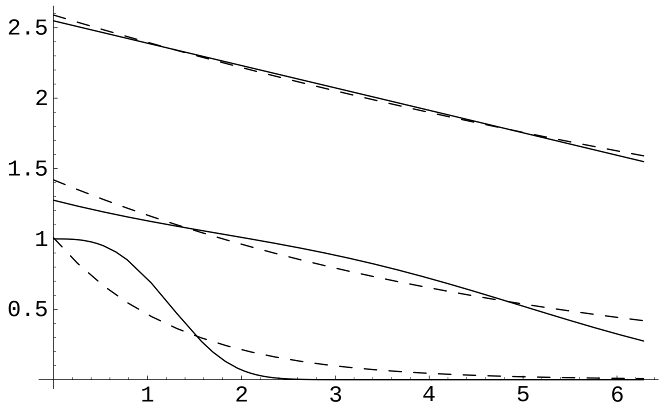

FIG. 4: $R_{2\pi}(\zeta)$ for various values of the resonance-broadening parameter $\Delta\omega$. The solid curves display the result for the generating function $R(\zeta) = H(\zeta)\exp[-\frac{1}{3}(\Delta\omega\zeta)^3]$; from bottom to top, $\Delta\omega = 1$, 0.25, and 0.1. Periodicity corrections are unimportant for $\Delta\omega = 1$. The dashed curves display the generating function $R(\zeta) = H(\zeta)\exp(-\eta_D\zeta)$ with the value of η_D chosen so that the areas under the generating functions agree.

$[0, 2\pi)$, $R_{2\pi}$ differs strongly from R (see Fig. 4), and periodicity constraints are important for the individual constituents of $\overline{G}_m$.

However, it is not necessarily the case that the Δ_m corrections find their way into the final solution for $\langle\mu\rangle$. In the translationally invariant exponential-damping model discussed in Appendix B, the contributions of $\overline{G}_m^{+>}$ and $\overline{G}_m^{+<}$ sum in just such a way that the Δ_m correction disappears; see Eq. (B10). The same behavior can be argued to occur here, at least approximately. Upon introducing $\omega_{mn} \doteq n - \iota m = -\kappa_{\|mn}$, Fourier transforming the generalization of Eq. (B15), then interchanging the order of integration, one obtains

$$\langle\widehat{\mu}\rangle^+_{mn} = \int_0^{2\pi} d\phi\, e^{in\phi}\left[\int_0^{\phi} d\overline{\phi}\, e^{-im\iota(\phi-\overline{\phi})}\left(\frac{K_{2\pi,m}(\phi)}{K_m(\overline{\phi})}\right) + \int_{\phi}^{2\pi} d\overline{\phi}\, e^{-im\iota(\phi-\overline{\phi})}\left(\frac{e^{-i\beta_m}K_{2\pi,m}(\phi+2\pi)}{K_m(\overline{\phi})}\right)\right]\langle\widehat{f}\rangle_m(\overline{\phi}) \tag{82a}$$

$$= \int_0^{2\pi} d\overline{\phi}\, e^{in\overline{\phi}}\langle\widehat{f}\rangle_m(\overline{\phi})\left[\int_{\overline{\phi}}^{2\pi} d\phi\, e^{i\omega_{mn}(\phi-\overline{\phi})}\left(\frac{K_{2\pi,m}(\phi)}{K_m(\overline{\phi})}\right) + \int_0^{\overline{\phi}} d\phi\, e^{i\omega_{mn}(\phi-\overline{\phi})}\left(\frac{e^{-i\beta_m}K_{2\pi,m}(\phi+2\pi)}{K_m(\overline{\phi})}\right)\right] \tag{82b}$$

$$= \int_0^{2\pi} d\overline{\phi}\, e^{in\overline{\phi}}\langle\widehat{f}\rangle_m(\overline{\phi})\int_{\overline{\phi}}^{\overline{\phi}+2\pi} d\phi\, e^{i\omega_{mn}(\phi-\overline{\phi})}\left(\frac{K_{2\pi,m}(\phi)}{K_m(\overline{\phi})}\right) \tag{82c}$$

$$= \int_0^{2\pi} d\overline{\phi}\, e^{in\overline{\phi}}\langle\widehat{f}\rangle_m(\overline{\phi})J_{mn}(\overline{\phi}), \tag{82d}$$

where

$$J_{mn}(\overline{\phi}) \doteq \int_0^{\infty} d\zeta\, e^{i\omega_{mn}\zeta}\left(\frac{K_m(\zeta+\overline{\phi})}{K_m(\overline{\phi})}\right). \tag{83}$$

In obtaining Eq. (82c), we used the substitution $\widehat{\phi} = \phi + 2\pi$ in the last integral of Eq. (82b) and noted that $\exp(-2\pi i\omega_{mn}) = \exp(i\beta_m)$. Equation (82d) then follows from the definition (76).

A qualitative understanding of Eq. (82d) may be obtained by modeling the $\exp(-\phi^3)$ dependence of $K(\phi)$ by $\exp(-\eta_D\phi)$, where η_D is chosen to match the $\omega = 0$ values of $R(\omega)$ (i.e., to match the area under the generating function); such an argument was first made by Dupree.[28] Note that with $\Delta\omega \doteq \zeta_D^{-1}$, one has

$$\widehat{R}_m(\omega) = \int_0^{\infty} d\zeta\, e^{i\omega\zeta}e^{-\frac{1}{3}(\Delta\omega\zeta)^3} \tag{84a}$$

$$= \pi\Delta\omega^{-1}\,\mathrm{Hi}(i\omega/\Delta\omega), \tag{84b}$$

where $\mathrm{Hi}(z)$ is the Airy function $w(z)$ that obeys[43]

$$w'' - zw = \pi^{-1} \tag{85a}$$

with the initial conditions

$$w(0) = \frac{2}{3}\,\mathrm{Bi}(0), \quad w'(0) = \frac{2}{3}\,\mathrm{Bi}'(0) \tag{85b}$$

(Bi being another Airy function). The real and imaginary parts of $\pi\,\mathrm{Hi}(i\omega)$ are displayed in Figs. 5 and 6. Thus $\eta_D/\Delta\omega = [\frac{2}{3}\pi\,\mathrm{Bi}(0)]^{-1} \approx 0.78 \equiv \overline{\eta}_D$. For the exponential-damping model, one has $K(\zeta+\overline{\phi})/K(\overline{\phi}) = K(\zeta) = \exp(-\eta_D\zeta)$ (happily independent of $\overline{\phi}$). The ϕ integral is then elementary and leads (cf. Sec. B 2 a) to the result $[-i(\omega_{mn}+i\eta_D)]^{-1}$. The $\overline{\phi}$ integral then takes the Fourier transform of $\langle\widehat{f}\rangle_{mn}$, and one obtains

$$\langle\widehat{\mu}\rangle^+_{mn} \approx \frac{\langle\widehat{f}\rangle_{mn}}{-i(\omega_{mn}+i\eta_D)}. \tag{86}$$

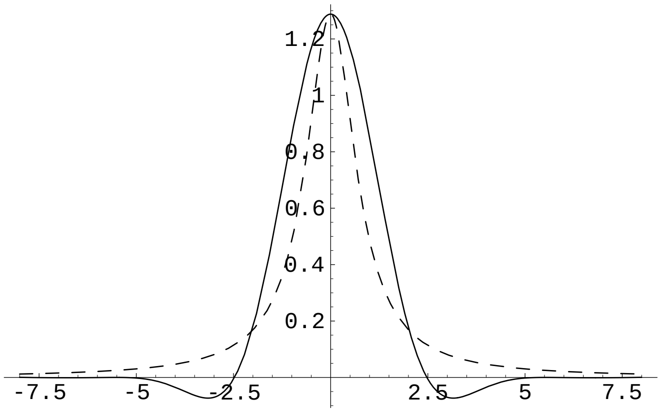

FIG. 5: Real parts of the functions $\pi\,\mathrm{Hi}(i\omega)$ (solid curve) and $[-i(\omega + i\overline{\eta}_D)]^{-1}$ (dashed curve). $\overline{\eta}_D$ is chosen so that the $\omega = 0$ values agree.

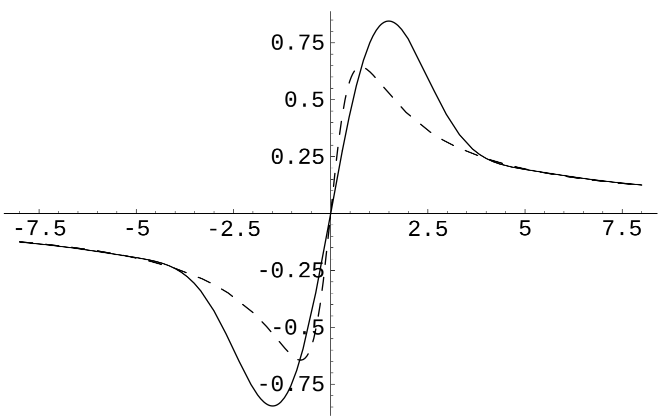

FIG. 6: Imaginary parts of the functions $\pi\,\mathrm{Hi}(i\omega)$ (solid curve) and $[-i(\omega+i\overline{\eta}_D)]^{-1}$ (dashed curve). The asymptotic behavior is ω^{-1} for large ω.

Upon working backwords through this approximation, one sees that

$$\langle\widehat{\mu}\rangle^{+}_{mn} \approx \widehat{R}_m(\omega_{mn})\langle\widehat{f}\rangle_{mn} \tag{87}$$

with $\widehat{R}_m(\omega)$ given by Eq. (84b). Figures 5 and 6 show that the approximation is not too bad even for nonzero values of ω.

In fact, formula (87) is more general than its derivation might suggest. Integration by parts of formula (83) shows that the nonresonant result $J_{mn} \sim (-i\omega_{mn})^{-1}$ emerges as the first term of an asymptotic expansion in ω^{-1}_{mn}, a useful consistency check. For resonant effects (small ω_{mn}), we consider the physical limit $2\pi\eta \ll 1$ (the periodicity constraint is essential). Now $\overline{\phi}$ is restricted to $[0, 2\pi)$, whereas contributions to the ζ integral occur even for $\zeta \gg 2\pi$. One can thus develop $J_{mn}(\overline{\phi})$ in a Taylor series $[J_{mn}(\overline{\phi}) = J^{(0)}_{mn} + O(\overline{\phi})]$, the lowest-order, $\overline{\phi}$-independent term of which is

$$J^{(0)}_{mn} = \int_0^\infty d\zeta\, e^{i\omega_{mn}\zeta}[K_m(\zeta)/K_m(0)] = \widehat{R}_m(\omega_{mn}), \tag{88}$$

since $K_m(0) = 1$. Thus formula (87) is exact for the exponential-damping model, and it correctly interpolates between the resonant and nonresonant limits in the important physical case of weak diffusion.

