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Zonal flows in toroidal systems

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An analytic study of the shielding and time evolution of zonal flows in tokamaks and stellarators is presented, using the action-angle formalism. This framework permits one to solve the kinetic equation without expansion of that equation in small parameters of radial excursions and timescale, resulting in more general expressions for the dielectric shielding, and with a scaling extended from that in earlier work. From these expressions, it is found that for each mechanism of collisional transport, there is a corresponding shielding mechanism, of closely related form and scaling. The effect of these generalized expressions on the evolution and size of zonal flows, and their implications for stellarator design are considered.

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I. INTRODUCTION

Since the early 1980s, a range of techniques for reducing the neoclassical (nc) transport in stellarators has been developed,[1] reducing the nc fluxes to levels subdominant to the turbulent fluxes over much of the plasma column, and a new generation of "transport-optimized" stellarator designs is now being implemented to test these techniques. As a result, new interest exists to also reduce the turbulent transport in stellarators. As for tokamaks, it is believed that an important mechanism for suppressing the turbulent fluxes in stellarators will be by having strong zonal flows (ZFs), primarily poloidal $E \times B$ flows due to a radially-varying electrostatic potential $\phi_Z(r,t)$ driven by the nonlinearities in the kinetic equation. It is thus of interest to understand how machine geometry will affect the strength of these flows.

A calculation of the ϕ_Z produced for a given nonlinear source S for tokamaks has been given by Rosenbluth and Hinton[2], and an analogous calculation for stellarators by Sugama and Watanabe.[3,4] These are basically linear response calculations, computing the dielectric response $\mathcal{D}$ in $k^2\phi_Z = 4\pi\delta\rho^{xt}/\mathcal{D}$, where $\delta\rho^{xt}$ is the external charge-density perturbation, driven by the assumed nonlinear source, $\delta\rho^{xt} \sim \int dt S(t)$. In Ref. 2, $\mathcal{D}$ is found to have a shielding contribution $\mathcal{D}^g \sim (k_r\rho_g)^2$ associated with the gyromotion (superscript g), corresponding to a "classical" polarization current $J^{p,g}$, and an analogous "nc", or "bounce" shielding $\mathcal{D}^b \sim F_t(k_r\rho_b)^2$ associated with the longer-timescale bounce motion (superscript b), with a corresponding bounce-polarization current $J^{p,b}$. (Here, k_r is the local radial wavevector of the ZF, ρ_g is the gyroradius, ρ_b is the banana width, and F_t is the fraction of toroidally-trapped particles.) In Ref. 3, it is found that for stellarators, there is a further contribution $\mathcal{D}^d \sim F_h$ due to motion on the still longer drift timescale (superscript d), which can appreciably modify this result. This term may in turn be associated with a drift polarization current $J^{p,d}$. (Here, $F_h \sim \epsilon_h^{1/2}$ is the fraction of helically-trapped particles.) As will be seen, this form is one particular limit of the drift shielding natural to the ordering adopted in Refs. 2,3 and 4, generalized in this paper. A second formal approach applied to studying ZFs in stellarators is one using the "time-dependent viscosity".[5] There, the kinetic equation is solved using a high- and low-frequency ordering, obtaining what is effectively the drift shielding contribution, in the "$1/\nu$" and banana regimes of stellarator transport. The results of both earlier lines of study are extended by the approach employed here (*cf.* Sec. III).

A formalism natural to treating particle motion on these different timescales, and in the relatively complex magnetic geometries of tokamaks and stellarators, is the "action-angle" (aa) formalism, originally formulated for tokamaks by Kaufman.[6] In it, one reparametrizes phase space points $\mathbf{z}$ from the more directly physical set $(\mathbf{r}, \mathbf{p})$ of real-space position $\mathbf{r}$ and conjugate momentum $\mathbf{p}$ to $(\boldsymbol{\theta}, \mathbf{J})$, with $\mathbf{J}$ the 3 invariant actions of the unperturbed motion and $\boldsymbol{\theta}$ their 3 conjugate angles. Using this formalism, solution of the kinetic equation can be carried out, and important quantities such as $\mathcal{D}$ can be computed, without having to introduce expansions in small parameters of radial and time scale, such as the ratios of ρ_g, ρ_b, or radial drift excursion ("super-banana width") ρ_d to system size L, or the frequency ω_Z of the ZF perturbation to the characteristic frequencies $\Omega_{g,b,d}$ of the particle motion. The resultant expressions for important quantities emerge in a form which is almost as simple as the more familiar forms for an unmagnetized homogeneous plasma. (Approximations may then be made in the description of the orbit, radial structure of the eigenmodes, and evaluation of integrals involving them.) The perspicuity of the aa expression for $\mathcal{D}$ permits one to more readily see parallels which exist among the different timescales, as will be seen.

The dielectric shielding computed here and other mechanisms affecting ZFs come together in the time evolution equation for the flux-surface averaged radial electric field $E_r \equiv \langle \nabla r \cdot \mathbf{E} \rangle$, obtained from the surface average of Ampere's law, plus an expression for the surface-averaged radial current J_r,

$$\partial_t E_r = -4\pi J_r, \quad (1)$$
$$J_r = (4\pi)^{-1}\chi\partial_t E_r + \sigma(E_r - E_a) + F_S/B.$$

The first term in J_r, proportional to the time derivative of E_r, represents the polarization current J^p, with χ containing the dielectric shielding contributions, the second term represents the nonambipolar radial current due to nc transport, where $E_a = -\langle\nabla r\cdot\nabla\Phi_a\rangle$ is the ambipolar value at which the ion and electron transport fluxes are equal, and F_S is the force, here assumed random, exerted by the turbulence within a magnetic surface normal to the magnetic field, which acts as a source driving E_r. Using Eq.(1b) in (1a) yields a Langevin-like equation, with drive F_S, and restoring term σE, where $E \equiv (E_r - E_a) = -\langle\nabla r\cdot\nabla\phi_Z\rangle$. Neglecting this latter term, as in the qualitative discussion in Ref. 2, results in the ensemble average $\langle E^2\rangle_p(t) \equiv \int dEp(E,t)E^2$ increasing without bound with t, corresponding to a $1/\omega$ divergence as $\omega \to 0$ in the spectral function $S^E(\omega) \equiv \langle E^2\rangle_p(\omega)$. [Here, $p(E)$ is the probability distribution function (pdf) for E.] As discussed in Sec. IV, refining this picture by including this term removes the divergence, resulting in a process where $E_r(t)$ evolves diffusively about $E_r = E_a$, reaching a bounded steady-state pdf.

In Sec. II, the aa formalism is used to obtain general expressions for the linear response, with $(\boldsymbol{\theta}, \mathbf{J})$ uncommitted to any particular magnetic geometry. In Sec. III this general form is specialized to toroidal geometries, and expressions for $\mathcal{D}$ and the response equation determining the size of ϕ_Z are obtained, valid for arbitrary ratios of $\rho_{g,b,d}/L$. These are then specialized to find special limits of the general expressions, and some of the results of earlier work are recovered, along with results in additional physically interesting limits. A close correspondence is found to exist between each collisional transport mechanism and a contribution to the polarization shielding. In Sec. IV we analyze the statistics of the ZF time evolution implied by Eqs.(1). In

Sec. V we summarize the results of the preceding sections.

