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Spectral Asymmetry Due to Magnetic Coordinates

Jong-kyu Park,, Allen H. Boozer, and Jonathan E. Menard

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Spectral asymmetry due to magnetic coordinates

Jong-kyu Park,[1,*] Allen H. Boozer,[2] and Jonathan E. Menard[1]

[1] *Princeton Plasma Physics Laboratory, Princeton, New Jersey, NJ 08543*

[2] *Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027*

Abstract

The use of magnetic coordinates is ubiquitous in toroidal plasma physics, but the distortion in Fourier spectra produced by these coordinates is not well known. A spatial symmetry of the field is not always represented by a symmetry in the Fourier spectrum when magnetic coordinates are used because of the distortion of the toroidal angle. The practical importance of spectral distortion is illustrated with a tokamak example.

[*] Electronic address: `jpark@pppl.gov`

Magnetic coordinates simplify and clarify the theory of toroidal plasmas confined by a magnetic field [1–5]. Assuming the existence of flux surfaces [6, 7], $\psi(\vec{x})$ with $\vec{B}\cdot\vec{\nabla}\psi = 0$, magnetic coordinates describe three-dimensional space by $(\psi, \vartheta, \varphi)$, where ψ is a radial coordinate, ϑ is a poloidal, and φ is a toroidal angle. The two angles and ψ are defined, so the magnetic field has the representation

$$\vec{B} = \vec{\nabla}\varphi \times \vec{\nabla}\psi + q(\psi)\vec{\nabla}\psi \times \vec{\nabla}\vartheta, \tag{1}$$

where $q(\psi)$ is called the safety factor. Since $\vec{B}\cdot\vec{\nabla}\psi = 0$ and $\vec{B}\cdot\vec{\nabla}(\vartheta - \varphi/q) = 0$, magnetic field lines are given in magnetic coordinates by $\psi = \psi_0$ and $\vartheta = \vartheta_0 + \varphi/q$, where ψ_0 and ϑ_0 are constants. The coordinate Jacobian is $\mathcal{J} \equiv (\vec{\nabla}\psi \times \vec{\nabla}\vartheta \cdot \vec{\nabla}\varphi)^{-1} = (\vec{B}\cdot\vec{\nabla}\vartheta)^{-1}$.

Magnetic coordinates are not uniquely defined for a given magnetic field. If (ϑ, φ) are the angular coordinates of one set of magnetic coordinates, then (θ, ϕ) are also magnetic coordinates [2] if

$$\vartheta = \theta + \frac{\omega}{q} \ \text{ and } \ \varphi = \phi + \omega, \tag{2}$$

where ω is an arbitrary function of position. In an axisymmetric tokamak, the arbitrary function ω is naturally restricted to having only ψ and ϑ dependence. If ϕ is the ordinary polar angle, distortions in the Fourier spectra arise from the ϑ dependent distortion in the toroidal angle if a transformation is made to a different set of magnetic coordinates. If the Jacobian of $(\psi, \vartheta, \varphi)$ coordinates is $\mathcal{J} = 1/\vec{B}\cdot\vec{\nabla}\vartheta$ and the Jacobian of (ψ, θ, ϕ) coordinates is $J = 1/\vec{B}\cdot\vec{\nabla}\theta$, then

$$\frac{\partial\omega}{\partial\vartheta} = \left(1 - \frac{\mathcal{J}}{J}\right) q. \tag{3}$$

If a tokamak is top-bottom symmetric, the Jacobians are even functions of ϑ and ω is an odd function of ϑ. It is the feature that ω is top-bottom antisymmetric when a tokamak is top-bottom symmetric that leads to the spectral distortion.

Suppose a top-bottom symmetric perturbation is applied to a top-bottom symmetric tokamak. If the ordinary polar angle of cylindrical coordinates is used as a magnetic coordinate, then the function that describes the perturbation obeys $f(\theta, \phi) = f(-\theta, \phi)$ and its Fourier spectrum is

$$F_{m,n} \equiv \frac{1}{(2\pi)^2} \oint f(\theta, \phi) e^{i(n\phi - m\theta)} d\theta d\phi = F_{-m,n}. \tag{4}$$

However, if the perturbation is described using $(\psi, \vartheta, \varphi)$ magnetic coordinates, Equation (2), the Fourier spectrum is (ϑ, φ)

$$\mathcal{F}_{m,n} \equiv \frac{1}{(2\pi)^2} \oint f(\theta, \phi) e^{i(n\varphi - m\vartheta)} d\vartheta d\varphi, \tag{5}$$

which need not have the symmetry $\mathcal{F}_{m,n} = \mathcal{F}_{-m,n}$.