E. Inclusion of backward motion; final periodic solution for $\langle\mu\rangle$

$\langle\mu\rangle^{+}(\theta, \phi)$ is doubly periodic. However, it does not obey the additional requirement that the physical solution must not depend on the direction of integration. That is,

$$\langle\mu\rangle^{+}(\theta, \phi) \neq \langle\mu\rangle^{-}(\theta, \phi) \tag{89}$$

in general, where $\langle\mu\rangle^{-}$ is obtained by integrating $\overline{G}$ backward along the lines. A general solution is

$$\langle\mu\rangle(\theta, \phi) = \alpha\langle\mu\rangle^{-}(\theta, \phi) + (1-\alpha)\langle\mu\rangle^{+}(\theta, \phi), \tag{90}$$

where α is to be determined. Direction independence requires, for example, that

$$\langle\mu\rangle^{\pm}(\theta, \phi) = \langle\mu\rangle^{\mp}(\theta, \phi + 2\pi). \tag{91}$$

Then the periodicity $\langle\mu\rangle(\theta, \phi) = \langle\mu\rangle(\theta, \phi + 2\pi)$ can be rearranged to

$$(1-2\alpha)[\langle\mu\rangle^{+}(\theta, \phi) - \langle\mu\rangle^{-}(\theta, \phi)] = 0. \tag{92}$$

In view of Eq. (89), the unique solution is $\alpha = \frac{1}{2}$, so[57]

$$\langle\mu\rangle(\theta, \phi) = \frac{1}{2}[\langle\mu\rangle^{+}(\theta, \phi) + \langle\mu\rangle^{-}(\theta, \phi)]. \tag{93}$$

It is unnecessary to do an independent calculation of $\langle\mu\rangle^{-}$ since it is built from the adjoint Green's function $R^\dagger$, which is simply related to R according to $R^\dagger(\theta, \phi; \theta', \phi') = R(\theta', \phi'; \theta, \phi)$. (See the analogous discussion of a simpler model in Sec. B 3 b.) This implies that the constituents $\overline{G}^{-\gt}_m$ and $\overline{G}^{-\lt}_m$ follow directly from $\overline{G}^{+\gt}_m$ and $\overline{G}^{+\lt}_m$; for example, $\overline{G}^{-\lt}_m(\zeta; \zeta') = \overline{G}^{+\gt}_m(\zeta'; \zeta)$.

Equation (93) can be simplified if the system possesses *stellarator symmetry*, which is the generalization of time-reversal invariance to the problem of magnetic field lines.[44] In the present context, that symmetry is

$$\langle\mu\rangle(\theta, \phi) = \langle\mu\rangle(-\theta, -\phi), \tag{94a}$$

$$\langle f\rangle(\theta, \phi) = -\langle f\rangle(-\theta, -\phi). \tag{94b}$$

Equation (94a) implies that $\langle\widehat{\mu}\rangle^{-}_{mn} = \langle\widehat{\mu}\rangle^{+*}_{mn}$, so $\langle\widehat{\mu}\rangle_{mn} = \mathrm{Re}\langle\widehat{\mu}\rangle^{+}_{mn}$. Equation (94b) implies that the $\langle\widehat{f}\rangle_{mn}$ are purely imaginary. To the extent that one can write $\langle\widehat{\mu}\rangle^{+}_{mn} = \widehat{G}^{+}_{mn}\langle\widehat{f}\rangle_{mn}$ for some $\widehat{G}^{+}$ [cf. Eq. (87)], the effect of the inclusion of backward traversal is thus to take the imaginary part of the forward Green's function:

$$\langle\widehat{\mu}\rangle_{mn} = i(\mathrm{Im}\,\widehat{G}^{+}_{mn})\langle\widehat{f}\rangle_{mn}. \tag{95}$$

For the exponential-damping model (86), one obtains

$$\langle\widehat{\mu}\rangle_{mn} \approx \left(\frac{-i\kappa_{\|mn}}{\kappa_{\|mn}^2+\eta_D^2}\right)\langle\widehat{f}\rangle_{mn}, \tag{96}$$

which is just the heuristic result (16) suggested at the beginning of Sec. III B. For nonresonant modes ($\kappa_{\|mn} \gg \eta_D$) formula (86) reduces to $\langle\widehat{\mu}\rangle_{mn} \approx \langle\widehat{f}\rangle_{mn}/i\kappa_{\|mn}$, in agreement with Eq. (14). η_D sets the width of the resonance, which scales as $D^{1/3} = O(\delta B^{2/3})$.

V. NUMERICAL CALCULATION OF THREE-DIMENSIONAL EQUILIBRIA WITH STOCHASTIC REGIONS

In this section we remark on a numerical procedure for finding the equilibrium $\boldsymbol{B}$. The discussion also serves as a summary of the ideas described in the previous sections.

For the numerical calculation of 3D equilibria with stochastic field lines, it is convenient to write the equilibrium equations in the form given by Eqs. (8)–(10), so the effects of the field-line stochasticity enter entirely through Eq. (9). This is the form in which the PIES 3D equilibrium code[13] casts the equations. The PIES code solves Ampère's equation by expressing its solution in terms of that of a Poisson equation, then by inverting a matrix to solve that equation using finite differences in the radial direction and Fourier decomposition in the angles. Equation (9) is solved on good flux surfaces by transforming to magnetic coordinates.[45,46] If the pressure is assumed to be flattened in stochastic regions, the solution of Eq. (9) is trivial in those regions. However, PIES calculations for the W7AS stellarator[4] have indicated the presence of a substantial stochastic region with a nonzero pressure gradient; for those calculations, the approach described below was adopted.

As discussed in Sec. III B 3, Eq. (9) cannot in practice be accurately solved along stochastic field-line trajectories if $\boldsymbol{\nabla} p \neq \boldsymbol{0}$. We do expect, however, that the field-line integration will preserve the statistical properties of appropriately averaged quantities, and this has led us to adopt statistical methods to study an appropriately averaged $\langle\mu\rangle$. If we write the magnetic field as a sum of two pieces, $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}$, where $\boldsymbol{B}_0$ has nested flux surfaces, Eq. (9) can be rewritten in terms of the magnetic coordinates of $\boldsymbol{B}_0$ as Eq. (19). (We have assumed here that $B_0^\phi \gg B_0^\theta$ and that the three components of $\delta\boldsymbol{B}$ are of the same order, so the term containing δB^ϕ in the magnetic differential equation can be neglected.) Fourier decomposing Eq. (19) on the flux surfaces of $\boldsymbol{B}_0$, we get

$$i[\iota(\psi)m-n]\widehat{\mu}_{mn}(\psi) + \left(\frac{\delta B^\psi}{B_0^\phi}\frac{\partial\mu}{\partial\psi}\right)_{mn} + \left(\frac{\delta B^\theta}{B_0^\phi}\frac{\partial\mu}{\partial\theta}\right)_{mn} = \widehat{f}_{mn}(\psi), \tag{97}$$

where f is defined in Eq. (12). Retaining a finite set of Fourier harmonics for the numerical calculation, $-N \leq n \leq N$ and $0 \leq m \leq M$ (i.e., setting to zero Fourier amplitudes where m or n lie outside these limits), amounts to a coarse graining, which we represent by the bracket average $\langle\ldots\rangle$:

$$i[\iota(\psi)m-n]\langle\widehat{\mu}\rangle_{mn}(\psi) + \left\langle\frac{\delta B^\psi}{B_0^\phi}\frac{\partial\mu}{\partial\psi}\right\rangle_{mn} + \left\langle\frac{\delta B^\theta}{B_0^\phi}\frac{\partial\mu}{\partial\theta}\right\rangle_{mn} = \langle\widehat{f}\rangle_{mn}(\psi). \tag{98}$$

For $\delta B/B$ sufficiently small, the $O(\delta B/B)$ terms can be neglected for nonresonant modes, recovering Eq. (14). For resonant modes, $\langle\widehat{\mu}\rangle_{mn}$ increases rapidly in magnitude as the resonance surface is approached, and the $\partial\mu/\partial\psi$ term must be retained in a narrow boundary layer near the rational surface. Only the resonant Fourier mode and its harmonics need to be retained in this term. The $\partial\mu/\partial\theta$ term remains negligible relative to the other terms. It is often a good approximation to retain only the lowest Fourier harmonic of the resonant mode.

In the preceding sections of this paper we have represented the solution for $\langle\mu\rangle$ in terms of Greens functions, and we have argued that if those functions are appropriately defined the term containing δB^ψ can be approximated by a diffusion term,

$$\left\langle\frac{\delta B^\psi}{B_0^\phi}\frac{\partial G^\pm}{\partial\psi}\right\rangle \approx \pm D^\psi\frac{\partial^2 G^\pm}{\partial\psi^2}, \tag{99}$$

where D^ψ is the magnetic field-line diffusion coefficient. The quasilinear approximation to D^ψ was discussed in Sec. III C 2. For the study described in Ref. 4, the diffusion coefficient was calculated numerically by field-line following.

If the resonant Fourier components of $\langle\mu\rangle$ have relatively high mode number, their amplitude is generally small, except possibly in a region near the resonant surface. In this context, the resonance broadening plays the role of keeping the amplitude of the resonant components small, and it is adequate to use a simplified model for the resonance broadening, such as the exponential damping model discussed in Sec. IV. The calculation is insensitive to the precise form of the broadening as long as the model preserves the width of the resonance-broadening region. In the more general case where low-order rational surfaces are present, the second-order ODEs determining $\langle\widehat{\mu}\rangle_{mn}$ can be solved numerically.

In stellarator equilibria, the toroidal mode number of the resonances is a multiple of the number of periods N_p. In the W7AS stellarator equilibria studied in Ref. 4, $N_p = 5$ and $\iota/N_p \approx 0.1$, and the resonant Fourier components of the equilibrium current density are small. As expected from the above considerations, the properties of the calculated equilibria (width of the stochastic region, etc.) were found to be insensitive to the form of the resonance-broadening model.

VI. FINAL REMARKS

We have described a theory and numerical procedure for calculating the plasma equilibrium in a toroidal device that includes regions of stochastic magnetic field lines with nonzero pressure gradient. The magnetic differential equations for pressure p and $\mu \doteq j_{\|}/B$ are singular at the rational surfaces. Resonance broadening by stochastic field-line diffusion permits finite solutions that do not require p or μ to be flattened in the stochastic regions. Technical problems that were addressed include the periodicity constraints in a torus, the invariance of the answer to the direction of traversal of the field lines, and the justification of certain approximations to a nonlinear Langevin equation for μ. We focused on calculation of the equilibrium magnetic field for specified pressure and current profiles. Although we discussed how those profiles are ultimately determined by solution of the transport problem, we did not treat that problem.