II. ACTION-ANGLE FORMALISM

As noted in Sec. I, in the aa formalism one parametrizes phase points $\mathbf{z}$ with the 3 invariant actions $\mathbf{J}$ of the unperturbed motion and their 3 conjugate angles $\boldsymbol{\theta}$. The collisionless motion is governed by a Hamiltonian $H(\mathbf{z},t) = H_0(\mathbf{J}) + h(\mathbf{z},t)$, with unperturbed and perturbing parts H_0 and h. Here we consider electrostatic perturbations only, $h(\mathbf{z},t) = e\delta\phi(\mathbf{r}(\mathbf{z}),t)$. The key feature of aa variables is that they make the description of particle motion very simple. Hamilton's equations are:

$$\dot{\boldsymbol{\theta}} \;=\; \partial_{\mathbf{J}} H = \boldsymbol{\Omega}(\mathbf{J}) + \partial_{\mathbf{J}} h \simeq \boldsymbol{\Omega}(\mathbf{J}), \tag{2}$$

$$\dot{\mathbf{J}} \;=\; -\partial_{\boldsymbol{\theta}} h = -i \sum_{\mathbf{l}} \mathbf{l} h_{\mathbf{l}}(\mathbf{J},t) \exp(i\mathbf{l}\cdot\boldsymbol{\theta}), \tag{3}$$

where $\partial_{\mathbf{J}}$ ($\partial_{\boldsymbol{\theta}}$) denotes a gradient in $\mathbf{J}$ ($\boldsymbol{\theta}$)-space, $\boldsymbol{\Omega}(\mathbf{J}) \equiv \partial_{\mathbf{J}} H_0$, and $\mathbf{l}$ is the 3-component vector index, specifying the harmonic of each component of $\boldsymbol{\theta}$ in the Fourier decomposition $h(\mathbf{z}) = \sum_{\mathbf{l}} h_{\mathbf{l}}(\mathbf{J}) \exp(i\mathbf{l}\cdot\boldsymbol{\theta})$.

The Vlasov equation may be written

$$(\partial_t + \hat{H}_0)\delta f(\mathbf{z},t) = -\delta\dot{\mathbf{J}}\cdot\partial_{\mathbf{J}} f_0 + S(\mathbf{z},t) f_0, \tag{4}$$

where $\hat{H}_0 \equiv \{,H_0\} = \boldsymbol{\Omega}\cdot\partial_{\boldsymbol{\theta}}$, with $\{,\}$ Poisson brackets, and we write distribution function $f(\mathbf{z},t) = f_0 + \delta f$, with $f_0(\mathbf{J})$ the unperturbed portion, satisfying $\hat{H}_0 f_0 = 0$, and $\delta f(\mathbf{z},t)$ the perturbed portion. Following Refs. 2,3, we take the nonlinear term $-\{\delta f, h\}$ equal to a specified source function $S(\mathbf{z},t) f_0$.

Laplace transforming in time and Fourier transforming in $\boldsymbol{\theta}$, one obtains

$$G_0^{-1}\delta f_{\mathbf{l}}(\mathbf{J},\omega) = i\mathbf{l}\cdot\partial_{\mathbf{J}} f_0 h_{\mathbf{l}}(\mathbf{J},\omega) + \delta f_{\mathbf{l}}(\mathbf{J},t=0) + S_{\mathbf{l}}(\mathbf{J},\omega) f_0, \tag{5}$$

with inverse propagator $G_0^{-1} \equiv (-i\omega + i\mathbf{l}\cdot\boldsymbol{\Omega} + \nu_f)$, in which we include an effective damping rate ν_f, to later consider the effect of collisions, which goes to a positive infinitessimal ϵ in the purely collisionless case. Eq.(5) is readily solved for $\delta f_{\mathbf{l}}(\omega)$, and the charge density at observation point $\mathbf{x}$ is then computed via (now showing species label s) $\delta\rho_s(\mathbf{x}) = \int d\mathbf{z}\rho(\mathbf{x}|\mathbf{z})\delta f_s(\mathbf{z})$, where $\rho(\mathbf{x}|\mathbf{z}) \equiv e_s\delta(\mathbf{x} - \mathbf{r}(\mathbf{z}))$ is the charge density kernel, e_s is the species charge, and $\delta()$ is the Dirac delta function. This yields 3 contributions, labelled A, B, and C, corresponding to the 3 terms on the right side of (5):

$$\delta\rho_{sA}(\mathbf{x},\omega) = \int d\mathbf{x}' K_s(\mathbf{x},\mathbf{x}',\omega)\delta\phi(\mathbf{x}',\omega) \tag{6}$$

$$\delta\rho_{s,B+C}(\mathbf{x},\omega) = (2\pi)^3 \int d\mathbf{J} \sum_{\mathbf{l}} \rho^*_{\mathbf{l}}(\mathbf{x}|\mathbf{J}) G_0[\delta f_{s\mathbf{l}}(\mathbf{J},t=0) + S_{s\mathbf{l}}(\mathbf{J},\omega) f_{s0}].$$

$\delta\rho_{sA}$, proportional to h or $\delta\phi$, gives the self-consistent response of the plasma, with response kernel K_s. $\delta\rho_{sB}$, due to the initial conditions of δf, gives the transient ballistic response, and the third term, $\delta\rho_{sC}$, is due to the nonlinear drive.

The electrostatic counterpart of the response kernel obtained in Ref. 6 is given by

$$\begin{aligned} K_s(\mathbf{x},\mathbf{x}',\omega) &= (2\pi)^3 \int d\mathbf{J} \sum_{\mathbf{l}} \rho^*_{\mathbf{l}}(\mathbf{x}|\mathbf{J}) \frac{\mathbf{l}\cdot\partial_{\mathbf{J}} f_{s0}}{\mathbf{l}\cdot\mathbf{\Omega} - \omega - i\nu_f} \rho_{\mathbf{l}}(\mathbf{x}'|\mathbf{J}) \\ &= K_s^{ad}(\mathbf{x},\mathbf{x}') + (2\pi)^3 \int d\mathbf{J} \sum_{\mathbf{l}} \rho^*_{\mathbf{l}}(\mathbf{x}|\mathbf{J}) \frac{\omega\partial_{H_0} f_{s0} + \mathbf{l}\cdot\partial_{\mathbf{J}})_{H_0} f_{s0}}{\mathbf{l}\cdot\mathbf{\Omega} - \omega - i\nu_f} \rho_{\mathbf{l}}(\mathbf{x}'|\mathbf{J}). \end{aligned} \tag{7}$$

In the second form here, we have separated out the (generalized) adiabatic term $K_s^{ad}(\mathbf{x},\mathbf{x}') \equiv e_s\delta(\mathbf{x}-\mathbf{x}') \int d\mathbf{z}\rho(\mathbf{x}|\mathbf{z})\partial_{H_0} f_{s0}$, by giving f_{s0} an explicit dependence on $H_0(\mathbf{J})$, so that $\partial_{\mathbf{J}} f_{s0}(H_0(\mathbf{J}),\mathbf{J}) = \partial_{H_0}\mathbf{\Omega} + \partial_{\mathbf{J}})_{H_0} f_{s0}$, where the $\partial_{\mathbf{J}})_{H_0}$ in the second term means $\partial_{\mathbf{J}}$ at constant H_0. Specializing f_0 to the local Maxwellian form

$$f_M(\mathbf{J}) \equiv \frac{n_0}{(2\pi MT)^{3/2}} \exp[-(H_0 - e\Phi_a)/T], \tag{8}$$

where density n_0, ambipolar radial potential Φ_a, and temperature T are functions of the drift-averaged minor radius $r_d(\mathbf{J})$, and M is the particle mass, one has $\partial_{H_0} f_{s0} = -T_s^{-1} f_{s0}$, and $K_s^{ad}(\mathbf{x},\mathbf{x}') = -1/(4\pi\lambda_s^2(\mathbf{x}))\delta(\mathbf{x}-\mathbf{x}')$, with $\lambda_s^2(\mathbf{x}) \equiv T_s/(4\pi n_{s0} e_s^2)$ the square of the local Debye length.