The asymmetry of the Fourier spectrum in $(\psi, \vartheta, \varphi)$ is proven by the simple example $f = \exp(-in\phi)$, which clearly satisfies $f(\theta, \phi) = f(-\theta, \phi)$. In (ψ, θ, ϕ) coordinates the only non-zero Fourier term is $F_{0,n} = 1$. However,

$$\begin{aligned}
\mathcal{F}_{m,n} &= \frac{1}{(2\pi)^2} \oint e^{i(n\varphi - n\phi - m\vartheta)} d\vartheta d\varphi \\
&= \frac{1}{(2\pi)} \oint e^{-i(n\omega + m\vartheta)} d\vartheta.
\end{aligned} \tag{6}$$

Therefore

$$\mathcal{F}_{m,n} - \mathcal{F}_{-m,n} = -\frac{i}{(2\pi)} \oint e^{-in\omega} \sin(m\vartheta) d\vartheta; \tag{7}$$

$$\mathcal{F}_{m,n} - \mathcal{F}_{-m,n} = \frac{\oint \sin(n\omega) \sin(m\vartheta) d\vartheta}{2\pi}, \tag{8}$$

which is non-zero since the integrand is the product of two odd functions of ϑ. In general $f(\theta, \phi) = \sum_\mu f_\mu \cos(\mu\theta) \exp(-in\phi)$, so the spectral shift is given by the Fourier decomposition of $\cos(\mu\theta) \exp(-in\phi)$, but $\cos(\mu\theta) = \cos\{\mu(\vartheta - \omega/q)\}$ is an even function, so

$$\mathcal{F}_{m,n} - \mathcal{F}_{-m,n} = \frac{\oint \cos\{\mu(\vartheta - \omega/q)\} \sin(n\omega) \sin(m\vartheta) d\vartheta}{(2\pi)}. \tag{9}$$

That is, the symmetry in (θ, ϕ), $F_{m,n} = F_{-m,n}$ is broken in (ϑ, φ), $\mathcal{F}_{m,n} \neq \mathcal{F}_{-m,n}$, if $n\omega \neq 0$. In a similar way, the antisymmetry in (θ, ϕ), $F_{m,n} = -F_{-m,n}$, for a top-bottom antisymmetric perturbation is broken in (ϑ, φ), $\mathcal{F}_{m,n} \neq -\mathcal{F}_{-m,n}$, if $n\omega \neq 0$. The extent of the breaking of the symmetry or the antisymmetry depends on the magnitude of $\omega(\vartheta)$. Note that the property of the symmetry is preserved for the $n = 0$ Fourier terms.

The spectral asymmetry is important in practical applications, such as studies of tokamaks subjected to magnetic perturbations [8–19]. Most of the external coils in a tokamak have the top-bottom symmetry or antisymmetry, so the components of the magnetic field are also often either top-bottom symmetric or antisymmetric unless the application is especially on the purpose of the correction for an intrinsic error field. Here only a top-bottom symmetric component of the field will be illustrated since an antisymmetric component can be discussed in a similar way. For a top-bottom symmetric component $f(\theta, \phi) = f(-\theta, \phi)$