At the level of fundamental nonlinear physics, the problem is conceptually challenging because in its most general form it involves all of the statistical closure issues of turbulence theory. We have not faced up to all of those. The true problem is statistically self-consistent, but we studied a passive model. In lieu of performing a fully statistical treatment of Ampère's law, we employed a hybrid statistical/Langevin approach, but then were forced to ignore nonlinear contributions to the fluctuating p and μ that enter on the right-hand side of $\boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{j}$. Although we believe that those terms are small in the practical applications of interest, a complete numerical implementation of such a hybrid scheme would be a major undertaking and has not been done. In general, the problem of plasma equilibrium in the presence of stochastic regions poses interesting and complex questions. Although we have addressed many of those in the present work, further research would be desirable.

The models discussed in this paper are consistent with a simple intuitive picture. For field lines diffusing radially, there is a scale length along the field line, ζ_D, such that the radial excursion becomes large enough to produce a significant change in the pitch of the field line, and in turn a significant change in phase. Parallel wavelengths short compared to ζ_D see little effect from the field-line stochasticity. Fourier modes with parallel wavelength long compared to ζ_D see a substantial reduction in amplitude due to phase mixing. This is the resonance-broadening effect. In many cases of practical interest, particularly those involving only high-order rational surfaces, the resonance broadening serves the role of preventing near-rational Pfirsch-Schlüter currents from having a significant effect on the equilibrium field. In those cases the equilibrium solution is not sensitive to the details of the resonance-broadening model, and it is mainly important to have a reasonably accurate estimate of ζ_D. This was the case for the equilibria of the W7AS stellarator discussed in Ref. 4. In other cases, such as those involving low-order rational surfaces embedded in $\boldsymbol{B}_0$ (a nearby field with good surfaces), the near-rational Pfirsch-Schlüter currents may play a significant role, and it may be desirable to fully implement the detailed solution discussed in Sec. IV, or even to develop the numerical model further, as discussed in the previous paragraph.

Acknowledgements

This work was supported by U.S. Department of Energy Contract No. DE–AC02–76–CHO–3073.

APPENDIX A: STATISTICAL CLOSURE THEORY FOR PASSIVE RANDOM MAGNETIC FIELDS

In this section we consider aspects of the statistical properties of Eq. (9), illustrating the issues with a simple stochastic model. To formulate that model, we recall from Sec. III B 3 that Eq. (9) can be written in terms of a time-like coordinate ϕ and two space-like coordinates ψ and θ. That is of course complicated, so we seek to derive a model that depends solely on a single variable, which we choose to be time t. (We write the model in terms of time to make contact with standard turbulence literature.) To that end, note that Eq. (9) can be written as

$$\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\mu+\delta\boldsymbol{B}\cdot\boldsymbol{\nabla}\mu=-\boldsymbol{\nabla}\cdot\boldsymbol{j}_{\perp}. \tag{A1}$$

If one divides this equation by $B_0 \doteq |\boldsymbol{B}_0|$ and defines $\boldsymbol{b}\doteq\delta\boldsymbol{B}/B_0$, it can be written as

$$\partial_\zeta\mu+\boldsymbol{b}\cdot\boldsymbol{\nabla}\mu=f^{\mathrm{ext}}, \tag{A2}$$

where ζ is the length along the unperturbed field lines and $f^{\mathrm{ext}} \doteq -\boldsymbol{\nabla}\cdot\boldsymbol{j}_{\perp}/B_0$. Of course, the simple form of Eq. (A2) is somewhat illusory; in field-line-following coordinates, the expressions of periodicity in the original angular coordinates becomes more complicated; see Eqs. (61). Nevertheless, Eq. (A2) is similar in form to the equation for forced 2D passive advection

$$\partial_t\widetilde{\psi}(\boldsymbol{x},t)+\widetilde{\boldsymbol{V}}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\widetilde{\psi}=\widetilde{f}^{\mathrm{ext}}(\boldsymbol{x},t). \tag{A3}$$

(See the discussion of self-consistency in the next paragraph.) To achieve a model dependent only on time, make the unwarranted assumption that $\widetilde{\boldsymbol{V}}$ is space-independent. Then Eq. (A3) can be rigorously Fourier-transformed to

$$\partial_t\widetilde{\psi}_{\boldsymbol{k}}(t)+i\boldsymbol{k}\cdot\widetilde{\boldsymbol{V}}(t)\widetilde{\psi}_{\boldsymbol{k}}(t)=\widetilde{f}^{\mathrm{ext}}_{\boldsymbol{k}}(t). \tag{A4}$$

Upon dropping the $\boldsymbol{k}$ subscript, defining

$$\widetilde{b}(t)\doteq\boldsymbol{k}\cdot\widetilde{\boldsymbol{V}}(t), \tag{A5}$$

and adding a damping term $\propto \nu$ that models the long-time dissipative effects contained in the transport problem and facilitates later discussion, one obtains the final stochastic equation

$$\partial_t \widetilde{\psi}(t) + \nu \widetilde{\psi} + i\widetilde{b}(t)\widetilde{\psi}(t) = \widetilde{f}^{\text{ext}}(t). \tag{A6}$$

To complete the specification of the model, one must state whether the problem is passive ($\delta\widetilde{b}/\delta\widetilde{\psi} = 0$, $\delta\widetilde{f}^{\text{ext}}/\delta\widetilde{\psi} = 0$) or self-consistent ($\widetilde{b} = \widetilde{b}[\widetilde{\psi}]$, $\widetilde{f}^{\text{ext}} = \widetilde{f}^{\text{ext}}[\widetilde{\psi}]$, where brackets denote functional dependence); if it is passive, one must also state the joint statistical distribution of $\widetilde{b}$ and $\widetilde{f}^{\text{ext}}$. The physics problem discussed in the body of the paper is actually self-consistent since the magnetic field determined by Ampère's law depends on μ, and that field feeds back into f^{ext}. We shall, however, ignore this complication and assume that the problem is passive. We shall assume jointly Gaussian statistics for $\widetilde{b}$ and $\widetilde{f}^{\text{ext}}$, taking $\langle\widetilde{b}\rangle = 0$ but allowing for a nonzero $\langle\widetilde{f}^{\text{ext}}\rangle$.

1. The direct-interaction approximation for passive statistics

With the restriction to passive statistics, Eq. (A6) becomes a generalization to include dissipation and random forcing of the famous *stochastic oscillator* model studied by Kubo[47] and Kraichnan[48]; see Ref. 3 for additional discussion and references. Although it is dynamically linear, it is nonlinear in random variables and thus exhibits the familiar *statistical closure problem* that ensemble averages of Eq. (A6) or of similar equations for products of $\widetilde{\psi}$, $\widetilde{b}$, and $\widetilde{f}^{\text{ext}}$ (involving at least one power of $\widetilde{\psi}$) lead to an unclosed hierarchy of equations coupling cumulants of various orders. Usually the stochastic oscillator is discussed in the absence of random forcing. Including a forcing that is statistically independent of $\widetilde{b}$ is not difficult; however, as we noted in Sec. III B 3, in the physics problem of interest one must allow for statistical correlations between $\widetilde{f}^{\text{ext}}$ and $\widetilde{b}$. This introduces additional complexity to the statistical analysis.

Let the correlation time of δb be called τ_{ac}^{b} and its rms level be called β. (τ_{ac}^{b} must be distinguished from the internal correlation time τ_{ac} that describes the decay of $\delta\psi$ correlations.) From β and τ_{ac}^{b}, one can build the dimensionless parameter $\mathcal{K} \doteq \beta\tau_{\text{ac}}^{b}$ known as the *Kubo number*. Another dimensionless parameter is the *Reynolds number* $\mathcal{R} \doteq \beta/\nu$, which is a measure of the relative importance of the (nondissipative) nonlinear term and the (dissipative) linear term. In general, the 2D $\mathcal{R}$–$\mathcal{K}$ parameter space contains a variety of regimes, as thoroughly discussed in Refs. 49 and 3. Here we shall assume that $\mathcal{R} \gg 1$. For arbitrary $\mathcal{K}$, the standard closure of Eq. (A6) is the direct-interaction approximation (DIA), originally proposed for the self-consistent Navier–Stokes equation by Kraichnan,[50] later studied in the context of the stochastic oscillator by Kraichnan[48] and in the context of Vlasov stochastic acceleration by Orszag and Kraichnan.[29] The DIA has an extensive literature; see Ref. 3 for references. Although the DIA was originally shown to be the exact description of a *random coupling model*,[48] it can also be obtained by an iteration-renormalization approach or from a stochastic Langevin equation. The DIA for passive advection is stated in Ref. 3 for the special case of vanishing mean field. In the presence of a mean field, the DIA for the stochastic oscillator is as follows. Following the notation of Ref. 3, p. 143, define

$$C(t,t') \doteq \langle\delta\psi(t)\delta\psi^*(t')\rangle, \tag{A7a}$$

$$V(t,t') \doteq \langle\delta\psi(t)\delta b^*(t')\rangle, \tag{A7b}$$

$$W(t,t') \doteq \langle\delta\psi(t)\delta f^{\text{ext}*}(t')\rangle, \tag{A7c}$$

$$X(t,t') \doteq \langle\delta b(t)\delta f^{\text{ext}*}(t')\rangle, \tag{A7d}$$

$$B(t,t') \doteq \langle\delta b(t)\delta b^*(t')\rangle, \tag{A7e}$$

$$F^{\text{ext}}(t,t') \doteq \langle\delta f^{\text{ext}}(t)\delta f^{\text{ext}*}(t')\rangle. \tag{A7f}$$

(In Ref. 3, B was called S. In the present model, δb is real, but it is useful to retain the complex conjugate for comparison with more general equations.) Also define the zeroth-order Green's function R_0, which obeys

$$(\partial_t + \nu)R_0(t;t') = \delta(t-t'). \tag{A8}$$

The solution of Eq. (A8) is $R_0(t;t') = H(\tau)e^{-\nu\tau}$, where $\tau \doteq t - t'$ and $H(\tau)$ is the Heaviside unit step function:

$$H(\tau) \doteq \begin{cases} 0 & (\tau < 0) \\ 1/2 & (\tau = 0) \\ 1 & (\tau > 0). \end{cases} \tag{A9}$$

Finally, define the mean response function $R \doteq \langle\widetilde{R}\rangle$, where

$$R_0^{-1}\widetilde{R}(t;t') + i\widetilde{b}(t)\widetilde{R}(t;t') = \delta(t-t'). \tag{A10}$$

Coupled equations for the two-point quantities and the mean field $\langle\psi\rangle$ define a particular statistical closure. The unknown quantities are R, C, V, and W; X, B, and F^{ext} are supposed to be given. In the stochastic oscillator, δb (hence B) is real. C enjoys the symmetry $C(t,t') = C^*(t',t)$, or in steady state $C(\tau) = C^*(-\tau)$. The exact equation for the mean field is

$$\partial_t\langle\psi\rangle + \nu\langle\psi\rangle + i\underbrace{\langle\delta b\,\delta\psi\rangle}_{V(t,t)} = \langle f^{\text{ext}}\rangle. \tag{A11}$$