III. TOROIDAL GEOMETRY

The expressions given thus far are valid for any system where the motion is "integrable", *i.e.*, where a complete set $\mathbf{J}$ of constants of the motion exists. We now specialize to toroidal geometries, including tokamaks and stellarators. Such a set $\mathbf{J}$ exists for systems with at least 1 symmetry direction, such as tokamaks and straight stellarators, manifested by their collisionless guiding-center orbits exactly closing on themselves in poloidal cross-section. An approximate set $\mathbf{J}$ exists for those classes of toroidal stellarators whose ripple has sufficient symmetry that "superbanana" orbits (those ripple trapped during at least part of their orbit) approximately close on themselves. Since devices without this feature have poor confinement, this includes most stellarators of interest. We represent position in terms of flux coordinates $\mathbf{r} = (\psi, \theta, \zeta)$, where $2\pi\psi$ is the toroidal flux within a flux surface, and θ and ζ are the poloidal and toroidal azimuths. In terms of these, the magnetic field may be written $\mathbf{B} = \nabla\psi \times \nabla\theta + \nabla\zeta \times \nabla\psi_p = \nabla\psi \times \nabla\alpha_p$, with $2\pi\psi_p$ the poloidal flux, Clebsch angle $\alpha_p \equiv \theta - \iota\zeta$, constant along a field line, and $\iota \equiv q^{-1} \equiv d\psi_p/d\psi$ the rotational transform. α_p and momentum $(e/c)\psi$ form a canonically conjugate pair for motion perpendicular to the field line. It is also useful to define an average minor radius $r(\psi)$ by $\psi \equiv \bar{B}_0 r^2/2$, with $\bar{B}_0 \equiv \bar{B}(r=0)$ the average magnetic field strength on axis. We consider toroidal systems with the nonaxisymmetric portion of magnetic field strength B dominated by a single helical phase $\eta_0 \equiv n_0\zeta - m_0\theta$,

$$B(\mathbf{x}) = \bar{B}(r)[1 - \epsilon_t(r)\cos\theta - \delta_h(\mathbf{x})\cos\eta_0], \tag{9}$$

with ripple strength $\delta_h(\mathbf{x})$ allowed to vary slowly over a flux surface, with flux-surface average $\epsilon_h(r) \equiv \langle\delta_h\rangle$.

A suitable choice for the aa variables is $\boldsymbol{\theta} = (\theta_g, \theta_b, \bar{\alpha}_p)$, $\mathbf{J} = (J_g, J_b, (e/c)\bar{\psi})$, with $J_g \equiv (Mc/e)\mu$ the gyroaction, μ the magnetic moment, θ_g the gyrophase, describing the fastest time scale of the motion, J_b the bounce action, θ_b its conjugate bounce phase, $\bar{\psi}$ the drift-orbit averaged value of ψ, and its conjugate phase $\bar{\alpha}_p$, the orbit-averaged Clebsch coordinate α_p, describing the slow, drift timescale. To make the periodicity of the drift angle 2π as for the other 2 phases, instead of $(\bar{\alpha}_p, (e/c)\bar{\psi})$ we use the closely related canonical pair $(\theta_d, J_d = (e/c)\bar{\psi}_d)$, with $\theta_d \equiv \bar{\alpha}_p/(1 - \iota q_{mn0})$, $\bar{\psi}_d \equiv \bar{\psi} - \bar{\psi}_p q_{mn0}$, where $q_{mn0} \equiv m_0/n_0$. For typical parameters, $\iota q_{mn0} \ll 1$, so that $(\theta_d \simeq \bar{\alpha}_p, \bar{\psi}_d \simeq \bar{\psi})$. Correspondingly one has the characteristic frequencies of motion $\mathbf{\Omega} \equiv (\Omega_g, \Omega_b, \Omega_d)$, with gyrofrequency Ω_g, bounce frequency Ω_b, and drift frequency Ω_d, and vector index $\mathbf{l} \equiv (l_g, l_b, l_d)$.

We adopt an eikonal form for the structure of any mode a,

$$\phi_a(\mathbf{x}) = \bar{\phi}_a(r) \exp i\eta_a(\mathbf{x}), \tag{10}$$

with wave phase $\eta_a(\mathbf{x}) \equiv [\int^r dr' \, k_r(r') + m\theta + n\zeta]$, and slowly-varying envelope $\bar{\phi}_a(r)$, assumed roughly constant over the radial excursion of a particle. Thus, mode a has local wavevector $\mathbf{k} \equiv \nabla\eta_a = k_r\nabla r + m\nabla\theta + n\nabla\zeta$. For the ZF potential $\phi_a \to \phi_Z$, one has $(m, n) = (0, 0)$.

Using form (10), one may evaluate the expression $h_{\mathbf{l}a}(\mathbf{J}) \equiv (2\pi)^{-3} \oint d\boldsymbol{\theta} \exp -i\mathbf{l} \cdot \boldsymbol{\theta} h_a(\mathbf{z})$ for the “coupling coefficient” of mode a to particles with actions $\mathbf{J}$. Writing $\eta_a(\mathbf{r}(\mathbf{z})) = \bar{\eta}_a + \delta\eta_a$, with $\delta\eta_a$ the portion of η_a oscillatory in $\boldsymbol{\theta}$ (so having zero $\boldsymbol{\theta}$ average), one finds $h_{\mathbf{l}a}(\mathbf{J}) = e_s\bar{\phi}_a(\bar{r}) \exp(i\bar{\eta}_a) G_{\mathbf{l}a}(\mathbf{J})$, with $G_{\mathbf{l}a} \equiv (2\pi)^{-3} \oint d\boldsymbol{\theta} \exp -i\mathbf{l} \cdot \boldsymbol{\theta} \exp i\delta\eta_a(\mathbf{z})$ the “orbit-averaging factor”. From Parseval’s theorem one may show these satisfy the important relation $1 = \sum_{\mathbf{l}} |G_{\mathbf{l}}|^2$, generalizing the much-used identity for Bessel functions $1 = \sum_l J_l^2(z)$.