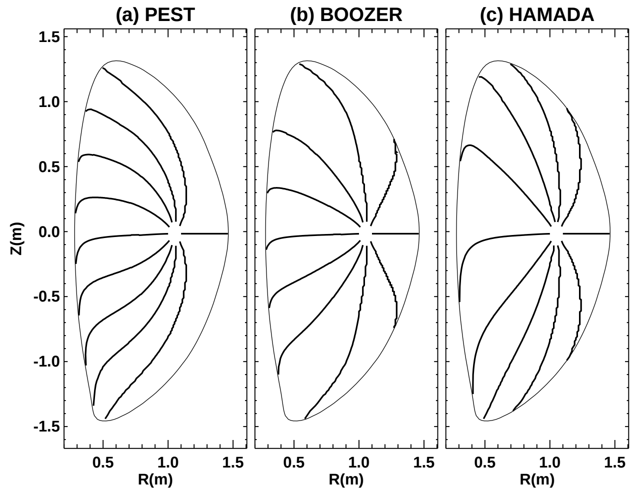

FIG. 1: Constant lines of the poloidal angles ϑ with intervals of 36° for (a) PEST (b) Boozer (c) Hamada magnetic coordinates in a NSTX plasma, with a single lower null divertor. Note the strong distortion of the poloidal angle when $\varphi = \phi$ in (a) compared with the other coordinate systems.

if the toroidal angle ϕ is the cylindrical angle. The symmetry implies the symmetry in the Fourier spectrum, $F_{m,n} = F_{-m,n}$, but the spectrum does not have this symmetry in other magnetic coordinate systems.

Magnetic coordinate systems can be defined by the Jacobian, which for the standard coordinate systems can be written as [20]

$$\mathcal{J} = \lambda(\psi) \frac{R^i}{|\vec{\nabla}\psi|^j B^k}, \tag{10}$$

where R is a major radius and $\lambda(\psi)$ is a normalizing function. The magnetic coordinates that use the cylindrical angle ϕ as the toroidal angle are called PEST [21] coordinates and have the Jacobian $\mathcal{J}_P \propto R^2$. The relation between PEST and cylindrical coordinates allows a simple description of external coils and conductors. More theoretically important coordinates are given by $\mathcal{J}_B \propto 1/B^2$ and $\mathcal{J}_H \propto 1$, called Boozer [3, 4] and Hamada [5] coordinates, respectively. These coordinates have toroidal angles φ that are distorted from the cylindrical angle ϕ. Using Equation (1) and $\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\phi + RB_\phi \vec{\nabla}\phi$, the deviation is shown to be

$$\omega = \phi - \varphi = \int_0^\vartheta \frac{\mathcal{J} B_\phi}{R} d\vartheta - q\vartheta. \tag{11}$$

Magnetic coordinates have a complicated relation to ordinary space. When one of the two angles (ϑ, φ) has a simplified relation to ordinary space, the other angle must become

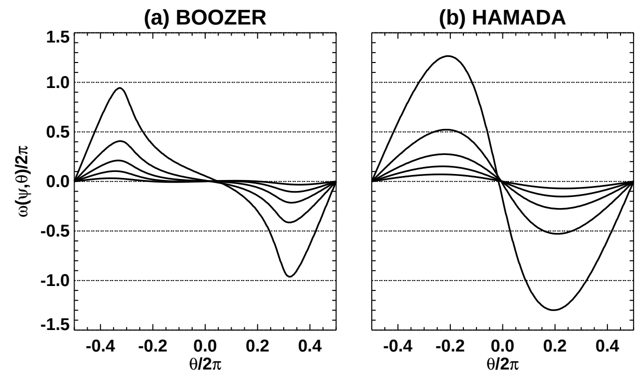

FIG. 2: The distortions of toroidal angle $\varphi - \phi = \omega(\psi, \vartheta)$ for (a) Boozer and (b) Hamada coordinates, at radial positions $\psi/\psi_{max} = 0.1$, 0.3, 0.5, 0.7 and 0.9. The amplitude of $\omega(\psi, \vartheta)$'s increase with ψ/ψ_{max}. The same NSTX plasma as in Figure 1 is used. Note $\omega(\psi, \vartheta) = -\omega(\psi, -\vartheta)$ and also the slight deviations from it due to the lower separatrix.

more complicated. This is illustrated by Figure 1. Figure 1 (a) shows the strong distortion of the poloidal angle of PEST coordinates in which the toroidal angle is the cylindrical angle. The example that is illustrated is the National Spherical Torus eXperiment (NSTX) device [22] and is almost top-bottom symmetric with a small deviation by a single lower null divertor. Boozer coordinates in (b) and Hamada coordinates in (c) represent poloidal angles that better represent the cross section, but the toroidal angles are complicated. Figure 2 shows the deviations $\omega(\vartheta)$ in five different radial positions $\psi/\psi_{max} = 0.1 \sim 0.9$ for the NSTX plasma. Although the example is illustrated with a spherical torus such as NSTX, the considerable variation of ω in the small ψ/ψ_{max}, where the aspect ratio of the surface becomes small, implies that the breaking of the symmetry can be significant also in a conventional tokamak.