The passive DIA is

$$R_0^{-1}R(t;t') + \Sigma\star R = \delta(t-t'), \tag{A12a}$$

$$\begin{aligned} R^{-1}C(t,t') &+ iV^\dagger(t,t')\langle\psi\rangle(t) + (\Sigma'\star V^\dagger)(t,t') \\ &= (F^{\text{int}}\star R^\dagger)(t,t') + W^\dagger(t,t'), \end{aligned} \tag{A12b}$$

$$\begin{aligned} R^{-1}V(t,t') &+ iB(t,t')\langle\psi\rangle(t) + (\Sigma'\star B)(t,t') \\ &= X^\dagger(t,t'), \end{aligned} \tag{A12c}$$

$$\begin{aligned} R^{-1}W(t,t') &+ iX(t,t')\langle\psi\rangle(t) + (\Sigma'\star X^\dagger)t,t') \\ &= F^{\text{ext}\dagger}(t,t'), \end{aligned} \tag{A12d}$$

where the adjoint of a two-point function A is defined by $A^\dagger(t,t') \doteq A^*(t',t)$ and the $\star$ denotes convolution:

$(A \star B)(t,t') \doteq \int_{-\infty}^{\infty} d\bar{t}\, A(t,\bar{t})B(\bar{t},t')$. The functions Σ, Σ', and $F^{\rm int}$ are defined by

$$\Sigma(t;\bar{t}) \doteq R(t;\bar{t})B(t,\bar{t}), \tag{A13a}$$

$$\Sigma'(t;\bar{t}) \doteq R(t;\bar{t})V(\bar{t},t), \tag{A13b}$$

$$F^{\rm int}(t,\bar{t}) \doteq C(t,\bar{t})B(t,\bar{t}) + V(t,\bar{t})V^{\dagger}(t,\bar{t}). \tag{A13c}$$

The superscript int denotes internal, i.e., a contribution arising from the stochastic nonlinearity.

This passive DIA has an *underlying Langevin representation.* That is, there exists a stochastic Langevin equation whose one- and two-point statistics are described by the DIA equations written above. This is important conceptually because it guarantees that the solution of the DIA is *realizable* (corresponds to a positive-semidefinite PDF). One consequence is that the steady-state fluctuation level remains positive. Kraichnan emphasized the importance of realizability throughout his long research career. The Langevin representation for the self-consistent DIA was given by Leith[51] and Kraichnan.[52] For the stochastic oscillator, the Langevin model adjoins to the exact mean-field equation (A11) the fluctuating equation

$$\partial_t\delta\psi + \nu\delta\psi + \Sigma\star\delta\psi + \Sigma'\star\delta b = -i\delta b\langle\psi\rangle + \delta f^{\rm int} + \delta f^{\rm ext}. \tag{A14}$$

Here Σ and Σ' are given by formulas (A13a) and (A13b) and are said to describe *coherent response.* The *incoherent noise* $\delta f^{\rm int}$ is realized by

$$\delta f^{\rm int}(t) \doteq -i\delta b(t)\delta\xi(t), \tag{A15}$$

where $\delta\xi$ is a centered Gaussian random variable, statistically independent of δb and $\delta f^{\rm ext}$, whose two-point correlation functions (with itself and with δb and $\delta f^{\rm ext}$) are constrained to be equal to the ones that follow from the statistical solution of the Langevin model. (It may be helpful to envision how realizations of $\delta\xi$ could be constructed in an explicit numerical time-stepping scheme.)

Because δb and $\delta\xi$ are statistically independent Gaussian fields, one can verify that

$$\langle\delta f^{\rm int}(t)\delta f^{\rm int *}(t')\rangle = B(t,t')C(t,t') + V(t,t')V^{\dagger}(t,t') \tag{A16a}$$

$$= F^{\rm int}(t,t'). \tag{A16b}$$

Thus $F^{\rm int}$ is the variance of the internal incoherent noise. This fact can be used to verify that Eq. (A14) leads to Eq. (A12b) for $C(t,t')$. To that end, multiply Eq. (A14) by $\delta\psi^*(t')$ and average. The result involves $\langle\delta f^{\rm int}(t)\delta\psi^*(t')\rangle$. That can be worked out by moving the Σ' term to the right-hand side of Eq. (A14), noting that R is the Green's function for the remaining left-hand side, solving for $\delta\psi$, and using $\langle\delta f^{\rm int}(t)\delta b(t')\rangle = 0$ and Eq. (A16b). That leads to the $F^{\rm int}$ term of Eq. (A12b); the other terms are reproduced directly from the definitions (A7). The V and W equations also follow directly.

As will become clearer in the next section, Dupree-style resonance-broadening theory ignores the incoherent noise $\delta f^{\rm int}$, and that is what we also propose to do in the problem of stochastic magnetic field lines discussed in the body of the text. However, justification of this approximation is not immediate, so we continue with a detailed examination of a special tractable case.

2. The white-noise limit

We are interested in the limit in which the magnetic fluctuations are small, i.e., $\mathcal{K} \ll 1$. In that limit, the problem becomes Markovian and diffusive, i.e., $\int_{t'}^{t} d\bar{t}\,\Sigma(t;\bar{t})R(\bar{t};t') \approx [\int_{-\infty}^{t} d\bar{t}\,\Sigma(t;\bar{t})]R(t;t')$. That is true because the $B(\tau)$ inside Σ [see Eq. (A13a)] falls to zero much more rapidly than R itself (at least for large $\mathcal{R}$). In steady state, the quantity $\int_0^{\infty} d\overline{\tau}\,\Sigma(\overline{\tau}) = \widehat{\Sigma}(0)$ defines a nonlinear frequency broadening η. [Here $\widehat{\Sigma}(\omega)$ is the integral Fourier transform; for conventions, see Eqs. (B2).] For $\mathcal{K} \ll 1$, one has

$$\eta \approx \int_0^{\infty} d\overline{\tau}\, B(\overline{\tau}) = \int_0^{\infty} d\overline{\tau}\,\beta^2 e^{-\overline{\tau}/\tau_{\rm ac}^b} = \beta^2\tau_{\rm ac}^b \equiv d. \tag{A17}$$

In terms of the random velocity of the passive-advection problem (A3), one finds, upon using Eq. (A5), that

$$d = k^2 D, \quad D \doteq \langle\delta\widetilde{V}^2\rangle\tau_{\rm ac}^b. \tag{A18}$$

D is a conventional diffusion coefficient that would follow from Fokker–Planck theory in the limit of small $\tau_{\rm ac}^b$. In this limit, $R(\tau) \approx \exp[-(\nu+\eta)\tau]$ or $\widehat{R}(\omega) = \{-i[\omega + i(\nu+\eta)]\}^{-1}$. Thus $\eta \approx d$ describes a resonance broadening, the "resonance" occurring, for this model that lacks linear waves, at $\omega = 0$.

The essence of the Markovian limit is adequately captured by the *white-noise limit*

$$B(\tau) \approx 2d\delta(\tau), \tag{A19}$$

and we shall examine the structure of the DIA in that limit in order to illustrate the role of the cross correlations between $\widetilde{b}$ and $\widetilde{f}^{\rm ext}$. To be consistent with Eq. (A19), we also assume

$$F^{\rm ext}(\tau) = 2F_0\delta(\tau), \quad X(\tau) = 2X_0\delta(\tau). \tag{A20}$$

Because the theory is causal, $\delta\psi(t)$ depends functionally only on $\delta b(\bar{t})$ and $\delta f^{\rm ext}(\bar{t})$ for $\bar{t} \le t$. Thus

$$V(t,t') = \langle\delta\psi[\delta b(\bar{t} \le t)]\delta b(t')\rangle = 0 \quad (t' > t). \tag{A21}$$

W is similarly causal. The fact that V and W are one-sided functions simplifies the theory considerably. Also, it can be seen by iteration that those functions are not singular at $t = t'$. Therefore, some terms vanish. For example, the $VV^{\dagger}$ terms in $F^{\rm int}$ [formula (A13c)] vanish, since one or the another of V or V^* vanishes in the construction $V(t,\bar{t})V^*(\bar{t},t)$. (Any possible contribution at the single point $t = \bar{t}$ is negligible since $\bar{t}$ is integrated over.) Similarly, Σ' vanishes because it involves the combination $R(t;\overline{t})V(\overline{t},t)$ and R is causal.

With these simplifications, the equations become

$$\partial_t\langle\psi\rangle + \nu\langle\psi\rangle + iV(t,t) = \langle f^{\mathrm{ext}}\rangle, \tag{A22a}$$

$$R^{-1}C(t,t') = -iV^*(t',t)\langle\psi\rangle(t) + W^*(t',t) + F^{\mathrm{int}}(t,t'), \tag{A22b}$$

$$R^{-1}V(t,t') = -i2d\delta(t-t')\langle\psi\rangle(t) + 2X_0^*\delta(t-t'), \tag{A22c}$$

$$R^{-1}W(t,t') = -2iX_0\delta(t-t')\langle\psi\rangle(t) + 2F_0\delta(t-t'), \tag{A22d}$$

where

$$R^{-1}(t;\overline{t}) \doteq (\partial_t + \nu + d)\delta(t-\overline{t}), \tag{A23a}$$

$$F^{\mathrm{int}}(t,\overline{t}) \doteq 2d\delta(t-\overline{t})C(t,t). \tag{A23b}$$

In general, solution of these equations from arbitrary initial conditions is complicated for $\langle\psi\rangle \neq 0$ and would best be done numerically. (The equations predict a nonvanishing mean field, even for $\langle f^{\mathrm{ext}}\rangle = 0$, in the presence of nonvanishing cross correlations X_0; for $X_0 = 0$, a self-consistent solution is $V = 0$ and $\langle\psi\rangle = 0$.) However, the steady-state solution can be obtained analytically. From Eq. (A22d),

$$W(t,t') = R(t;t')(-2iX_0\langle\psi\rangle + 2F_0). \tag{A24}$$

Because $R(\tau)$ is real, $\widehat{R}^\dagger(\omega) = \widehat{R}(-\omega) = \widehat{R}^*(\omega)$. Similarly,

$$V(t,t') = R(t,t')(-2id\langle\psi\rangle + 2X_0^*), \tag{A25}$$

One can obtain $V(t,t)$ by again noting that $R(t;t) = H(0) = \frac{1}{2}$. Thus

$$V(t,t) = -id\langle\psi\rangle + X_0^*. \tag{A26}$$

Upon inserting this into the steady-state version of Eq. (A22a) and solving, one gets

$$\langle\psi\rangle = (\langle f\rangle - iX_0^*)/(\nu + d). \tag{A27}$$

The steady-state two-point equations can be solved by Fourier transformation. Define $C_0 \doteq C(t,t)$. The Fourier transform of the C equation is

$$\widehat{R}^{-1}(\omega)\widehat{C}(\omega) = -i\widehat{V}^\dagger(\omega)\langle\psi\rangle + \widehat{\Omega}^\dagger(\omega) + \widehat{F}^{\mathrm{int}}(\omega) \tag{A28a}$$

$$= -i\widehat{R}^\dagger(\omega)(2id\langle\psi\rangle^* + 2X_0)\langle\psi\rangle + \widehat{R}^\dagger(\omega)(2iX_0^*\langle\psi\rangle^* + 2F_0) + 2dC_0\widehat{R}^\dagger(\omega), \tag{A28b}$$

$$= 2[d\,|\langle\psi\rangle|^2 + 2\,\mathrm{Im}(X_0\langle\psi\rangle) + dC_0 + F_0] \times \widehat{R}^*(\omega), \tag{A28c}$$

or

$$\widehat{C}(\omega) = 2|\widehat{R}(\omega)|^2[d\,|\langle\psi\rangle|^2 + 2\,\mathrm{Im}(X_0\langle\psi\rangle) + dC_0 + F_0]. \tag{A29}$$

One can solve for the fluctuation level C_0 by integrating over ω and using $\int_{-\infty}^{\infty} d\omega\, a/(\omega^2 + a^2) = \pi$. One finds

$$C_0 = \nu^{-1}[d\,|\langle\psi\rangle|^2 + 2\,\mathrm{Im}(X_0\langle\psi\rangle) + F_0]. \tag{A30}$$

For $\langle\psi\rangle = 0$, this reduces to $C_0 = F_0/\nu$, which describes the balance between external forcing and linear dissipation. In the context of the classical Brownian motion of an unmagnetized test particle in a Gibbsian thermal bath, this result is equivalent to the Einstein relation between the velocity-space diffusion coefficient ($\sim F_0$) and polarization drag (ν).