Multiplying Poisson’s equation $-\nabla^2\phi_a(\mathbf{x}, \omega) = 4\pi \sum_s \delta\rho_s(\mathbf{x}, \omega)$ by $\phi_a^*(\mathbf{x}, \omega)$, putting Eq.(10) in Eqs.(6) and (7) and using $d^6\mathbf{z} = d\boldsymbol{\theta} d\mathbf{J} = d\mathbf{r} d\mathbf{p}$, one obtains the radial integral of the radially-local response equation $\mathcal{E}(r)$:

$\int drV'\bar{\phi}^*_a(r)\mathcal{E}(r)$, with $\mathcal{E}$ given by

$$k^2\mathcal{D}(\mathbf{k},\omega)\frac{e_i\bar{\phi}_a(r)}{T_i} = \sum_s \lambda_{si}^{-2}\sum_{\mathbf{l}}\langle G^*_{\mathbf{l}a}(\mathbf{J})\frac{i[\delta f_{s\mathbf{l}}(t=0)/f_{s0}+S_{s\mathbf{l}}(\omega)]}{(\omega-\mathbf{l}\cdot\mathbf{\Omega}+i\nu_{fs})}\rangle \qquad (11)$$

Here, $V' \equiv dV/dr$ is the radial derivative of the volume $V(r)$ enclosed by the flux surface r or ψ, $\lambda_{si}^2 \equiv T_i/(4\pi n_{s0}e_se_i)$, $k^2 \equiv |\mathbf{k}|^2$, and $\langle\ldots\rangle \equiv (2\pi)^{-2}\oint d\theta d\zeta\int d\mathbf{p}(f_0/n_0)\ldots$ is the flux surface and momentum-space average over the unperturbed distribution function f_0. Dielectric function $\mathcal{D}$ is given by $\mathcal{D}(\mathbf{k},\omega) \equiv 1+\sum_s\chi_s(\mathbf{k},\omega)$, with susceptibility $\chi_s(\mathbf{k},\omega) = (k\lambda_s)^{-2}g_s(\mathbf{k},\omega))$, and

$$g_s(\mathbf{k},\omega) = 1-\sum_{\mathbf{l}}\langle|G_{\mathbf{l}a}(\mathbf{J})|^2\frac{\omega-\omega^f_{*s}}{\omega-\mathbf{l}\cdot\mathbf{\Omega}+i\nu_{fs}}\rangle. \qquad (12)$$

Here, $\omega^f_* \equiv \omega_*[1+\eta(u^2-3)/2]$, with $\omega_* \equiv -k_\alpha cT/(eBL_n)$ the diamagnetic drift frequency, $\eta \equiv d\ln T/d\ln n$, $u \equiv v/v_s$ the particle velocity, normalized to the thermal speed v_s, $k_\alpha \equiv l_d/r$, and $L_n^{-1} \equiv -\partial\ln n_0/\partial r$. As usual, the 1 in $\mathcal{D}$ is the vacuum term from the left side of the Poisson equation, negligible in comparison with the χ_s, which correspond to $\delta\rho_{sA}$ in Eq.(6). The 1 in g_s comes from the adiabatic term K^{ad}_s in Eq.(7). The 2 terms on the right side of Eq.(11) arise from $\delta\rho_{s,B+C}$. This response equation is of essentially the same form as that obtained in Refs. 2 and 3, or of any linear response calculation. The differences lie in the form of the dielectric $\mathcal{D}$, and in the use of the aa form, which facilitates dealing with the range of timescales and of orbit-averaging effects in complex geometries in a general manner.

We now evaluate the $G_{\mathbf{l}}$. As discussed in previous applications[10–14] of the aa framework, to evaluate these one needs a description of the particle position $\mathbf{r}(\mathbf{z})$, to evaluate the required $\boldsymbol{\theta}$ integrations. The 3 trapping states (passing, toroidally-trapped, and helically-trapped) are indicated by trapping index $\tau = p, t$ and h, respectively. Then an approximate description of $\mathbf{r}(\boldsymbol{\theta})$ is

$$r = r_d+\delta r^{(d)}(\theta_d)+\delta r^{(b)}(\theta_b)+\delta r^{(g)}(\theta_g) \qquad (13)$$

$$\theta = \sigma_h \theta_d + \sigma_p \theta_b + \delta\theta^{(b)}(\theta_b) + \delta\theta^{(g)}(\theta_g)$$
$$\zeta = \zeta_{d0} + (\sigma_h q_{mn0} + \sigma_{tp} q)\theta_d + \delta\zeta^{(b)}(\theta_b) + \delta\zeta^{(g)}(\theta_g),$$

where we use trapping-state "switch" σ_τ to describe the behavior for different states τ in a single expression: $\sigma_\tau = 1$ for a particle in trapping-state τ, and 0 otherwise. Thus, $1 = \sigma_h + \sigma_t + \sigma_p$. Also, $\sigma_{tp} \equiv \sigma_t + \sigma_p$ equals 1 if a particle has $\tau = t$ or p, and 0 for $\tau = h$.

Eqs.(13) manifest 2 kinds of dependence on the phases θ_i ($i \to g, b, d$), a secular, linear dependence, and oscillatory dependences, held in functions $\delta x^{(i)}(\theta_i)$, with $x \to r, \theta, \zeta$. Here we approximate each of the latter by a harmonic, (co)sinusoidal form, *e.g.*, $\delta r^{(i)}(\theta_i) \simeq \rho_i \cos\theta_i$, with amplitude ρ_i. This is a very good approximation for gyromotion (with ρ_g the gyroradius), and a good approximation for bounce motion not too near a trapping-state boundary (with ρ_b the banana width). For simplicity, we assume that superbanana ($\tau = h$) particles do not detrap, but precess poloidally dominated by $E \times B$ poloidal drift, $\Omega_d \simeq \Omega_{dE}$, which is roughly constant on a given orbit, while drifting radially as $v_{Bt} \sin\theta$, as usual. [Here, $v_{Bt} = \epsilon_t \mu \bar{B}/(M\Omega_g r)$.] This produces superbananas which are displaced circles, with superbanana width $\rho_d = \sigma_h v_{Bt}/\Omega_{dE}$, a common approximation in stellarator transport theory. The radial drift motion is thus also harmonic in θ_d. For simplicity, we have neglected from this orbit description a second type of superbanana width, the finite radial excursions ρ_{dt} made by $\tau = t$ particles on the drift timescale, which give rise to the "banana-drift" transport branch.[7–9] Inclusion of this additional mechanism presents no difficulty for the basic formalism.

The h_1 or G_1 have been evaluated previously[10,13,14] for perturbations with nonzero m and n, but neglecting the effect of finite ρ_d. For the current application to ZFs, we keep finite ρ_d, but set $m = 0 = n$, making only the first of Eqs.(13) necessary. Using the Bessel identity $J_l(z) = (2\pi)^{-1} \oint d\theta e^{-il\theta} e^{iz\sin\theta}$ and Eq.(10), one finds

$$G_{\mathbf{l}a}(\mathbf{J}) = J_{l_g}(z_g) J_{l_b}(z_b) J_{l_d}(z_d) e^{-i\xi_a}, \tag{14}$$

with $z_{g,b,d} = k_r \rho_{g,b,d}$, and ξ_a a phase factor. Since $G_\mathbf{l}$ appears only as $|G_\mathbf{l}|^2$ in the theory here, the value of ξ_a does not enter.