The breaking of the symmetry of the Fourier spectrum that occurs when a magnetic perturbation is described in magnetic coordinates with distorted toroidal angles is illustrated in Figure 3 and 4. Here an $n = 3$ external field is applied from the midplane Error Field Control (EFC) coils [23] to the NSTX plasma shown in Figure 1. The external flux (Figure 3) and field (Figure 4) is decomposed at $q = m/n = 12/3 = 4$ rational surface, but as in any flux surface, the symmetry in PEST coordinates is not preserved in other magnetic coordinates with $\varphi \neq \phi$, and the deviation from the symmetry increases with $\omega(\vartheta)$, or with the deviation of Jacobian from the PEST Jacobian $\mathcal{J}_P \propto R^2$. Boozer coordinates have

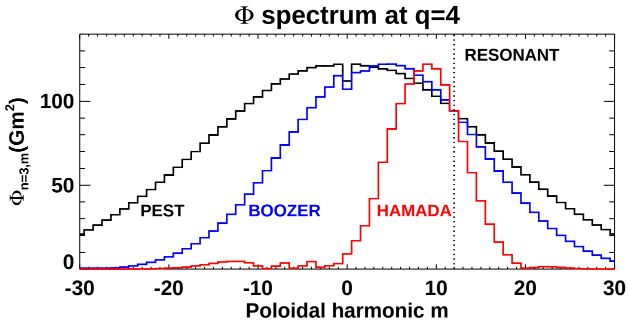

FIG. 3: Comparison of poloidal Fourier spectra of the flux $\Phi \equiv \delta\vec{B}\cdot\vec{\nabla}\psi/\vec{B}\cdot\vec{\nabla}\vartheta$ decomposed on $q = 12/3 = 4$ surface using PEST (black), Boozer (blue) and Hamada (red) magnetic coordinates in a NSTX plasma with $n = 3$ applied field. The symmetry in PEST magnetic coordinates is not preserved. The dotted-line shows shows the resonant Fourier harmonics of Φ are independent of the choice of magnetic coordinates.

$\mathcal{J}_B \propto 1/B^2 \sim R^2$, so the deviation is weak, but Hamada with $\mathcal{J}_H \propto 1$ strongly distorts the spectrum.

Although different magnetic coordinate systems have different spectra, physical results must remain unchanged. To illustrate this, consider the important problem of the breaking of magnetic surfaces by a perturbing magnetic field $\delta\vec{B}$. The perturbed magnetic surfaces are the constant-p_s surfaces where $p_s(\vec{x}) = p + \delta p$, $(\vec{B} + \delta\vec{B})\cdot\vec{\nabla}p_s = 0$, and $\vec{B}\cdot\vec{\nabla}p = 0$. The unperturbed magnetic field $\vec{B}$ is assumed to be given in magnetic coordinates, Equation (1), so p is a function of ψ alone. To lowest order in the perturbation,

$$\left(\frac{\partial}{\partial\vartheta} + q(\psi)\frac{\partial}{\partial\varphi}\right)\delta p = -\frac{\delta\vec{B}\cdot\vec{\nabla}\psi}{\vec{B}\cdot\vec{\nabla}\vartheta}\frac{dp}{d\psi}. \tag{12}$$

This equation is trivially solved if $\Phi(\vec{x}) \equiv \delta\vec{B}\cdot\vec{\nabla}\psi/\vec{B}\cdot\vec{\nabla}\vartheta$ is given as a Fourier series, $\Phi = \sum \Phi_{mn}\exp\{i(m\vartheta - n\varphi)\}$, where Φ has units of flux. Although Φ and its Fourier series depend on the choice of magnetic coordinates, a resonant Fourier harmonic, which means Φ_{mn} with $m = nq$, does not. Equation (12) is singular when Φ_{mn} is resonant and the resolution of this singularity (III. A. in Ref. [24]) is a magnetic island with a width that scales as $\sqrt{|\Phi_{mn}|}$.