One can also arrive at the result (A30) by directly considering the steady-state form of the spectral balance equation for $C(t,t)$: $0 = \partial_t C(t,t) = \cdots$.

It is important to note that in the manipulations leading to Eq. (A30) a near-cancellation occurred between the size of the C term on the left-hand side of Eq. (A29) and the dC_0 term on the right-hand side, the size of the latter term being $[d/(\nu + d)]C_0$ after integration over ω. That term arose from F^{int}. Had that term been arbitrarily neglected, the result for $\langle\psi\rangle = 0$ would have been $C_0 = F_0/d$, i.e., the external forcing would have been incorrectly balanced by the nonlinear diffusion rather than the linear dissipation. The correct formalism understands that the diffusive effect arises from a time-*reversible* (Hamiltonian) term in the primitive amplitude equation. Although diffusion leads to the decay of two-time correlation and response functions (*via* what is sometimes called *nonlinear scrambling*), the stochastic nonlinearity does not dissipate the mean-square fluctuation level. Thus, the simplified equation $\partial_t\delta\psi + i(\delta b\,\delta\psi - \langle\delta b\,\delta\psi\rangle) = 0$ leads to

$$\partial_t\langle\tfrac{1}{2}\,|\delta\psi|^2\rangle = \mathrm{Re}\langle\delta\dot{\psi}\,\delta\psi^*\rangle = \mathrm{Re}(-i\delta b\,|\delta\psi|^2) = 0. \tag{A31}$$

In more complicated physical problems, this nonlinear conservation of scalar variance generalizes to the conservation of quadratic quantities like the fluctuation energy or enstrophy, well known to be crucial in the theory of spectral cascades.

For $\langle\psi\rangle \neq 0$ and nonvanishing cross correlation X_0, the right-hand side of Eq. (A30) is not manifestly positive-definite, yet that must be so if the realizability argument based on the Langevin representation is correct. In order to check that, note that without loss of generality one can write the Gaussian white-noise fluctuation δf^{ext} as

$$\delta f^{\mathrm{ext}}(t) = \alpha\delta b(t) + \delta\widehat{f}(t), \tag{A32}$$

where $\delta\widehat{f}$ and δb are uncorrelated. (Since they are Gaussian, they are in fact statistically independent.) The coefficient α sets the size of the cross correlation X, i.e.,

$$X(t,t') = \langle\delta b(t)\delta f^{\mathrm{ext}*}(t')\rangle \tag{A33a}$$

$$= \alpha^*\langle\delta b(t)\delta b^*(t')\rangle \tag{A33b}$$

$$= 2\alpha^* d\,\delta(t-t'). \tag{A33c}$$

Thus $X_0 = \alpha^* d$ and, from Eq. (A27), $\langle\psi\rangle = (\langle f^{\text{ext}}\rangle - i\alpha)/(\nu + d)$. Straightforward manipulations using these results reduce Eq. (A30) to

$$C_0 = \frac{1}{\nu}\left(\frac{d}{(\nu+d)^2}\left|\langle f^{\text{ext}}\rangle + i\alpha\nu\right|^2 + \widehat{F}_0\right), \tag{A34}$$

which is positive-definite. This is an important consistency check. Note that for $\langle f^{\text{ext}}\rangle = 0$ the contribution of the cross correlation to Eq. (A34) is $O(\mathcal{R}^{-1})$ and is thus very small.

If time dependence were retained in the equation for $\partial_t C(t,t)$, one would learn that C_0 evolves to its steady state on the long, dissipative time scale. For times intermediate between the turbulent autocorrelation time and the dissipative time, it evolves linearly. This is just the physics of a diffusive random walk transposed to the present model. An exact analogy is to the evolution of the mean-square weights in collisionless δf particle simulation, which were observed to increase linearly in simulations. Those observations were explained by Krommes and Hu[53] with the aid of both statistical modeling and physical arguments; that reference provides useful background for the present discussion. An important point made by those authors was that the turbulent transport can saturate on the autocorrelation time scale even in the face of the short-time increase in the mean-squared weights (that both can happen simultaneously was called the *entropy paradox* by Krommes and Hu); if that were not true, the entire rationale for δf simulations of microturbulence would be in jeopardy. In the present passive model, the turbulent transport (d) is stationary on the short time scale to the extent that $\langle\delta b^2\rangle$ is.

Although the incoherent noise must be retained for discussion of the long-time balances, it is possible that it is unimportant on the short time scale. The criterion involves a comparison of the dC_0 and $\widehat{F}_0$ terms in equations like Eq. (A29) on the time scale on which the diffusion (and mean field) saturate. In terms of an arbitrary source $S(t)$, consider the solution of

$$(\partial_t + 2\nu)C_0(t) = 2S(t). \tag{A35}$$

At short times, the ν term is negligible and one finds $C_0(t) \approx 2tS(0)$. At the short saturation time τ^b_{ac}, this is $C_0 \approx 2\tau^b_{\text{ac}}S(0)$. The size of the dC_0 term at short times is thus relatively $dC_0 = O(d\tau^b_{\text{ac}}) = O(\mathcal{K}^2)$, where Eq. (A17) was used. Thus the incoherent noise is bounded and small on the short time scale.

The principal conclusions of this Appendix are (i) the magnetic differential equation for $j_\parallel/B$ can be approximately treated as a problem of passive advection; (ii) a white-noise model captures the essence of the statistical closure problem; (iii) the statistical closure can be represented in terms of a nonlinear Langevin model; (iv) incoherent noise is unimportant on the short time scale on which the magnetic diffusion coefficient saturates, although it is crucial on the longer, dissipative transport time scale (which is included in the white-noise model studied here but is not considered in the treatment of plasma equilibrium in the body of the paper).

APPENDIX B: GREEN'S FUNCTIONS

In this section we discuss Green's formalism for solving linear ODEs, placing particular emphasis on periodicity constraints. Various formulations are shown to be mathematically equivalent.

1. Fourier transforms and the shifted-sum representation of periodic functions

In the discussion we will need to refer to functions periodically extended from the semiclosed fundamental domain $\mathcal{I} \doteq [0, 2\pi)$ to the real line $\mathcal{R} \doteq (-\infty, \infty)$. We shall indicate the T-periodic extension by the subscript T (e.g., $T = 2\pi$). Periodic functions can be created by a discrete Fourier-series representation; we shall use the transform pair (with a time-like convention)

$$F_{2\pi}(\phi) = \sum_{n=-\infty}^{\infty} \widehat{F}_n e^{-in\phi}, \tag{B1a}$$

$$\widehat{F}_n = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{in\phi} F(\phi). \tag{B1b}$$

We will also require the Fourier integral representation

$$F(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\tau}\widehat{F}(\omega), \tag{B2a}$$

$$\widehat{F}(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} F(\tau). \tag{B2b}$$

In Eq. (B1b), $F(\phi)$ may be viewed as having support only on $\mathcal{I}$; the inverse transform then replicates $F(\phi)$ on each of the intervals $\mathcal{I}_l \doteq [2\pi l, 2\pi(l+1))$ (note that $\mathcal{I} \equiv \mathcal{I}_0$) to produce the periodic extension $F_{2\pi}(\phi)$. The identity underlying this transformation is the expression for a periodic delta function

$$\delta_{2\pi}(\phi) \doteq \sum_{l=-\infty}^{\infty} \delta(\phi - 2\pi l) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-in\phi}. \tag{B3}$$

Given an arbitrary function $F(\phi)$ defined on $\mathcal{I}$, or equivalently an arbitrary set of Fourier coefficients $\widehat{F}_n$, the extension from $\mathcal{I}$ to $\mathcal{R}$ *via* the inverse Fourier transform (B1a) obeys $F_{2\pi}(\phi+2\pi) = F_{2\pi}(\phi)$; however, it is not necessarily continuous at the boundary points $2\pi l$. That is, $F_{2\pi}(0_-) \neq F_{2\pi}(0_+)$ in general. We shall call such possibly discontinuous but periodic solutions *"periodic,"* using the quotation marks for emphasis. An example of a "periodic" function is shown in Fig. 7. Of course, in physical applications involving toroidal topology the boundary points $\phi = 0$ and $\phi = 2\pi$ are identical, so physical solutions must be continuous and differentiable there. We shall call those solutions *periodic*; they are a special case of "periodic" solutions.

An alternate approach generates a periodic representation from the *shifted-sum representation*

$$F_{2\pi}(\phi) = \sum_{l=-\infty}^{\infty} R(\phi - 2\pi l), \tag{B4}$$

where the *generating function* $R(\phi)$ is absolutely integrable on $\mathcal{R}$. Fourier-series analysis of formula (B4) shows that

$$\widehat{F}_n = (2\pi)^{-1}\widehat{R}(\omega = n). \tag{B5}$$

[The Fourier-series transform pair is a special case of this construction in which the support for $F(\phi)$ lies only in $\mathcal{I}$; one obtains Eq. (B3) from $R(\tau) = \delta(\tau)$, $\widehat{R}(\omega) = 1$.] This use of generating functions defined on $\mathcal{R}$ underpins the ballooning transformation frequently used for the linear eigenmode analysis of toroidal systems; a useful pedagogical article that cites the original references is by Thyagaraja.[54] Here we are not interested in eigenvalues *per se* but rather in the construction of periodic Green's functions. (Of course, representations of Green's functions in terms of infinite sums over eigenvalues are well known, but we will not have to use those explicitly.)