For drift turbulence, which is driving the ZFs, one typically has $k_\perp^d \rho_{gi} \sim 0.3$, and frequencies $\omega_d \sim \omega_*(k_\perp^d)$. For ZFs, one has much smaller k_r and frequencies ω_Z, down by an order of magnitude, perhaps by the "mesoscale" ratio, $k_r^Z/k_\perp^d \sim \sqrt{\rho_{gi}/a}$. Thus, for both species, one has the ordering $\omega_Z, \Omega_d \ll \Omega_b \ll \Omega_g$, and $z_g < z_b < 1$. For the moment we leave the relative sizes of ω_Z and Ω_d unspecified. Also, one may have $z_d \gtrsim 1$ for trapped particles, for ions and also, notably, for electrons, as noted in Ref. 3. Thus, as opposed to tokamaks, in stellarators electrons can participate in orbit averaging, because their radial excursions on the drift timescale can be comparable with those of ions.

Because $z_{g,b} \ll 1$, the factors $J_{l_{g,b}}^2$ in $|G_\mathbf{l}|^2$ in Eq.(12) are negligible unless $l_{g,b} = 0$, reducing the triple sum there to a single sum $\sum_{l_d}$. In that sum, if one has $\omega \gg \Omega_d$, then over the l_d–range $\Delta l_d \sim z_d$ over which $J_{l_d}^2$ in Eq.(12) is appreciable the integrand does not change greatly, so that one can perform the summation, using the identity $\sum_l J_l^2 = 1$, which eliminates the $J_{l_d}^2$ factor, leaving only the factor $J_{l_g}^2 J_{l_b}^2$. In the other limit $\omega \ll \Omega_d$, the sum is dominated by the $l_d = 0$ term, and the effect of $J_{l_d}^2$ survives. Thus, for $\omega \ll \Omega_d$, all of gyro-, bounce- and drift-averaging contribute. Neglecting ν_{fs}, Eq.(12) becomes

$$g_s(\mathbf{k},\omega) \simeq 1 - \Lambda_{0b}(b_g, b_b), (\omega \gg \Omega_d), \tag{15}$$

$$g_s(\mathbf{k},\omega) \simeq 1 - \Lambda_{0d}(b_g, b_b, b_d), (\omega \ll \Omega_d),$$

where $\Lambda_{0d}(b_g, b_b, b_d) \equiv \langle J_g^2 J_b^2 J_d^2 \rangle$, $\Lambda_{0b}(b_g, b_b) \equiv \Lambda_{0d}(b_g, b_b, b_d = 0) \equiv \langle J_g^2 J_b^2 \rangle$, $J_{g,b,d}^2 \equiv J_0^2(z_{g,b,d})$, $b_g \equiv k_r^2 \rho_{gT}^2$, $b_b = b_g q^2/\epsilon_t^{1/2}$, and $b_d \equiv k_r^2 \rho_{dT}^2$, with $\rho_{gT} \equiv v_T/\Omega_g$, v_T the species thermal velocity, and $\rho_{dT} \equiv \rho_d(v = v_T) \propto v_T^2$.

The physics represented by Eqs.(15) is that if the the ZF drive in a stellarator has a time variation slow compared with Ω_d [*cf.* Eq.(15b)], $\tau = h$ particles have time to partially shield out ϕ_Z by drifting along their collisionless superbanana orbits, an averaging mechanism not available to tokamaks. If the ZF drive varies rapidly compared with Ω_d [Eq.(15a)], this new mechanism for radial averaging is lost. Eq.(15a) also holds in the tokamak limit ($\epsilon_h \to 0$), where one has $z_d = 0$. And in the cylindrical limit ($\epsilon_t \to 0$) of a large-aspect ratio tokamak, z_b vanishes, and the Λ's in Eqs.(15) are replaced by the more familiar $\Lambda_0(b_g) \equiv \Lambda_{0b}(b_g, b_b = 0) \equiv \langle J_g^2 \rangle = I_0(b_g)e^{-b_g}$, with $I_0(b)$ the modified Bessel function of the first kind. For $b_g < 1$, one has $\Lambda_0(b_g) \simeq 1 - b_g$, and thus $g_s \simeq b_g$, the contribution from the classical polarization current $J^{p,g}$. The functions Λ_{0b} and Λ_{0d} succinctly describe the additional contributions from finite b_b, corresponding to shielding due to the "bounce" polarization current $J^{p,b}$ computed in Refs. 2 and 3, and from finite b_d, corresponding to a "drift" polarization current $J^{p,d}$, extending the result in Ref. 3, as noted in Sec. I.

We approximately evaluate Λ_{0b} and Λ_{0d} using the small–argument expansion $J_0(z) \simeq 1 - (z/2)^2$ for the Bessel functions. (While $z_{g,b} < 1$ is a good assumption, one may have $z_d < 1$ or $z_d \gtrsim 1$. The above expressions for $\Lambda_{0d}, \Lambda_{0b}$ are valid for arbitrary values of $z_{g,b,d}$.) First taking $z_d < 1$, one has $\Lambda_{0d}(b_g, b_b, b_d) \simeq 1 - \frac{1}{2}\langle z_g^2 \rangle - \frac{1}{2}\langle z_b^2 \rangle - \frac{1}{2}\langle z_d^2 \rangle$. Evaluating these averages, one finds $\frac{1}{2}\langle z_g^2 \rangle = b_g$, $\frac{1}{2}\langle z_b^2 \rangle = c_b b_b$, and $\frac{1}{2}\langle z_d^2 \rangle = c_d b_d$. Here, $c_b \simeq 3\sqrt{2}/\pi \simeq 1.4$, and $c_d \simeq (15/2)F_h$, with $F_h = (2/\pi)\sqrt{2\epsilon_h}$ the fraction of particles with $\tau = h$, here assuming ripple strength δ_h is constant on a flux surface. The factor c_b has been evaluated here for a tokamak, approximately agreeing with the value 1.6 found in Ref. 2. Its value for a stellarator is computed in Ref. 3. Coefficient c_d is proportional to F_h because only $\tau = h$ particles have superbanana excursions ρ_d, and the large factor 15/2 there enters because of the strong energy weighting from $\rho_d^2 \propto v^4$. Eqs.(15) then yield

$$g_s(\mathbf{k},\omega) \simeq b_g + c_b b_b, (\omega \gg \Omega_d), \tag{16}$$
$$g_s(\mathbf{k},\omega) \simeq b_g + c_b b_b + c_d b_d, (\omega \ll \Omega_d).$$

Assuming the source terms on the right side of Eq.(11) remain unchanged, one sees that ZFs in a stellarator with $\omega_Z \ll \Omega_d$ will be appreciably reduced below those in a stellarator with $\omega_Z \gg \Omega_d$ or in a tokamak, due to the addition contribution from $J^{p,d}$, to which not only ions, but also electrons, may contribute.