To prove the island width is a coordinate independent quantity, the Fourier harmonics of the flux,

$$\Phi_{mn} = \frac{1}{(2\pi)^2}\oint e^{i(n\varphi - m\vartheta)}\frac{\delta\vec{B}\cdot\vec{\nabla}\psi}{\vec{B}\cdot\vec{\nabla}\vartheta}d\vartheta d\varphi, \tag{13}$$

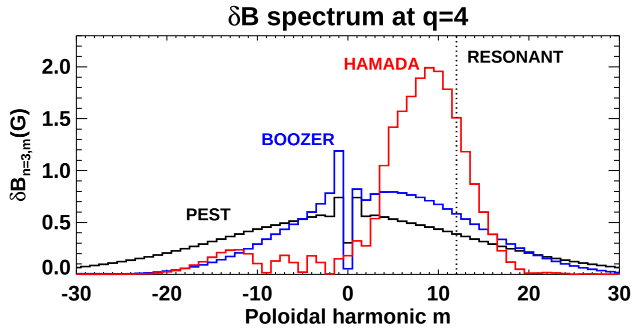

FIG. 4: Comparison of poloidal Fourier spectra of the normal field $\delta B \equiv \delta\vec{B}\cdot\hat{n}$ decomposed on $q = 12/3 = 4$ surface using PEST (black), Boozer (blue) and Hamada (red) magnetic coordinates in a NSTX plasma with $n = 3$ applied field. The symmetry in PEST magnetic coordinates is not preserved. The dotted-line shows the strong dependence of the resonant Fourier harmonics of δB on the choice of magnetic coordinates.

will be shown to be the same in all magnetic coordinate systems if $m = nq$. The area element on a constant-ψ surface is $d\vec{a} = (\vec{\nabla}\psi)\mathcal{J} d\vartheta d\varphi$, so using Equation (2) for the relation between different sets of magnetic coordinates

$$\Phi_{mn} = \frac{1}{(2\pi)^2}\oint e^{i(n\phi - m\theta)} e^{i(n-m/q)\omega}\delta\vec{B}\cdot d\vec{a}. \tag{14}$$

Since ω drops out of this equation for a resonant Fourier harmonic, $m = nq$, and the area element $d\vec{a}$ is a coordinate system invariant, resonant Fourier harmonics are the same in all sets of magnetic coordinates as illustrated at the dotted line in Figure 3.

A number of papers have discussed the resonant Fourier harmonics of magnetic field $\delta B \equiv \delta\vec{B}\cdot\hat{n}$ instead of Φ, assuming the Fourier spectrum is little changed by going to magnetic coordinates [11–13, 15–19]. But this assumption can be very inaccurate in toroidal plasmas. The magnetic field is decomposed as

$$\delta B_{mn} = \frac{1}{(2\pi)^2}\oint e^{i(n\varphi - m\vartheta)}(\delta\vec{B}\cdot\hat{n}) d\vartheta d\varphi. \tag{15}$$

Using $e^{i(n\varphi - m\vartheta)} = e^{i(n\phi - m\theta)}$ at the rational surface,

$$\delta B_{mn} = \frac{1}{(2\pi)^2}\oint e^{i(n\phi - m\theta)}\frac{(\delta\vec{B}\cdot d\vec{a})(\vec{B}\cdot\vec{\nabla}\vartheta)}{|\vec{\nabla}\psi|}. \tag{16}$$

Equation (16) shows that the resonant Fourier harmonics of the magnetic field depend on the different weighting from $(\vec{B}\cdot\vec{\nabla}\vartheta)/|\vec{\nabla}\psi|$ and differ between coordinate systems. The resonant Fourier harmonics of the flux $\Phi = (\delta\vec{B}\cdot\vec{\nabla}\psi)/(\vec{B}\cdot\vec{\nabla}\vartheta)$, however, are independent of the magnetic coordinate system. The dotted line in Figure 4, shows the resonant Fourier harmonics, $\delta B_{m=12,n=3}$, at the rational surface $q = 12/3$ are strongly dependent on magnetic coordinates in toroidal plasmas.

In summary, the distortion of the toroidal angle that can occur in magnetic coordinates produces a shift in Fourier spectra. When the geometry has top-bottom symmetry, this symmetry need not be apparent in the Fourier spectra. However, physical quantities, such as the width of islands, cannot depend on the choice of coordinate system, and this is connected with the invariance of certain features of the spectra to the choice of magnetic coordinates.

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[1] M. D. Kruskal and R. M. Kulsrud, Phys. Fluids **1**, 265 (1958).

[2] J. M. Greene and J. L. Johnson, Phys. Fluids **5**, 510 (1962).

[3] A. H. Boozer, Phys. Fluids **23**, 904 (1980).

[4] A. H. Boozer, Phys. Fluids **24**, 1999 (1981).

[5] S. Hamada, Nucl. Fusion **2**, 23 (1962).

[6] A. I. Morosov and L. S. Solov'ev, *Reviews of Plasma Physics*, vol. 2.1 (Consultants Bureau, New York, 1966).

[7] H. Grad, Phys. Fluids **10**, 137 (1967).

[8] J.-K. Park, M. J. Schaffer, J. E. Menard, and A. H. Boozer, Phys. Rev. Lett. **99**, 195003 (2007).

[9] H. Reimerdes, T. C. Hender, S. A. Sabbagh, J. M. Bialek, M. S. Chu, A. M. Garofalo, M. P. Gryaznevich, D. F. Howell, G. L. Jackson, R. J. La Haye, et al., Phys. Plasmas **13**, 056107 (2006).

[10] W. Zhu, S. A. Sabbagh, R. E. Bell, J. M. Bialek, B. P. LeBlanc, S. M. Kaye, F. M. Levinton, J. E. Menard, K. C. Shaing, A. C. Sontag, et al., Rhys. Rev. Lett. **96**, 225002 (2006).

[11] R. J. Buttery, M. D. Benedetti, D. Gates, Y. Gribov, T. Hender, R. La Haye, P. Leahy,

J. Leuer, A. Morris, A. Santagiustina, et al., Nucl. Fusion **39**, 1827 (1999).

[12] J. T. Scoville and R. J. LaHaye, Nucl. Fusion **43**, 250 (2003).

[13] S. M. Wolfe, I. H. Hutchinson, R. S. Granetz, J. Rice, A. Hubbard, A. Lynn, P. Phillips, T. C. Hender, D. F. Howell, R. J. La Haye, et al., Phys. Plasmas **12**, 056110 (2005).

[14] T. E. Evans, R. A. Moyer, P. R. Thomas, J. G. Watkins, T. H. Osborne, J. A. Boedo, E. J. Doyle, M. E. Fenstermacher, K. H. Finken, R. J. Groebner, et al., Phys. Rev. Lett. **92**, 235003 (2004).

[15] T. C. Hender, R. Fitzpatrick, A. W. Morris, P. G. Carolan, R. D. Durst, T. Eddlington, J. Ferreira, S. J. Fielding, P. S. Haynes, J. Hugill, et al., Nucl. Fusion **32**, 2091 (1992).

[16] R. J. La Haye, R. Fitzpatrick, T. C. Hender, A. W. Morris, J. T. Scoville, and T. N. Todd, Phys. Fluids B **4**, 2098 (1992).

[17] R. Fitzpatrick, Nucl. Fusion **33**, 1049 (1993).

[18] R. Fitzpatrick, Phys. Plasmas **5**, 3325 (1998).

[19] A. J. Cole, C. C. Hegna, and J. D. Callen, Phys. Plasmas **15**, 056102 (2008).

[20] C. Z. Cheng, Phys. Rep. **211**, 1 (1992).

[21] R. C. Grimm, R. L. Dewar, and J. Manickam, J. Comput. Phys. **49**, 94 (1983).

[22] J. Spitzer, M. Ono, M. Peng, D. Bashore, T. Bigelow, A. Brooks, J. Chrzanowski, H. M. Fan, P. Heitzenroeder, and T. Jarboe, Fusion Technology **30**, 1337 (1996).

[23] J. E. Menard, M. G. Bell, R. E. Bell, E. D. Fredrickson, D. A. Gates, S. M. Kaye, B. P. LeBlanc, R. Maingi, D. Mueller, S. A. Sabbagh, et al., Nucl. Fusion **43**, 330 (2003).

[24] A. H. Boozer, Rev. Mod. Phys. **76**, 1071 (2004).

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Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2750
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
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