2. Green's functions for periodic systems

Let L be a linear operator and ϕ be a 2π-periodic angle, and consider the solution of $Lu(\phi) = f_{2\pi}(\phi)$, where u is required to be both periodic and continuous. The simplest formal solution of this problem introduces the function $G_{2\pi}(\phi;\phi')$ that obeys

$$LG_{2\pi}(\phi;\phi') = \delta_{2\pi}(\phi - \phi') \tag{B6}$$

subject to the periodicity constraint

$$G_{2\pi}(0;\phi') = G_{2\pi}(2\pi;\phi'). \tag{B7}$$

Then

$$u(\phi) = \int_{\mathcal{I}_l} d\overline{\phi}\, G_{2\pi}(\phi;\overline{\phi}) f_{2\pi}(\overline{\phi}) \quad (\phi \in \mathcal{I}_l). \tag{B8}$$

Usually this formula is written for $l = 0$, i.e., for the fundamental domain $\mathcal{I}$.

a. *An example problem*

As an example, consider the problem

$$\partial_\phi u + \eta u = f(\phi) \tag{B9}$$

on $\mathcal{I}$. This example is related to the magnetic-field-line problem discussed in the main text; ϕ is a proxy for the distance along a field line (here having rotational transform $\iota = 0$), and η is a measure of the irreversible diffusion induced by field-line stochasticity. The utility of this model is that it can be immediately solved by Fourier transformation:

$$u(\phi) = \sum_{n=-\infty}^{\infty} \frac{\widehat{f}_n}{-i(n + i\eta)} e^{-in\phi}. \tag{B10}$$

We now show how to recover this result from Green's formalism. We shall use the standard notation

$$G(\phi;\phi') = \begin{cases} G^{>}(\phi;\phi') & (\phi > \phi') \\ G^{<}(\phi;\phi') & (\phi < \phi'). \end{cases} \tag{B11}$$

From Eq. (B7) and the jump condition

$$G^{>}(\phi';\phi') - G^{<}(\phi';\phi') = 1, \tag{B12}$$

one finds

$$G^{>}_{2\pi}(\phi;\phi') = e^{-\eta(\phi-\phi')}/\mathcal{D}^{-}, \tag{B13a}$$

$$G^{<}_{2\pi}(\phi;\phi') = e^{-2\pi\eta} e^{-\eta(\phi-\phi')}/\mathcal{D}^{-}, \tag{B13b}$$

where

$$\mathcal{D}^{\pm} \doteq 1 \pm e^{-2\pi\eta}. \tag{B14}$$

On $\mathcal{I}$, formula (B8) is explicitly

$$u(\phi) = \int_0^\phi d\overline{\phi}\, G^{>}_{2\pi}(\phi;\overline{\phi}) f(\overline{\phi}) + \int_\phi^{2\pi} d\overline{\phi}\, G^{<}_{2\pi}(\phi;\overline{\phi}) f(\overline{\phi}). \tag{B15}$$

If f is expanded according to Eq. (B1a), the integrals can readily be performed and one recovers Eq. (B10). [The simple form (B10) arises because Eq. (B9) contains no angle-dependent coefficients, so each of $G^{>}$ and $G^{<}$ is translationally invariant. In more general cases, the representation (B15) may be more useful.] Note that although the correction term $e^{-2\pi\eta}$ controls the ratio of $G^{<}_{2\pi}$ to $G^{>}_{2\pi}$, it disappeared from the final solution.

b. *Interpretion of the example periodic Green's function using the method of shifted sums*

Instead of solving directly for the periodic Green's function, an alternate approach exploits a shifted-sum representation to construct a periodic function from a generating function defined on $\mathcal{R}$. We shall illustrate the technique by reconsidering the previous example.

The calculation to follow will involve the familiar causal response function $G^{+}(\tau)$ that obeys

$$(\partial_t + \eta)G^{+}(t;t') = \delta(t - t') \tag{B16}$$

on $\mathcal{R}$, subject to $G^{+}(-\infty) = 0$. With $\tau \doteq t - t'$, one has

$$G^{+}(\tau) \doteq H(\tau)K(\tau), \quad K(\tau) \doteq e^{-\eta\tau}, \tag{B17}$$

where $H(\tau)$ is the unit step function [see Eq. (A9)].

The periodic Green's function that obeys Eq. (B6) may be constructed by using $G^+(\phi)$ as the generating function for the shifted-sum representation (B4). Within $\mathcal{I}$, the value of the causal $G^>_{2\pi}$ is then the sum of contributions from all domains $\mathcal{I}_l$ with $l \le 0$:

$$G^>_{2\pi}(\phi;\phi') = \sum_{l\le 0} K(\phi - \phi' - 2\pi l) \tag{B18a}$$

$$= e^{-\eta(\phi-\phi')} \sum_{l\ge 0} (e^{-2\pi\eta})^l \tag{B18b}$$

$$= e^{-\eta(\phi-\phi')}/(1 - e^{-2\pi\eta}). \tag{B18c}$$

Furthermore, one has $G^<_{2\pi}(\phi;\phi') = G^>_{2\pi}(\phi + 2\pi;\phi')$, as illustrated in Fig. 3. These results agree with Eqs. (B13).

Since the periodicity constraint can be enforced directly, as was done in Sec. B 2 a, it is not necessary to employ a shifted-sum representation for this example. However, that approach does have an intuitive physical interpretation as a superposition of contributions, it clearly demonstrates the origin of the denominator $\mathcal{D}^-$, and it shows why the periodicity constraint is important for $\eta \ll (2\pi)^{-1}$. For the problem of magnetic field lines with nonzero rotational transform discussed in the body of the paper, the periodicity constraint is more complicated and the shifted-sum representation is very useful.

c. *Remarks on the direct construction of periodic Green's functions*

The mathematics of the previous two subsections is elementary. However, it masks several important issues: (1) When applied to the physical problem of magnetic field lines, how do Eq. (B9) and the resulting periodic solution relate to the fact that the solution should be irreversible no matter whether one moves forward or backward along the lines? (2) For situations in which the integrals required in Eq. (B15) cannot be performed analytically (either because the forms of $G^>$ and $G^<$ are too complicated or because they are only known numerically), are the separate $G^>$ and $G^<$ integrals numerically well-behaved? Note that in the example the denominator $\mathcal{D}^-$ enters; at small η, that function nearly vanishes (the periodicity constraint is important), so that each of the $G^>$ and $G^<$ integrals is nearly infinite. To arrive at Eq. (B10), a cancellation occurred between the $G^>$ and $G^<$ terms such that the remaining terms involved the ratio $\mathcal{D}^-/\mathcal{D}^- = 1$, i.e., the ratio of two very small terms.

Both of these issues can be discussed within the more general framework of boundary-value problems, the traditional way in which Green's functions are usually presented.[55] This helps with the discussion of issue (1) because it permits treatment of initial-value problems. It also enables one to address issue (2) because in this method a periodic and continuous solution can be found as a special case of more general "periodic" ones. Because homogeneous rather than periodic boundary conditions are used for Green's function in this approach, it turns out that the separate $G^>$ and $G^<$ pieces of the forcing integral remains $O(1)$ in the limit $\eta \to 0$.

3. General formalism for boundary-value problems

Let L be a linear operator and x be a generic independent variable (later we examine the particular cases where x is either time t or a periodic angle ϕ) and consider the solution of $Lu(x) = f(x)$ in the *open* domain $\mathcal{I} \doteq (a, b)$ subject to the *boundary-value constraints* $u(a) = u_a$ and $u(b) = u_b$. Green's solution of this problem introduces the function $G(x; x')$ that obeys

$$LG(x;x') = \delta(x - x') \quad (x, x' \in \mathcal{I}) \tag{B19}$$

subject to the *homogeneous* boundary conditions

$$G(a;x') = G(b;x') = 0; \tag{B20}$$

it also refers to the *bilinear concomitant* or *conjunct* $\mathcal{C}(u, v)$ defined by

$$\mathcal{C}(u,v) = \int dx\,(uLv - vL^\dagger u) \tag{B21}$$

($L^\dagger$ being the adjoint operator), i.e., it is the boundary term left over after an integration by parts. Then it can be shown[55] that in $\mathcal{I}$

$$u(x) = -\,\mathcal{C}\big(u(\overline{x}), G(x;\overline{x})\big)\big|_a^b + \int_a^b d\overline{x}\, G(x;\overline{x}) f(\overline{x}). \tag{B22}$$

For second-order equations, one usually has no difficulty constructing Green's function. For first-order equations, the construction is not generally possible, as the problem is overconstrained by the two boundary conditions. Two exceptions are semi-infinite domains, e.g., $t \in (0, \infty)$, and "periodic" solutions on $\phi \in \mathcal{R}$. As an important example of the first type that will ultimately be relevant to the problem of magnetic lines, consider

$$L \equiv \partial_t + \eta \tag{B23}$$

with the initial condition $u(a) = u_a$. One readily deduces

$$L^\dagger = -\partial_t + \eta, \tag{B24a}$$

$$\mathcal{C}(u,v) = uv. \tag{B24b}$$

The natural Green's function to use for the initial-value problem is the causal one G^+ introduced in Sec. B 2 b, for then the conjunct term is $-u(\overline{t})G^+(t;\overline{t})\big|_a^\infty = G^+(t;a)u_a$. The solution

$$u(t) = G^+(t;a)u_a + \int_a^t d\overline{t}\, G^+(t;\overline{t}) f(\overline{t}) \tag{B25}$$

coincides with the solution of the initial-value problem obtained with the aid of an integrating factor.

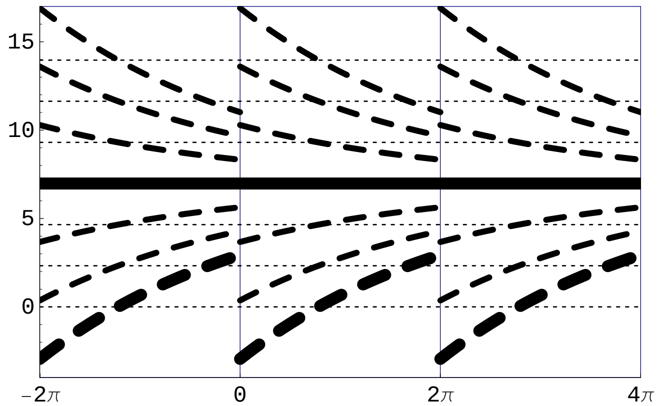

FIG. 7: Illustration of a family of "periodic" solutions (dashed curves) that includes a periodic and continuous one (heavy solid line). The family is generated from the solution to $Lu = f$ with $L \doteq \partial_t + \eta$, $\eta = (2\pi)^{-1}$, and constant forcing $f(t) = A = 1$. It is parametrized by the value $\overline{u} \doteq \frac{1}{2}[u(0_+) + u(2\pi_-)]$ (horizontal dotted lines). As $\eta \to 0$, the periodic solution $u(\phi) = u_* = A/\eta$ approaches ∞ while the "periodic" solution with $\overline{u} = 0$ (heavy dashed line) approaches a periodically extended ramp, a solution to $\partial_\phi u(\phi) = A$.

a. The relationship between periodic solutions and the initial-value problem

Suppose one wants to solve the previous problem in the domain (a, b) subject to the periodicity constraint $u_b = u_a$. The solution (B25) defines a family of solutions parametrized by u_a. The final value u_b is given by

$$u_b = G^+(b; a)u_a + \int_a^b d\overline{t}\, G^+(b; \overline{t})f(\overline{t}). \tag{B26}$$

Now the special solution satisfying $u_b = u_a$ can be found by setting $u_a = u_b = u_*$; thus

$$u_* = [1 - G^+(b; a)]^{-1} \int_a^b d\overline{t}\, G^+(b; \overline{t})f(\overline{t}). \tag{B27}$$

If this solution were extended to $\mathcal{R}$ by a shifted-sum formula, it would be both T-periodic ($T \doteq b - a$) and continuous. (An example of this construction is shown in Fig. 7.) Also, by calculating the derivative of formula (B25) at $u_a + \epsilon$ and $u_b - \epsilon$, one finds that the first derivative is continuous as well provided that $f(t)$ is periodic.