One notes that the drift contribution $g^d = c_d b_d \simeq F_h (k_r \rho_d)^2$ in Eq.(16b) has a form analogous to the bounce and gyro contributions, as opposed to the scaling $g^d \simeq F_h$ found in Refs. 3,4, noted in Sec. I. In that work, the term $\Omega_d \partial_{\theta_d}$ was neglected in their counterpart of kinetic equation (4). Since $\rho_d \sim 1/\Omega_d$, that work is accordingly done in the limit of very large superbanana width $\rho_d(\tau = h) \to \infty$ [but $\rho_d(\tau = t, p) = 0$, as before]. In that limit, $J_d^2 \equiv J_0^2(z_d)$ equals 0 (1) for $\tau = h(t,p)$, and one finds $\Lambda_{0d}(b_g, b_b, b_d) \simeq 1 - b_g - c_b b_b - F_h$, yielding the limit $g^d = F_h$ found in Refs. 3,4, instead of the $z_{g,b,d} < 1$ form given in Eq.(16b) above.

The ZF time evolution as the successive shielding mechanisms set in is thus as follows. [The longer-time, diffusive ZF evolution is discussed in Sec. IV.] For times short compared to a gyroperiod ($t < \Omega_g^{-1}$), none of the 3 shielding mechanisms has time to be established. For $\Omega_g^{-1} < t < \Omega_b^{-1}$, the classical polarization term $g^g \simeq b_g$ begins to shield ZF potential ϕ_Z, while the gyrophase-dependent ($l_g \neq 0$) portions of the ballistic term in Eq.(11) phase mix away. This very early phase is not captured by gyrokinetic simulations, which carry no gyrophase dynamics. An analogous, bounce-related phase is then entered for $\Omega_b^{-1} < t < \Omega_d^{-1}$, in which the additional bounce-shielding term $g^b \simeq c_b b_b$ produces a further damping of ϕ_Z, superposed on which are oscillations at $\omega \sim \Omega_b$ from the $l_b \neq 0$ ballistic terms, which phase mix away, as seen in simulations.[3,4] If the low-frequency ($\omega < \Omega_d$) portion of the ZF source $S(\omega)$

is dominant, then in the interval $\Omega_d^{-1} < t$, the drift contribution $g^d \simeq c_d b_d$ establishes itself, further damping the ZF to its shielded value.

The expressions given thus far have assumed very low collisionality, $\nu_f < \Omega_d$. Such an ordering pertains to helically–trapped particles in the so-called "superbanana regime",[15,16] with $\nu_f \to \nu_h \equiv \nu/(2\epsilon_h)$, the frequency for $\tau = h$ particles to scatter out of a ripple well. Ions may satisfy such an ordering for realistic parameters. For electrons, being much more collisional, this is less common, but can occur for very large Ω_{dE}, such as sometimes produced at the electron root.[17] More typically, electrons collisionally detrap having completed only a radial excursion $\Delta r_{d\nu} \simeq v_{Bt}/\nu_h$ smaller than their full superbanana excursion $\rho_d = v_{Bt}/\Omega_d$, producing transport in the $1/\nu$ regime.[18,19] In this case the electron contribution to g^d will be reduced. Mathematically, increasing ν_f in Eq.(12) broadens the nearby resonances at successive values of l_d, for each value of (l_g, l_b). When ν_f becomes larger than $\Delta l_d \Omega_d$ (where as above, $\Delta l_d \sim z_d$), then the resonances add to form a single (l_g, l_b) resonance, and as in the case of $\omega > \Omega_d$, the l_d resonances may be summed over, eliminating the $J^2_{l_d}$ factor. [An analogous coalescence of l_b-resonances may be expected at still higher ν_f, where ν_f becomes larger than Ω_b. In this case, the separate l_b resonances coalesce, removing the $J^2_{l_b}$ factor, and the bounce contribution g^b to g.]

In Ref. 5 the "time-dependent viscosity" is computed in the $1/\nu$ and banana regimes. In the moment method formulation of collisional transport used there, the radial fluxes Γ_s giving $J_r = \sum_s e_s \Gamma_s$ in Eq.(1b) are proportional to the averaged toroidal viscosity $\langle \mathbf{B}_t \cdot \nabla \cdot \boldsymbol{\pi} \rangle$ (with $\boldsymbol{\pi}$ the viscosity tensor). Thus, in that formal approach, the polarization contributions to J_r, corresponding to the term in χ in Eq.(1b), are those coming from the high-frequency limit of $\boldsymbol{\pi}$, and so of perturbed distribution function δf. In the linear-response approach adopted in Refs. 3,4 and the present work, the same δf instead is used to determine the dielectric response $\mathcal{D}$.

The bounce-averaged kinetic equation used in Ref. 5 may be written

$$\partial_t \delta f \simeq -\bar{\dot{r}} \partial_r f_0 + C\delta f, \qquad (17)$$

which may be obtained from Eq.(4) here, but replacing source term Sf_0 there with collision term $C\delta f$, and neglecting the bounce-average of the convective term $\hat{H}_0$, valid in the $1/\nu$ regime. (Here, $\bar{\dot{r}}$ is the bounce-averaged radial drift velocity.) In the low-frequency limit, $\partial_t \delta f$ is neglected in comparison with $C\delta f$, and (17) reduces to the usual equation used to compute δf and the flux in the $1/\nu$ regime. For the high-frequency limit, $C\delta f$ is instead neglected, and δf has essentially the same form as at low-frequency, but with ν_h replaced by a flow-damping rate γ, *i.e.*, with $C\delta f \simeq -\nu_h \delta f$ replaced by $\partial_t \delta f \simeq \gamma \delta f$. These low- and high-frequency limits are both captured by the aa-solution δf_1 obtained as in Eq.(5), dropping the source and initial-value terms there. Taking the gyro- and bounce-averaged portions of this ($l_{g,b} = 0$), we write $\bar{\dot{r}}(\theta_d) = -i \sum_{l_d=1}^{\infty} v_{l_d} \exp(i l_d \theta_d)$, with $v_{l_d} = -v_{-l_d}$ real, one has $\delta f_{l_d} = i G_0 v_{l_d} \partial_r f_0$, with $\nu_f \to \nu_h$ in G_0 as defined following Eq.(5). Taking $\omega = i\gamma$, and the lowest nonvanishing drift harmonic ($l_d = \pm 1$) for simplicity, gives $\delta f \simeq v_1 \partial_r f_0 e^{i\theta_d}/[\Omega_d - i(\gamma + \nu_h)] + c.c.$, where $v_1 \simeq v_{Bt}/2$. In the $1/\nu$ regime, Ω_d is neglected in the denominator, and this expression approximates the form obtained in Ref. 5 combining its high ($\nu_h \to 0$) and low ($\gamma \to 0$) results. This form is also valid in the lower-ν superbanana regime considered above, and is readily generalized to one keeping all drift-harmonics l_d.

One may also consider the effect on $\mathcal{D}$ or g of techniques developed to minimize stellarator nc transport. It has been argued[3,20–22] that neoclassically-optimized stellarators should also have lower turbulent transport, due to less damping of ZFs. The basic idea of most nc optimization techniques has been to reduce ripple transport by reducing $\rho_d \simeq v_{Bt}/\Omega_d$, either by diminishing the radial drift velocity amplitude v_{Bt}, or by enhancing the poloidal precession frequency Ω_d.[1] One sees from the above expressions for Λ_{0d}, characterized by the argument $\frac{1}{2}\langle z_d^2\rangle = \frac{1}{2}k_r^2\langle\rho_d^2\rangle$, that this is just what is needed to diminish the low-ω shielding from g^d.