The representation (B25) with $u_a = u_*$ has a form different from Eq. (B8), although it must be equivalent (as we will show in the next paragraph). One advantage of formula (B25) is that the convolution integral is well behaved in the limit $\eta \to 0$, unlike the individual pieces $G^>_{2\pi}$ and $G^<_{2\pi}$ of formula Eq. (B15). It is true that in that limit u_* involves the small denominator $1 - G^+(b; a) \doteq \mathcal{D}^-$, but that is unavoidable; $\mathcal{D}^-$ ($\sim 2\pi\eta$ for small η) must appear in the solution somewhere because the periodic solution for constant forcing ($f = \widehat{f}_0 = A$) is $u = A/\eta$. The present representation removes division by a possibly small quantity from the convolution integral. In Sec. B 3 c we discuss a variation of this construction that may be even more convenient numerically.

With u_a replaced by u_*, Eq. (B25) can be rearranged to have the form of a convolution between a *periodic* Green's function G_T and the source, i.e.,

$$u(t) = \int_a^t d\overline{t}\, G^>_T(t; \overline{t})f(\overline{t}) + \int_t^b d\overline{t}\, G^<_T(t; \overline{t})f(\overline{t}). \tag{B28}$$

To do so, note that for first-order problems the causal Green's function enjoys the semigroup property $G^+(t; t') = G^+(t; \overline{t})G(\overline{t}; t')$ for arbitrary $\overline{t}$ intermediate between t' and t. If one writes $G^+(t; t') = H(t - t')K(t; t')$, the semigroup property holds also for the K's without restriction on t'. This is obvious for the specific, constant-η solution (B17), but it holds also for time-dependent generalizations. Upon defining

$$\mathcal{D}^- \doteq 1 - K(b; a), \tag{B29}$$

one has

$$u(t) = \frac{1}{\mathcal{D}^-}\Big(K(t; a)\int_a^b d\overline{t}\, K(b; \overline{t})f(\overline{t}) + [1 - K(b; a)]\int_a^t d\overline{t}\, K(t; \overline{t})f(\overline{t})\Big) \tag{B30a}$$

$$= \frac{1}{\mathcal{D}^-}\Big([K(t; a)K(b; t) + 1 - K(b; a)] \times \int_a^t d\overline{t}\, K(t; \overline{t})f(\overline{t}) + K(t; a)K(b; t)\int_t^b d\overline{t}\, K(t; \overline{t})f(\overline{t})\Big), \tag{B30b}$$

where $K(b; \overline{t}) = K(b; t)K(t; \overline{t})$ was used. (One needs to use K rather than G because in the last integral one has $t \le \overline{t}$ and G^+ itself is causal.) Upon using $K(t; a)K(b; t) = K(b; a)$, the formula (B30b) is seen to have the form (B28) with

$$G^>_T(t; t') = K(t; t')/\mathcal{D}^-, \tag{B31a}$$

$$G^<_T(t; t') = K(b; a)K(t; t')/\mathcal{D}^-. \tag{B31b}$$

This generalizes the specific example solution (B13).

b. The role of anti-causal Green's functions

Although we have shown how to construct a periodic solution, we have not yet addressed the consequences of an important symmetry: microscopic dynamics are time-reversible. Consider $a = 0$ and $b = T$ for simplicity. If one views the equation $(\partial_t + \eta)u = f$ as a Langevin equation coarse-grained on a microscopic autocorrelation time $\tau_{\rm ac}$, then the positive dissipation η represents the effects of statistical phase mixing that becomes complete for $t > \tau_{\rm ac}$ and damps an initial disturbance as time increases. In a time-reversed world, disturbances should damp as one moves *backwards* in time. Thus, with $t = 0$ being the arbitrary origin at which the velocity of a test particle is specified, a Langevin equation valid for both signs of time is[56]

$$[\partial_t + \mathrm{sgn}(t)\eta]u = f(t), \tag{B32}$$

where

$$f(-t) = -f(t). \tag{B33}$$

The latter symmetry is required so that a delta-function kick produces the same jump Δu moving either forward or backward in time: $\Delta u = u(t + \Delta t) - u(t) = u\big(-(t + \Delta t)\big) - u(-t)$.

For $t < 0$, Eq. (B32) can be solved in terms of the adjoint Green's function $G^\dagger$ that obeys [see Eq. (B24a)]

$$-\partial_t G^\dagger(t; t') + \eta G^\dagger = \delta(t - t'). \tag{B34}$$

It is well known and can be easily proven that $G^\dagger(t; t') = G(t'; t)$. For the final-value problem, it is natural to use the boundary condition $G^\dagger(t'_+; t') = 0$; then $G^\dagger$ is the anti-causal (or *advanced*) Green's function G^- such that $G^-(t; t') = G^+(t'; t)$ or $\widehat{G}^\dagger(\omega) = G^*(\omega)$. For $t < 0$, one then finds

$$u(t) = G^-(t; 0)u_0 + \int_0^t d\bar{t}\, G^-(t; \bar{t})f(\bar{t}) \quad (t < 0) \tag{B35}$$

and it can be readily verified that with the symmetry (B33) one has

$$u(-t) = u(t). \tag{B36}$$

(The analogous symmetry in magnetic-field problems is called *stellarator symmetry*[44]; see Sec. IV E.) Of course, the solution Eq. (B35) can be made periodic as well by the proper choice of u_0.

Motivated by the problem of magnetic lines, now suppose that the interval $[-T, 0)$ is physically identified with $[0, T)$. That is, imagine picking up the interval $[-T, 0)$ and laying it on top of $[0, T)$ *without* reversing it left and right (this operation is distinct from time reversal). With $u^\pm \doteq H(\pm t)u(t)$, the most general periodic solution is

$$u(t) = (1 - \alpha)u^+(t) + \alpha u^-(t), \tag{B37}$$

where α (the fraction of the advanced solution that is included) is to be determined. Imposing the condition (B36) leads to the condition

$$(1 - 2\alpha)[u^+(t) - u^-(t)] = 0. \tag{B38}$$

Since $u^-(t) \neq u^+(t)$ in general, the unique solution is $\alpha = \frac{1}{2}$; thus[57]

$$u(t) = \frac{1}{2}[u^+(t) + u^-(t)] = \frac{1}{2}[u^+(t) + u^+(-t)]. \tag{B39}$$

In terms of the Fourier components, this becomes

$$\widehat{u}_n = \mathrm{Re}\,\widehat{u}_n^+ = \mathrm{Re}\left(\frac{\widehat{f}_n}{-i(\omega_n + i\eta)}\right). \tag{B40}$$

Equation (94b) implies that $\widehat{f}_n$ is purely imaginary. Thus

$$\widehat{u}_n = \left(\frac{i\omega_n}{\omega_n^2 + \eta^2}\right)\widehat{f}_n. \tag{B41}$$

Compare the analogous result (96).

c. Construction of periodic solutions from the boundary-value formulation

The boundary-value formulation may be used to construct periodic solutions, both discontinuous and, as an important special case, continuous. This provides an alternate route to the directly periodic approach discussed in Sec. B 2 that may be advantageous in situations in which the periodicity constraint is strong.

In the presence of discontinuities, one may define the value of a function at a point by its mean value: $u(\phi) = \frac{1}{2}[u(\phi_-) + u(\phi_+)]$. The boundary condition (B20) becomes

$$G(0_+; \phi') + G(2\pi_-; \phi') = 0, \tag{B42}$$

and one also has the jump condition

$$G(\phi'_+; \phi') - G(\phi'_-; \phi') = 1. \tag{B43}$$

Green's solution in $\mathcal{I}$ is

$$\begin{aligned} u(\phi) &= -\,u(\overline{\phi})G(\phi; \overline{\phi})\Big|_{0_+}^{2\pi_-} + \int_0^{2\pi} d\overline{\phi}\, G(\phi; \overline{\phi})f(\overline{\phi}) && \text{(B44a)} \\ &= G^>(\phi; 0)u(0_+) - G^<(\phi; 2\pi)u(2\pi_-) + F(\phi), && \text{(B44b)} \end{aligned}$$

where the convolution integral has been called $F(\phi)$. $u(\phi)$ may be periodically extended to $\mathcal{R}$. In general, that extension is discontinuous at $\phi = 2\pi l$. To determine the size of the jumps, one must solve Eq. (B44b) for $u_+ \doteq u(0_+)$ and $u_- \doteq u(2\pi_-)$. Those follow from the self-consistency conditions

$$u_+ = G^>(0; 0)u_+ - G^<(0; 2\pi)u_- + F(0), \tag{B45a}$$

$$u_- = G^>(2\pi; 0)u_+ - G^<(2\pi; 2\pi)u_- + F(2\pi). \tag{B45b}$$

This is a linear system to be solved for $\boldsymbol{u}_0 \doteq (u_+,\, u_-)^T$:

$$\mathsf{M} \cdot \boldsymbol{u}_0 = \boldsymbol{F}. \tag{B46}$$

Manipulations using the two constraints (B42) and (B43) show that M can be written as

$$\mathsf{M} = \begin{pmatrix} -G^< & -G^> \\ G^< & G^> \end{pmatrix}, \tag{B47}$$

where

$$G^< \equiv G^<(0; 0), \quad G^> \equiv G^>(2\pi; 2\pi). \tag{B48}$$

The nature of the solution depends on the structure of M. It is straightforward to find that the eigenvalues λ, the (unnormalized) left eigenvectors $\boldsymbol{l}$, and the right eigenvectors $\boldsymbol{r}$ of M are

$$\lambda = 0\colon \quad \boldsymbol{l}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{r}_0 = \begin{pmatrix} G^> \\ -G^< \end{pmatrix}; \tag{B49a}$$

$$\lambda = 1\colon \quad \boldsymbol{l}_1 = \begin{pmatrix} G^< \\ G^> \end{pmatrix}, \quad \boldsymbol{r}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{B49b}$$