One notes that associated with each of the 3 polarization contributions in Eq.(16b) is a collisional (classical+nc) transport mechanism: the gyromotion producing the classical polarization term g^g also gives rise to classical transport, the bounce motion producing g^b gives rise to axisymmetric nc transport, and the drift motion yielding g^d also produces the "superbanana" branch of transport, dominant in conventional stellarators. As indicated above, for simplicity we have not included in the calculations leading to Eqs.(15) and (16) two additional contributions, one coming from the radial drift excursion ρ_{dt} made by $\tau = t$ particles in a nonsymmetric torus, and one from the finite banana widths ρ_{bh} from $\tau = h$ particles. Each of these makes a contribution to the shielding from g, and also corresponds to a transport mechanism, the former to the banana-drift branch of transport,[7–9] and the latter to the nc transport in a straight (helically symmetric) stellarator. Thus, instead of the 3 contributions to ZF shielding in Eq.(16b), a full description would include 5, each corresponding to one of the 5 branches of collisional transport.[1]

The form of the polarization shielding contributions to g is close to the form of the radial transport coefficient D for each mechanism. For each mechanism j, one may use the heuristic form $D^j \simeq F_j\nu_{fj}(\Delta r_j)^2$, with F_j the fraction of particles participating in that mechanism, Δr_j the radial step in the random walk process, and ν_{fj} the effective stepping frequency in that random walk. For example, for the axisymmetric banana regime, one has $F_j \rightarrow F_t \simeq \epsilon_t^{1/2}$, $\Delta r_j \rightarrow \rho_b \simeq q\rho_g/\epsilon_t^{1/2}$, and $\nu_{fj} \rightarrow \nu_t \simeq \nu/\epsilon_t$, yielding the usual banana diffusion expression $D^{bn} \simeq \nu q^2\rho_g^2\epsilon_t^{3/2}$. For the $1/\nu$ superbanana regime, one has $F_j \rightarrow F_h \simeq \epsilon_h^{1/2}$, $\Delta r_j \rightarrow v_{Bt}/\nu_h$, and $\nu_{fj} \rightarrow \nu_h \simeq \nu/\epsilon_h$. On the other hand, the small-argument, low-ν contribution to g in Eqs.(16) is $g^j \simeq \frac{1}{2}k_r^2\langle\rho_j^2\rangle$. We approximately include the ν-dependence in g^j described above by replacing ρ_j with Δr_j (which, as discussed above, becomes less than ρ_j for larger ν), and a factor F_j arises from doing the indicated average. We then have approximately $g^j \simeq F_j(k_r\Delta r_j)^2$, and thus $g^{j'}/g^j \simeq (D^{j'}/D^j)(\nu_{fj}/\nu_{fj'})$. Therefore, taking $j \to g$ and $j' \to b$, one expects the gyro- contribution g^g in Eqs.(16) to be smaller than the bounce contribution g^b, because classical diffusion D^g is subdominant to banana diffusion D^b. Similarly, taking $j \to b$, $j' \to d$, one expects the drift contribution g^d to dominate g^b in Eq.(16b) approximately when superbanana transport D^d becomes large compared with D^b.

The main goal of nc transport optimization has been to reduce D^d below the anomalous level D^{an}, typically larger than D^b by an order of magnitude. Using the above relations, this yields the approximate criterion $g^d/g^b \lesssim D^{an}/D^b$ for acceptably low g^d. However, since one expects D^{an} to be an increasing function of g^d due to reducing ZFs, this criterion is somewhat indeterminate, requiring a specific description of this functional dependence.

IV. STATISTICS OF ZF EVOLUTION

As noted in Sec. I the time evolution of the ZFs is governed by a Langevin-like equation, given by inserting Eq.(1b) into (1a). In the ω domain, this may be written

$$-i\omega E(\omega) + \gamma_E E(\omega) = c_S(\omega), \tag{18}$$

where $\mathcal{D}(\omega) \equiv 1 + \chi(\omega)$ as before, $\gamma_E(\omega) \equiv 4\pi\sigma/\mathcal{D}(\omega)$, and $c_S(\omega) \equiv -4\pi F_S/B\mathcal{D}(\omega)$. We analyze this for the longer-time diffusive behavior of E.

Assuming first that $\mathcal{D}(\omega) = \mathcal{D}_0$ is ω-independent, then $\gamma_E = \gamma_{E0}$ is also ω-independent, and in the time domain Eq.(18) reduces to a standard Langevin equation for E,

$$\partial_t E(t) + \gamma_E E(t) = c_S(t). \tag{19}$$

The source c_S that drives the zonal fbws is approximated as random. Thus, ensemble averaging (19), one has

$$\partial_t \langle E \rangle_p = -\gamma_E \langle E \rangle_p. \tag{20}$$

If γ_E is sufficiently small compared to the inverse correlation time $\nu_S = 1/\tau_S$ of c_S, the short time response of E to c_S is $E(t) = \int_{-\infty}^{t} c_S(t')dt'$. Thus, ensemble averaging $\partial_t E^2$, one finds $\partial_t \langle E^2 \rangle_p = S^{c0} \equiv \int_{-\infty}^{\infty} d\tau C^c(\tau)$, where $C^c(\tau) \equiv \langle c_S(t) c_S(t-\tau) \rangle_p$ is the correlation function for c_S. Its Fourier transform is the spectral function $S^c(\omega)$, and $S^{c0} \equiv S^c(\omega = 0)$. Thus the random force F_S causes diffusion in E, with diffusion coefficient $S^{c0}/2$. The corresponding pdf $p(E,t)$ for E obeys

$$\partial_t p = \partial_E (\frac{1}{2} S^{c0} \partial_E p + \gamma_E E p), \tag{21}$$

again satisfying Eq.(20), while $\langle E^2 \rangle_p = \int E^2 p dE$ obeys

$$\partial_t \langle E^2 \rangle_p = S^{c0} - 2\gamma_E \langle E^2 \rangle_p. \tag{22}$$

Fourier transforming this, one finds an expression for the spectrum in terms of the driving source S^{c0}, $\langle E^2(\omega) \rangle_p = S^{c0}/(-i\omega + 2\gamma_E)$. Neglecting the restoring term predicts a purely diffusive $\langle E^2(t) \rangle_p$, increasing without bound, corresponding to $\langle E^2(\omega) \rangle_p \simeq S^{c0}/(-i\omega)$. The restoring term removes the $1/\omega$ divergence for $\omega \lesssim \gamma_E$. In the steady state, Eqs.(22) and (21) yield $\langle E^2 \rangle_p = S^{c0}/2\gamma_E$, and $p(E) = p_0 \exp(-\gamma_E E^2/S^{c0})$. Since $\gamma_E \sim \mathcal{D}^{-1}$ and $S^{c0} \sim \mathcal{D}^{-2}$, one has $\langle E^2 \rangle_p \sim \mathcal{D}^{-1}$. Thus, assuming the turbulent forces F_S driving the ZFs are unaffected, the larger $\mathcal{D}$ implied at low-ω by the drift-polarization shielding would reduce γ_E, but reduce the diffusion S^{c0} even more, resulting in a smaller ZF amplitude $\langle E^2 \rangle_p^{1/2}$.

The flux through E space is represented in Eq.(21) as $\mathcal{F} = -\frac{1}{2} S^{c0} \partial_E p - \gamma_E E p$. A cross field viscosity acting on $E \times B$ flow could also be included with an additional term in the flux, $\mathcal{F} = -\frac{1}{2} S^{c0} \partial_E p - \gamma_E E p + \nu \nabla^2 E$.

We have seen in Sec. IV [*e.g.*, Eqs.(16)] that $\mathcal{D}$ has an ω-dependence, making γ_E ω-dependent as well, so the constant-γ_E Langevin treatment just given is only approximately valid. Eq.(18) is more easily treated in the ω-domain. Solving it for $E(\omega)$, one finds an expression for its spectral function,

$$S^E(\omega) = S^c(\omega)/(\omega^2 + \gamma_E^2(\omega)), \quad (23)$$

Taking the usual model for C^c, $C^c(\tau) = \langle c_S^2(t)\rangle_p \exp(-\nu_S|\tau|)$, one has $S^c(\omega) = \langle c_S^2(t)\rangle_p 2\nu_S/((\omega^2 + \nu_S^2(\omega))$. Because $\gamma_E \ll \nu_S$, the falloff with ω of S^E in Eq.(23) is controlled by the factor $(\omega^2 + \gamma_E^2(\omega))$, and one may take $\omega \simeq 0$ in the factor $S^c(\omega)$ there. Thus, $S^E(\omega) \simeq S^{c0}/(\omega^2 + \gamma_E^2(\omega))$, and

$$C^E(\tau) \simeq S^{c0} \int \frac{d\omega}{2\pi} \frac{\exp(-i\omega\tau)}{\omega^2 + \gamma_E^2(\omega)} \simeq S^{c0} \frac{\exp(-\gamma_E|\tau|)}{2\gamma_E}, \quad (24)$$

where the final form strictly holds only for γ_E independent of ω. Setting $\tau = 0$ in this expression recovers the steady-state result given above for $\langle E^2\rangle_p$. The first form for C^E in Eq.(24) is valid for an ω-dependent γ_E.

V. SUMMARY

In this work, we have used the action-angle formalism to study the shielding of ZFs, obtaining general expressions for their polarization shielding, and the timescales on which they develop. The expressions are valid for arbitrary radial excursion sizes (gyroradius ρ_g, bounce/banana width ρ_b, and radial drift excursion ρ_d) on each of the 3 timescales of the collisionless motion, and show that the drift polarization shielding yields a contribution of a form analogous to those from shielding on the gyro- and bounce- timescales, extending earlier results for this contribution, which can be the dominant contribution to the polarization shielding.

The evolution of ZFs on a longer, diffusive timescale is governed by a Langevin-like equation, with radial electric field $E_r(t)$ moving diffusively about roots E_a of the ambipolarity equation. The resultant probability distribution function is bounded, a balance between the turbulent fluctuations inducing diffusion, and the nc fluxes providing a restoring force to $E \equiv E_r - E_a = 0$. Expressions for the restoring force, diffusion coefficient, and steady-state distribution function have been obtained. The linear polarization contributions enter into each of these. The larger drift-polarization shielding predicted for stellarators should cause a smaller restoring force, weaker diffusion, and smaller ZF amplitude $\langle E^2 \rangle_p^{1/2}$, assuming the turbulent forces F_S are unchanged.

We have noted that each contribution g^j ($j \rightarrow g, b, d$) to the shielding function g [Eq.(12), (15), or (16)] corresponds to a particular collisional transport mechanism, and moreover, that the scalings and relative sizes of the g^j are quite similar to those of the radial transport coefficients D^j. Thus, stellarators with neoclassically-optimized designs (reduced D^d) also have reduced drift-polarization shielding g^d, and thus, a larger ZF amplitude. Assuming the amplitude of the source [S_{I} in Eq.(11) or F_S in Eq.(1b)] is unchanged, this implies the tendency suggested in earlier work, that neoclassically-optimized designs will have larger ZFs, and consequently lower turbulent transport as well. However, such an assumption about the source has not yet been demonstrated, and further study is needed to clarify the variation with machine design of these source terms, and of the consequent level of turbulent transport.

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[1] H.E. Mynick, *Phys. Plasmas* **13**, 058102 (2006).

[2] M.N. Rosenbluth, F.L. Hinton, *Phys. Rev. Letters* **80**, 724 (1998).

[3] H. Sugama, T.H. Watanabe, *Phys. Rev. Letters* **94**, 115001 (2005).

[4] H. Sugama, T.H. Watanabe, *Phys. Plasmas* **13**, 012501 (2006).

[5] K.C. Shaing, *Phys. Plasmas* **13**, 052505 (2005).

[6] A. N. Kaufman, *Phys. Fluids* **15**, 1063 (1972).

[7] A.H. Boozer, *Phys. Fluids* **23**, 2283 (1980).

[8] R. J. Goldston, R. B. White and A. H. Boozer, *Phys. Rev. Lett.* **47**, 647 (1981).

[9] R. Linsker, A.H. Boozer, *Phys. Fluids* **25**, 143 (1982).

[10] H.E. Mynick,J.A. Krommes, *Phys.Fluids* **23**, 1229-1237 (1980).

[11] R. D. Hazeltine, S. M. Mahajan and D. A. Hitchcock, *Phys. Fluids* **24**, 1164 (1981).

[12] I.B. Bernstein and K. Molvig, *Phys. Fluids* **26**, 1488 (1983).

[13] H.E. Mynick and R.E. Duvall, *Phys. Fluids*–B 1, 750 (1989).

[14] H.E. Mynick, A.H. Boozer, *Phys. Plasmas* **12**, 062513 (2005).

[15] A.A. Galeev, R.Z. Sagdeev, H.P. Furth, M.N. Rosenbluth, *Phys. Rev. Letters* **22**, 511 (1969).

[16] A.A. Galeev,R.Z. Sagdeev, Sov. Phys. Usp. **12**, 810 (1970).

[17] H.E. Mynick, W.N.G. Hitchon, *Nucl. Fusion* **23**, 1053 (1983).

[18] A.A. Galeev, R.Z. Sagdeev, H.P. Furth, *Zh.Prikl.Mekh. i Tekhn.Fiz.*, 3 (1968).

[19] J.W. Connor, R.J. Hastie, *Phys. Fluids* **17**, 114 (1974).

[20] H. Yamada, J.H. Harris, A.Dinklage, E.Escasibar, F.Sano, S.Okamura, U.Stroth, A.Kus, J.Talmadge, S.Murakami, M.Yokoyama, C.Beidler, V.Tribaldos, K.Y.Watanabe, *Proc. 31st EPS Conference on Plasma Physics, P5-9,* (London, June 29-July 2, 2004).

[21] K.C. Shaing, *Physics of Plasmas* **12**, 082508 (2005).

[22] K. Watanabe, H. Sugama, *Proceedings of the 21st International Conference on Plasma*

Physics and Controlled Nuclear Fusion Research (Chengdu, China, October, 2006) paper EX/5-4, (International Atomic Energy Agency, Vienna, Austria, 2007).

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