(These are properly orthogonal.) The nonzero eigenvalue is actually $G^> - G^<$, which equals 1 from the jump condition (B43). The fact that M possesses a nonempty null eigenspace means that one must consider the Fredholm alternative. In order that Eq. (B46) be solvable, $\boldsymbol{F}$ must be orthogonal to $\boldsymbol{l}_0$, i.e., it must be proportional to $\boldsymbol{r}_1$. That follows from the boundary condition (B42). Thus one can write $\boldsymbol{F} = F\boldsymbol{r}_1$ $[F = F(2\pi)]$ and expand $\boldsymbol{u}_0$ as

$$\boldsymbol{u}_0 = 2\overline{u}\boldsymbol{r}_0 + b\boldsymbol{r}_1, \tag{B50}$$

where $\overline{u}$ and b are constants to be determined. By adding the components of $\boldsymbol{u}_0$, one sees that $\overline{u} = \frac{1}{2}[\boldsymbol{u}(0_+) + \boldsymbol{u}(2\pi_-)]$. By applying M to Eq. (B50) and equating the result to $\boldsymbol{F}$, one finds that $b = F$ whereas $\overline{u}$ is undetermined. Thus the formalism generates a family of "periodic" solutions parametrized by $\overline{u}$ (see Fig. 7). The special continuous solution can therefore be found by adjusting $\overline{u}$ to the special value u_* that makes the jump $\Delta u \doteq u_+ - u_-$ vanish; one finds

$$u_* = F/(G^< + G^>), \tag{B51}$$

and one can readily check that Eq. (B50) reduces to $u_+ = u_- = u_*$, i.e., $\boldsymbol{u}_0 = u_*(1,\, 1)^T$.

As an example, again consider the problem (B9). For homogeneous boundary conditions, one finds

$$G^>(\phi;\phi') = e^{-\eta(\phi-\phi')}/\mathcal{D}^+, \tag{B52a}$$

$$G^<(\phi;\phi') = -e^{-2\pi\eta}e^{-\eta(\phi-\phi')}/\mathcal{D}^+, \tag{B52b}$$

where $\mathcal{D}^+$ is defined by formula (B14). Thus

$$G^> + G^< = (1 - e^{-2\pi\eta})/\mathcal{D}^+. \tag{B53}$$

In conjunction with the expression for $F(2\pi)$, formula (B51) can be seen to be identical to the result (B27).

For identical endpoints, formula Eq. (B44b) can be written as

$$u(\phi) = A(\phi)u_* + F(\phi), \tag{B54}$$

where $A(\phi) \doteq G^>(\phi;0) - G^<(\phi;2\pi)$. Theoretically, with u_* given by Eq. (B51), the representation (B54) should be periodic. However, in cases where $F(\phi)$ cannot be manipulated analytically, and particularly for small η, it may be better to determine u_* by requiring that the values of $u(0)$ and $u(2\pi)$ are identical, i.e.,

$$u_* = -\left(\frac{F(2\pi) - F(0)}{A(2\pi) - A(0)}\right). \tag{B55}$$

This guarantees the periodicity even in the face of numerical errors, and the construction again works with integrals whose individual pieces are $O(1)$ in the limit of small η.

---

* Electronic address: `krommes@princeton.edu`
† Electronic address: `reiman@pppl.gov`

1. A. Rechester and M. N. Rosenbluth, Phys. Rev. Lett. **40**, 38 (1978).
2. J. A. Krommes, C. Oberman, and R. G. Kleva, J. Plasma Phys. **30**, 11 (1983).
3. J. A. Krommes, Phys. Rep. **360**, 1 (2002).
4. A. Reiman, M. C. Zarnstorff, D. Monticello, A. Weller, J. Geiger, and the W7–AS Team, Nucl. Fusion **47**, 572 (2007).
5. M. N. Rosenbluth, R. Z. Sagdeev, J. B. Taylor, and G. M. Zaslavskiı, Nucl. Fusion **6**, 297 (1966).
6. R. Jaspers, N. J. Lopes Cardozo, K. H. Finken, B. C. Schokker, G. Mank, G. Fuchs, and F. C. Schüller, Phys. Rev. Lett. **72**, 4093 (1994).
7. M. C. Zarnstorff, A. Weller, J. Geiger, E. Fredrickson, S. Hudson, J. P. Knauer, A. Reiman, A. Dinklage, G.-Y. Fu, L. P. Ku, et al., in *Fusion Energy 2004 (Proc. 20th Int. Conf. Vilamoura)* (International Atomic Energy Agency, Vienna, 2004), CD-ROM EX/3–4; `http://www-naweb.iaea.org/napc/physics/fec/fec2004/datasets/index.html`.
8. A. Weller, S. Sakakibara, K. Y. Watanabe, K. Toi, J. Geiger, M. C. Zarnstorff, S. R. Hudson, A. Reiman, A. Werner, C. Nührenberg, et al., Fusion Science and Technology **50**, 158 (2006).
9. Y. Suzuki, K. Y. Watanabe, S. Sakakibara, N. Nakajima, and N. Ohyabu, in *22nd IAEA Fusion Energy Conference* (International Atomic Energy Agency, Vienna, 2008), IAEA–TH/P9-19.
10. T. E. Evans, R. A. Moyer, K. H. Burrell, M. E. Fenstermacher, I. Joseph, A. W. Leonard, T. H. Osborne, G. D. Porter, M. J. Schaffer, P. B. Snyder, et al., Nature Phys. **2**, 419 (2006).
11. K. L. Wong, S. Kaye, D. R. Mikkelsen, J. A. Krommes, K. Hill, R. Bell, and B. LeBlanc, Phys. Rev. Lett. **99**, 135003/1 (2007).
12. S. P. Hirshman and D. K. Lee, Comput. Phys. Comm. **39**, 143 (1986).
13. A. H. Reiman and H. Greenside, Comput. Phys. Commun. **43**, 157 (1986).
14. S. Hudson, Phys. Plasmas **16**, 010701 (2009).
15. L. Spitzer, Phys. Plasmas **1**, 253 (1958).
16. H. Grad and H. Rubin, in *Proc. Second Intern. Conf. on*

the Peaceful Uses of Atomic Energy (1958), vol. 31, p. 190.
[17] D. Raburn, A. Reiman, D. Monticello, and R. Samtaney, Bull. Am. Phys. Soc. **52(16)**, 255 (2007).
[18] N. G. van Kampen, Phys. Rep. **24**, 171 (1976).
[19] K. L. Wong, S. Kaye, D. R. Mikkelsen, J. A. Krommes, K. Hill, R. Bell, and B. LeBlanc, Phys. Plasmas **15**, 056108 (2008).
[20] W. A. Newcomb, Phys. Fluids **2**, 362 (1959).
[21] A. H. Boozer, Phys. Fluids **24**, 1999 (1981).
[22] A. H. Reiman, N. Pomphrey, and A. H. Boozer, Phys. Fluids B **1**, 555 (1989).
[23] T. H. Stix, Nucl. Fusion **18**, 353 (1978).
[24] J. M. Finn, Phys. Plasmas **12** (2005).
[25] J. A. Krommes, Prog. Theor. Phys. Suppl. **64**, 137 (1978).
[26] M. Vlad, F. Spineanu, J. H. Misguich, and R. Balescu, Phys. Rev. E **67**, 026406 (2003).
[27] P. A. Sturrock, Phys. Rev. **141**, 186 (1966).
[28] T. H. Dupree, Phys. Fluids **9**, 1773 (1966).
[29] S. A. Orszag and R. H. Kraichnan, Phys. Fluids **10**, 1720 (1967).
[30] S. A. Orszag, in *Turbulence of Fluids and Plasmas*, edited by J. Fox (Polytechnic Press, Brooklyn, New York, 1969), p. 17.
[31] G. Benford and J. J. Thomson, Phys. Fluids **15**, 1496 (1972).
[32] R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley, New York, 1975), reprinted by Krieger Publishing Co., Malabar, Florida, 1991.
[33] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
[34] W. D. D'haeseleer, W. N. G. Hitchon, J. D. Callen, and J. L. Shohet, *Flux Coordinates and Magnetic Field Structure* (Springer, Berlin, 1991).
[35] J. A. Krommes, in *Handbook of Plasma Physics*, edited by A. A. Galeev and R. N. Sudan (North–Holland, Amsterdam, 1984), vol. 2, chap. 5.5, pp. 183–268.
[36] D. J. Tetreault, Ph.D. thesis, Massachusetts Institute of Technology (1976).
[37] G. I. Taylor, Proc. London Math. Soc., Ser. 2 **20**, 196 (1921), reprinted in *Turbulence: Classic Papers on Statistical Theory*, edited by S. K. Friedlander and L. Topper (Interscience, New York, 1961), p. 1.
[38] J. A. Krommes, Phys. Plasmas **7**, 1148 (2000).
[39] M. A. Beer, S. C. Cowley, and G. W. Hammett, Phys. Plasmas **2**, 2687 (1995).
[40] R. L. Dewar and A. H. Glasser, Phys. Fluids **26**, 3038 (1983).
[41] S. P. Hirshman and K. Molvig, Phys. Rev. Lett. **42**, 648 (1979).
[42] H. Biglari, P. H. Diamond, and P. W. Terry, Phys. Fluids B **2**, 1 (1990).
[43] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions* (U.S. Government Printing Office, Washington, D.C., 1964).
[44] R. L. Dewar and S. R. Hudson, Physica D **112**, 275 (1998).
[45] A. H. Reiman and H. S. Greenside, J. Comput. Phys. **75**, 423 (1988).
[46] A. H. Reiman and N. Pomphrey, J. Comput. Phys. **94**, 225 (1991).
[47] R. Kubo, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Interscience, New York, 1959), vol. I, p. 181.
[48] R. H. Kraichnan, J. Math. Phys. **2**, 124 (1961), erratum: J. Math. Phys. **3**, 205 (1962).
[49] J. A. Krommes and R. A. Smith, Ann. Phys. (N.Y.) **177**, 246 (1987).
[50] R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959).
[51] C. E. Leith, J. Atmos. Sci. **28**, 145 (1971).
[52] R. H. Kraichnan, J. Fluid Mech. **41**, 189 (1970).
[53] J. A. Krommes and G. Hu, Phys. Plasmas **1**, 3211 (1994).
[54] A. Thyagaraja, J. Plasma Phys. **59**, 367 (1998).
[55] I. Stakgold, *Green's Functions and Boundary Value Problems* (Wiley, New York, 1979).
[56] J. A. Krommes and G. Hu, Phys. Fluids B **5**, 3908 (1993).
[57] An analogous superposition of causal and anti-causal solutions was discussed by J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945) in their famous discussion of the mechanism of radiation.

The Princeton Plasma Physics Laboratory is operated
by Princeton University under contract
with the U.S. Department of Energy.

Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2750
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov