Princeton Plasma Physics Laboratory

PPPL-4224

PPPL-4224

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H. Qin, R.H. Cohen,
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Preprint
(February 2007)

Prepared for the U.S. Department of Energy under Contract DE-AC02-76CH03073.

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H. Qin

Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543

R. H. Cohen, W. M. Nevins, and X. Q. Xu

Lawrence Livermore National Laboratory, Livermore, CA 94550

Abstract

It turns out that gyrokinetic theory can be geometrically formulated as a special case of a geometrically generalized Vlasov-Maxwell system. It is proposed that the phase space of the spacetime is a 7-dimensional fiber bundle P over the 4-dimensional spacetime M, and that a Poincaré-Cartan-Einstein 1-form γ on the 7-dimensional phase space determines particles' worldlines in the phase space. Through Liouville 6-form Ω and fiber integral, the 1-form γ also uniquely defines a geometrically generalized Vlasov-Maxwell system as a field theory for the collective electromagnetic field. The geometric gyrokinetic theory is then developed as a special case of the geometrically generalized Vlasov-Maxwell system. In its most general form, gyrokinetic theory is about a symmetry, called gyro-symmetry, for magnetized plasmas, and the 1-form γ again uniquely defines the gyro-symmetry. The objective is to decouple the gyro-phase dynamics from the rest of particle dynamics by finding the gyro-symmetry in γ. Compared with other methods of deriving the gyrokinetic equations, the advantage of the geometric approach is that it allows any approximation based on mathematical simplification or physical intuition to be made at the 1-form level, and yet the field theories still have the desirable exact conservation properties such as phase space volume conservation and energy-momentum conservation if the 1-form does not depend on the spacetime coordinate explicitly. A set of generalized gyrokinetic equations valid for the edge plasmas is then derived using this geometric method. This formalism allows large-amplitude, time-dependent background electromagnetic fields to be developed fully nonlinearly in addition to small-amplitude, short-wavelength electromagnetic perturbations. The fact that we adopted the geometric method in the present study does not necessarily imply that the major results reported here can not be achieved using classical methods. What the geometric method offers is a systematic treatment and simplified calculations.

PACS numbers: 52.20.Dq,52.30.Gz,45.50.-j

I. INTRODUCTION

The kinetic equation system that is most analytically and algorithmically suitable for studying the dynamics of edge plasma in magnetic fusion devices is the gyrokinetic equation system [1–30]. The origin of gyrokinetic theory can be traced back to the early work of extending the Chew-Goldberger-Low theory [31] to higher orders by Frieman, Davidson, and Langdon [1, 2]. The introduction of guiding-center coordinates by Catto [5] and the Lie perturbation methods by Cary [32, 33] and Littlejohn [34] played important role in the development of gyrokinetic theory. Littlejohn developed the theory of guiding center using the non-canonical coordinate perturbation method [6, 9, 11, 12]. Lee [35] first realized that the gyrokinetic Poisson equation is nontrivially different from the regular Poisson equation. The most important difference is the "polarization drift density", which surprisingly has exactly the same form as an "extra" term discovered early by Friedman *et al* [36] in the Poisson equation for implicit schemes under different context and motivation. Soon, Dubin *et al* [13] applied Hamiltonian Lie perturbation method to the derivation of the gyrokinetic equation. The Lagrangian Lie perturbation method suitable for plasma kinetic theories using guiding center coordinates was introduced by Littlejohn [14] and Boghosian [37]. Hahm [15, 18] and Brizard [16] used the Lagrangian non-canonical perturbation method in their derivation of gyrokinetic equations. Subsequently, many aspects [17, 19–26, 28, 29] of the modern gyrokinetic theory, such as the concept of gyro-center gauge [24], high frequency gyrokinetics [21, 24], and gyro-center pull-back transformation [23, 28] have been worked out. The variational gyrokinetic formalisms were developed by Sugama [25] and Brizard [26], and similar work were previously done by Similon [38] and Boghosian [37]. The terminology of "gyrokinetic field theory" was first introduced by Sugama [25]. Gyrokinetic theory has become the foundation for modern large scale computer simulation studies of tokamak physics [35, 36, 39–48]. However, it is difficult to apply previously derived gyrokinetic system to the edge plasmas due to the unique features of their dynamics. In the pedestal cycle for H-modes, there exists a long-term dynamics for the pedestal build-up when the plasma is heated by neutral beam injections. The exact dynamics of the pedestal build-up is determined by the short time-scale, nonlinearly saturated microturbulence. The continuous build-up of pedestal eventually will drive edge localized mode (ELM) unstable [49, 50], which is also short time-scale. The nonlinearly evolved ELM reduce the height of the pedestal by a large portion and the pedestal starts to grow again, which marks the beginning of another pedestal cycle. In the present study, we develop a general gyrokinetic system, where the long-term pedestal dynamics is described by a time-dependent background, and the microturbulence and ELMs are described by nonlinear perturbations on the dynamic background. Such a split between dynamic background and perturbations is also convenient when studying the physics associated with the electric field in the radial direction $\mathbf{E}_r$ in the edge. Because the pedestal width L_p is much smaller than the minor radius, the $\mathbf{E}_r$ developed is much bigger than that in the core region. Since the pedestal is time-dependent, so is $\mathbf{E}_r$. It is therefore necessary to allow a large background electric field $\mathbf{E}_0(t)$ to nonlinearly evolve in the gyrokinetic equation system. The background magnetic field $\mathbf{B}_0(t)$ is allowed to be time-dependent as well, which will conveniently include the change of magnetic equilibrium during the pedestal cycle or the ramp-up phase of the toroidal current. In previous gyrokinetic systems, the nonlinear dynamics of the background electromagnetic field was not treated.

The most important new feature of the present study is that a geometric method is adopted. We first developed a geometrically generalized Vlasov-Maxwell system which is valid for any particle-field interaction model and applies to a wide range of kinetic systems such as gyrokinetic models for magnetized plasmas and kinetic descriptions for high intensity charged particle beams. We propose that the phase space of the spacetime is a 7-dimensional fiber bundle P over the 4-dimensional spacetime M, and that a Poincaré-Cartan-Einstein 1-form γ on the 7D phase space determines particles' worldlines in the phase space. Through the construction of Liouville 6-form Ω and fiber integral, the 1-form γ also elegantly and uniquely defines the geometrically generalized Vlasov-Maxwell system as a field theory for the collective electromagnetic field. The geometric gyrokinetic theory is then developed as a special case of the geometrically generalized Vlasov-Maxwell system. In its most general form, gyrokinetic theory is about a symmetry, called gyro-symmetry, for magnetized plasmas. Our objective is to decouple the gyro-phase dynamics from the rest of particle dynamics by finding the gyro-symmetry. Obviously, this is fundamentally different from the conventional gyrokinetic concept of "averaging out" the "fast gyro-motion". This objective is accomplished by asymptotically constructing a good coordinate system, which is of course a nontrivial task. The fact that we adopted the geometric method in the present study does not necessarily imply that the major results reported here can not be achieved using classical methods. What the geometric method offers is a systematic treatment and simplified calculations. Indeed, the perturbative calculation is greatly simplified by using the Lie coordinate perturbation method [11, 32, 33] enabled by the geometric nature of the phase space dynamics. Compared with other methods of deriving the gyrokinetic equations, the advantage of the geometric approach is that it allows any approximation based on mathematical simplification or physical intuition to be made at the 1-form level, and yet the equation system still has the desirable exact conservation properties such as phase space volume conservation and energy-momentum conservation.

II. GEOMETRICALLY GENERALIZED VLASOV-MAXWELL EQUATIONS

Because it turns out that the geometry of the Vlasov-Maxwell equations is best manifested in the spacetime of relativity, we will start from the phase space for spacetime. The phase space where the Vlasov-Maxwell equations reside is a 7-dimensional manifold

$$P = \{(x,p) \mid x \in M,\ p \in T_x^* M,\ g^{-1}(p,p) = -m^2c^2\} \ , \tag{1}$$

where M is the 4-dimensional spacetime, T^*M is the 8-dimensional cotangent bundle of M, and g^{-1} is the inverse of the metric tensor of M defined by

$$(g^{-1})^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma \ . \tag{2}$$

The phase space is a fiber bundle over spacetime M (see Fig. 1),

$$\pi :\ P \longrightarrow M \ . \tag{3}$$

The worldlines of particles on P are determined by a given Poincaré-Cartan-Einstein 1-form γ on P through the Hamilton's equation

$$i_\tau d\gamma = 0 \ , \tag{4}$$

where τ is a vector field, whose integrals are particles' worldlines on the 7D phase space P (including time). Here $d\gamma$ is the exterior derivative of γ and $i_\tau d\gamma$ is the inner product

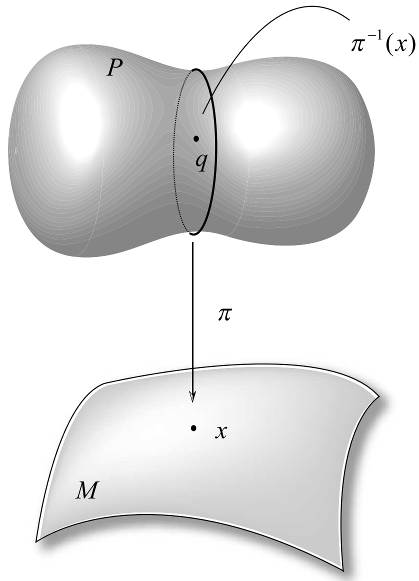

FIG. 1: Phase space and fiber integral.

between $d\gamma$ and τ. The collective electromagnetic field is given by the potential 1-form A (normalized by c/e) on M. In a Cartesian inertial coordinate system x^μ ($\mu = 0, 1, 2, 3$),

$$x^0 = ct \ , \ A_0 = -\phi \ , \ \text{and } A = (-\phi, \mathbf{A}) \ , \tag{5}$$

where ϕ and $\mathbf{A}$ are the scalar and vector potential of the electromagnetic field. The interaction between particles and the field is completely determined by the dependence of γ on A.

Very elegantly, the Poincaré-Cartan-Einstein 1-form γ also geometrically defines a field theory for the interaction between particles and the collective electromagnetic field. Define the Liouville 6-form Ω on the 7D phase space P as

$$\Omega = -\frac{1}{3!} d\gamma \wedge d\gamma \wedge d\gamma \, . \tag{6}$$

In the 7D phase space, the Liouville Theorem of phase space volume conservation is replaced by

$$L_\tau \Omega = i_\tau d\Omega + d\left(i_\tau \Omega\right) = 0 \, , \tag{7}$$

where L_τ is the Lie derivative along the vector field τ of a particle's worldline on P. The geometrically generalized Vlasov equation for the particle distribution function f in the phase space is

$$L_\tau f = i_\tau df = 0\,. \tag{8}$$

A simple but important property for f and Ω is

$$L_\tau \left(f\Omega\right) = \left(L_\tau f\right)\Omega + \left(L_\tau \Omega\right) f = 0\,, \tag{9}$$

from which we can derive the conservative version of the Vlasov equation as

$$d\left[i_{\tau f}\left(\Omega \wedge dt\right)\right] = 0\,. \tag{10}$$

The dynamics of the collective electromagnetic field A is described by the classical field theory specified by the action

$$S = \int_x \mathcal{L}\,, \tag{11}$$

where the Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L}(x) = -\frac{1}{2} dA \wedge *dA + 4\pi \int_{\pi^{-1}(x)} f\Omega \wedge \gamma\,. \tag{12}$$

Here $\int_{\pi^{-1}(x)}$ is the fiber integral [51] at the point x on the spacetime M (see Fig. 1), and $*\alpha$ is the Hodge-dual of α on M. We have normalized γ by m, $\mathbf{A}$ by mc/e, and ϕ by m/e. These normalizations will be used thereafter, unless it is explicitly stated otherwise. To be more general, the 1-form γ is allowed to be a non-local function on M, $\gamma = \gamma\left(g(x)\right)$, where $g(x)$ is a function of x. The field equation for A is obtained through the variational procedure,

$$\frac{\delta S}{\delta A} = E\left(\mathcal{L}\right) = 0\,, \tag{13}$$

where $\delta S/\delta A$ is the variational derivative and $E\left(\mathcal{L}\right)$ is the Euler derivative. Carrying out the variational derivative, we have

$$d * dA = 4\pi * j\,, \tag{14}$$

$$j^\alpha(x) = \int_{x'} \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\delta\gamma\left(g(x')\right)}{\delta A_\alpha(x)}\ ,\ \left(\alpha = 0, 1, 2, 3\right), \tag{15}$$

$$\frac{\delta\gamma(g)}{\delta A_{\alpha}(x)}=\delta\left[x-g(x')\right]\left\{\frac{\partial\gamma(g)}{\partial A_{\alpha}(g)}-\frac{\partial}{\partial x^{\beta}}\left[\frac{\partial\gamma(g)}{\partial A_{\alpha,\beta}(g)}\right]\right\}\ ,\ (\alpha,\beta=0,1,2,3)\,. \tag{16}$$

Here $A_{\alpha,\beta}(x)$ represents $\partial A_{\alpha}(x)/\partial x^{\beta}$, and it is assumed that γ does not depend on derivatives of A higher than the first order. When evaluating the 4-current in Eq. (15) by variation with respect to A_{α}, the term $f\Omega$ is kept fixed. This is because the action in Eq. (11) has a mixed representation. The field is Eulerian and the particles are Lagrangian, which is the same as Low's first variational principle for the Vlasov-Maxwell system [52]. The action principle given by Eqs. (11)-(13) can be transformed into pure Eulerian through the reduction associated with the particle re-labeling symmetry. For the current purpose of deriving governing equations which adopt various physical assumptions and mathematical simplifications, but still possess good conservation properties, the action principle given by Eqs. (11)-(13) with mixed representation is easier to work with. The 4-current $j^{\alpha}(x)$ is therefore

$$\begin{aligned}j^{\alpha}(x)=&\int_{x'}\int_{\pi^{-1}(x')}\delta\left[x-g(x')\right]f\Omega\wedge\frac{\partial\gamma(g)}{\partial A_{\alpha}(g)}\\&-\frac{\partial}{\partial x^{\beta}}\left[\int_{x'}\int_{\pi^{-1}(x')}\delta\left[x-g(x')\right]f\Omega\wedge\frac{\partial\gamma(g)}{\partial A_{\alpha,\beta}(g)}\right]\ ,\ (\alpha=0,1,2,3)\,.\end{aligned} \tag{17}$$

The second term in Eq. (17) is the 4-magnetization-current, whose 0-th component is the polarization density. If γ is a local function of x, *i.e.*, $g(x)=x$, then Eq. (17) reduces to

$$j^{\alpha}(x)=\int_{\pi^{-1}(x)}f\Omega\wedge\frac{\partial\gamma(x)}{\partial A_{\alpha}(x)}-\frac{\partial}{\partial x^{\beta}}\left[\int_{\pi^{-1}(x)}f\Omega\wedge\frac{\partial\gamma(x)}{\partial A_{\alpha,\beta}(x)}\right]\ ,\ (\alpha=0,1,2,3)\,.$$

For example, for a classical particle interacting with the field through the Lorentz force, we can construct γ as follows. First, take the only two geometric objects related to the dynamics of charged particles, the momentum 1-form p and the potential 1-form A on M, then perform the only nontrivial operation, i.e., addition with the right units, to let particles interact with fields,

$$\hat{\gamma}=A+p\ . \tag{18}$$

$\hat{\gamma}$ is what we call Poincaré-Cartan-Einstein 1-form on the spacetime M. The Poincaré-

Cartan-Einstein 1-form on the phase space P is obtained by pulling back $\hat{\gamma}$,

$$\gamma = \pi^* \hat{\gamma} \quad . \tag{19}$$

Eqs. (15) and (17) reduce to

$$*j(x) = \int_{\pi^{-1}(x)} f\Omega \ , \tag{20}$$

which implies that the "velocity integrals" in kinetic theory are geometrically fiber integrals. The fact that $*j(x)$ is the conventional 3-form flux can be verified by expressing Ω in a coordinate system composed of inertial coordinates x^μ ($\mu = 0, 1, 2, 3$) for M and three corresponding coordinate p_i with $i = 1, 2$, and 3 for $T_x M$. In this coordinate system we have the following expressions in the phase space P,

$$p_0 = -\sqrt{m^2c^2 + p^2} \ , \tag{21}$$

$$d\gamma = \frac{e}{c} A_{i,j} dx^j \wedge dx^i + dp_i \wedge dx^i - e\phi_{,j} dx^j \wedge dt - c\frac{\partial}{\partial p_i}\sqrt{m^2c^2 + p^2} dp_i \wedge dt \ , \tag{22}$$

$$\begin{aligned} \Omega = &\left(dx^1 \wedge dx^2 \wedge dx^3 - \frac{p_1}{m\gamma_r} dt \wedge dx^2 \wedge dx^3 \right. \\ &\left. - \frac{p_2}{m\gamma_r} dx^1 \wedge dt \wedge dx^3 - \frac{p_3}{m\gamma_r} dx^1 \wedge dx^2 \wedge dt \right) \wedge dp_1 \wedge dp_2 \wedge dp_3 \,, \end{aligned} \tag{23}$$

where terms in Eq. (23) that vanish upon fiber integration have been dropped, and

$$\gamma_r = \sqrt{1 + \frac{p^2}{m^2c^2}} \ . \tag{24}$$

Overall, the Vlasov-Maxwell equations for classical particles interacting with filed through Lorentz force on the 7D phase space P can be geometrically written as

$$df(\tau) = 0, \ i_\tau d\gamma = 0 \ , \ \text{and } d * dA = 4\pi \int_{\pi^{-1}(x)} f\Omega \ . \tag{25}$$

As discussed before, the 1-form γ in the geometrically generalized Vlasov-Maxwell system is completely general. It can take any form based on physical intuition and mathematical simplification. For particles interact with field through mechanism other than Lorentz force,

γ will assume different form from Eq. (19). For example, a neutron or a charge-neutral virus interacts with magnetic field through their magnetic moments. In the context of present study, the particles of interest are the gyrocenters, whose 1-form is different from Eq. (19), but was derived from Eq. (19) under certain approximations and coordinate transformations. One prominent feature in the 1-form for the gyrocenter that does not exist in Eq. (19) is the dependence of γ on the derivatives of A, which according to Eq. (17) will induces 4-magnetization-current, whose 0-th component is a polarization density term in the Poisson equation. It is indeed a new revelation that the well-known polarization density in the gyrokinetic theory is fundamentally the consequence of the dependence of the gyrocenter 1-form on the field strength, i.e., the first derivatives of A. In the next section, we will systematically derive the gyrocenter 1-form.

III. GYRO-SYMMETRY AND LIE COORDINATE PERTURBATION METHOD

We start from the Poincaré-Cartan-Einstein 1-form for a classical, non-relativistic charged particle interacting with electromagnetic field through Lorentz force

$$\gamma = A + p = (\mathbf{A} + \mathbf{v}) \cdot d\mathbf{x} - \left[\frac{v^2}{2} + \phi \right] dt \ . \tag{26}$$

Here, the bold mathematical symbols $\mathbf{A}$ and $\mathbf{p}$ represent the $i = 1, 2, 3$ components of the 1-forms A and p, $d\mathbf{x}$ represents dx^i $(i = 1, 2, 3)$, and $(\mathbf{A} + \mathbf{v}) \cdot d\mathbf{x}$ is just a shorthand notation for $\sum_{i=1,2,3}(A_i + v_i)dx^i$. Particles' dynamics is determined by Hamilton's equation (4).

Gyrokinetic theory is about a symmetry called gyro-symmetry. A symmetry vector field η (infinitesimal generator) of γ is defined to be a vector field that satisfies

$$L_\eta \gamma = ds \tag{27}$$

for some function s on the phase space, where L_η is the Lie derivative along η. Vector field η generates a 1-parameter symmetry group for γ. The symmetry for γ that we are interested is an approximate one. It is an exact symmetry when the electromagnetic fields are constant in spacetime, and in this case it is given by

$$\eta = v_x \left(\frac{1}{B} \frac{\partial}{\partial x} + \frac{\partial}{\partial v_y} \right) + v_y \left(\frac{1}{B} \frac{\partial}{\partial y} - \frac{\partial}{\partial v_x} \right) \ . \tag{28}$$

For any symmetry vector field η, we can apply Cartan's formula $L_\eta \gamma = d(i_\eta \gamma) + i_\eta d\gamma$ to obtain

$$d(i_\eta \gamma) + i_\eta d\gamma = ds \ . \tag{29}$$

Taking the inner product between the vector field τ of a worldline and Eq. (29) gives

$$d(\gamma \cdot \eta) \cdot \tau = ds \cdot \tau \ , \tag{30}$$

which implies that $\gamma \cdot \eta - s$ is an invariant. This is the well-known Noether's theorem which links symmetries and invariants. Applying Noether's theorem, we can verify that the corresponding invariant is the expected magnetic moment

$$\mu = \frac{v_x^2 + v_y^2}{2B} \ , \tag{31}$$

as expected. Eq. (28) indicates that the gyro-symmetry η is neither a pure rotation in the momentum space, nor a pure rotation in the configuration space. It is desirable to construct a new coordinate such that η is a coordinate base

$$\eta = \frac{\partial}{\partial \theta} \ , \tag{32}$$

where θ is the gyrophase coordinate. When the fields are not constant in spacetime, the gyro-symmetry η in Eq. (28) is broken. We then assume the spacetime inhomogeneity is weak and seek an asymptotic symmetry. To achieve this goal, we first construct a non-canonical phase space coordinate system $\bar{Z} = (t, \bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta})$ where γ can be expanded into an asymptotic series

$$\gamma = \bar{\gamma}_0 + \bar{\gamma}_1 + \bar{\gamma}_2 + \ \dots \ , \tag{33}$$

where $\bar{\gamma}_1 \sim \varepsilon \bar{\gamma}_0$, $\bar{\gamma}_2 \sim \varepsilon \bar{\gamma}_1$, and $\varepsilon \ll 1$. By the construction of $\bar{Z}$, $\bar{\gamma}_0$ admits the gyro-symmetry $\eta = \partial / \partial \bar{\theta}$, but $\bar{\gamma}_1$ and $\bar{\gamma}_2$ do not necessarily. $\bar{Z}$ is therefore called the zeroth order gyrocenter coordinate. Then, a coordinate perturbation transformation $g : \bar{Z} \to Z = g(\bar{Z})$

is introduced such that in the new coordinates $Z = (t, \mathbf{X}, u, w, \theta)$, γ_1 and/or γ_2 admit the gyro-symmetry $\eta = \partial/\partial\theta$. In the present study, we seek a stronger and sufficient symmetry condition

$$\partial\gamma/\partial\theta = 0 \ .$$

Naturally, Z is the called the first and/or second-order gyrocenter coordinate. The small parameter ε is a measure of the weakness of spacetime inhomogeneity of the fields. The coordinate perturbation procedure itself shows that the most relaxed conditions for the existence of an asymptotic gyro-symmetry is

$$\mathbf{B} \equiv \mathbf{B}_0 + \mathbf{B}_1, \ \mathbf{E} \equiv \mathbf{E}_0 + \mathbf{E}_1, \tag{34}$$

$$\mathbf{B}_1 \sim \varepsilon_1 \ \mathbf{B}_0, \ \mathbf{E}_0 \sim \frac{\mathbf{v} \times \mathbf{B}_0}{c} \ , \ \ \mathbf{E}_1 \sim \varepsilon_1 \ \frac{\mathbf{v} \times \mathbf{B}_0}{c} \ , \tag{35}$$

$$\left(|\rho| \ \frac{\nabla B_0}{B_0}, \frac{1}{\Omega B_0} \frac{\partial B_0}{\partial t} \right) \sim \left(|\rho| \ \frac{\nabla E_0}{E_0}, \frac{1}{\Omega E_0} \frac{\partial E_0}{\partial t} \right) \sim \varepsilon_0 \ , \tag{36}$$

$$\left(|\rho| \ \frac{\nabla B_1}{B_1}, \frac{1}{\Omega B_1} \frac{\partial B_1}{\partial t} \right) \sim \left(|\rho| \ \frac{\nabla E_1}{E_1}, \frac{1}{\Omega E_1} \frac{\partial E_1}{\partial t} \right) \sim 1 \ . \tag{37}$$

Here the fields were split into two parts. The leading order fields $(\mathbf{E}_0, \mathbf{B}_0)$ are the time-dependent background fields with long spacetime scale length compared with the spacetime gyroradius $\rho = (\rho, 1/\Omega)$. The small parameter ε_0 measures the weak spacetime inhomogeneities of the background fields. For edge plasmas, the background electric field can be large. The order of $\mathbf{E}_0$ implies that the potential drop of background field can be comparable to the thermal energy of the particles, *i.e.*, $e\mathbf{E}_0 \cdot \rho \sim 1$. The next order fields $(\mathbf{E}_1, \mathbf{B}_1)$ are the perturbed parts with spacetime scale length comparable to the spacetime gyroradius. The perturbation amplitude is measured by the small parameter ε_1. Both ε_0 and ε_1 measure the weak spacetime inhomogeneities of the overall fields. In general, we assume $\varepsilon \sim \varepsilon_0 \sim \varepsilon_1$.

The coordinate perturbation method adopted here belongs to the class of perturbation techniques generally referred as the Lie perturbation method [11, 32, 33], where the coordinate transformation g is a continuous group generated by a vector field G with $g : z \longmapsto Z = g(z, \varepsilon)$ and $G = dg/d\varepsilon|_{\varepsilon=0}$. Under the coordinate transformation g, γ is pulled-back.

$$\Gamma(Z) = g^{-1*}\gamma(z) = \gamma\left[g^{-1}(Z)\right] = \gamma(Z) - L_{G(Z)}\gamma(Z) + O(\varepsilon^2)$$
$$= \gamma(Z) - i_{G(Z)}d\gamma(Z) - d\left[\gamma \cdot G(Z)\right] + O(\varepsilon^2)\ , \tag{38}$$

where use has been made of $-G = dg^{-1}/d\varepsilon|_{\varepsilon=0}$. In our case, γ is an asymptotic series as in Eq. (33). Let $Z = g_1(z, \varepsilon)$ and we have

$$\Gamma(Z) = \Gamma_0(Z) + \Gamma_1(Z) + O(\varepsilon^2)\ , \tag{39}$$

$$\Gamma_0(Z) = \gamma_0(Z)\ , \tag{40}$$

$$\Gamma_1(Z) = \gamma_1(Z) - i_{G_1(Z)}d\gamma_0(Z) - d\left[\gamma_0 \cdot G_1(Z)\right]\ . \tag{41}$$

By using another coordinate transformation, the perturbation procedure can be straightforwardly carried out to the second order. Let $Z{=}g_2(g_1(z,\varepsilon),\delta)$ and $\delta \sim \varepsilon^2$. The second order transformed 1-form is

$$\Gamma_2(Z) = \gamma_2(Z) - L_{G_1(Z)}\gamma_1(Z) + \left(\frac{1}{2}L^2_{G_1(Z)} - L_{G_2(Z)}\right)\gamma_0(Z)\ , \tag{42}$$

where $G_2 = dg_2/d\delta|_{\delta=0}$.

IV. GYROCENTER COORDINATES

In order to construct the zeroth order gyrocenter coordinate $\bar{Z} = (t, \bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta})$, we first define two vector fields on spacetime M,

$$\mathbf{D}(y) \equiv \frac{\mathbf{E}_0(y) \times \mathbf{B}_0(y)}{\left[B_0(y)\right]^2}\ ,\quad \mathbf{b}(y) \equiv \frac{\mathbf{B}_0(y)}{B_0(y)}\ , \tag{43}$$

where $y \in M$. To decompose particle's velocity at the gyrocenter, it is necessary to define, at every y, the following vector fields which also depend on $\mathbf{v}_x$, particle's velocity at another spacetime position $x \in M$,

$$w(y, \mathbf{v}_x)\mathbf{c}(y, \mathbf{v}_x) \equiv [\mathbf{v}_x - \mathbf{D}(y)] \times \mathbf{b}(y) \times \mathbf{b}(y) \ , \tag{44}$$

$$\mathbf{c}(y, \mathbf{v}_x) \cdot \mathbf{c}(y, \mathbf{v}_x) = 1 \, , \tag{45}$$

$$\rho(y, \mathbf{v}_x) \equiv \frac{\mathbf{b}(y) \times [\mathbf{v}_x(y) - \mathbf{D}(y)]}{B_0(y)} \, , \tag{46}$$

$$u(y, \mathbf{v}_x)\mathbf{b}(y) \equiv [\mathbf{v}_x - \mathbf{D}(y)] \cdot \ \mathbf{b}(y) \ \mathbf{b}(y) \ , \tag{47}$$

$$\mathbf{a}(y, \mathbf{v}_x) \equiv \mathbf{b}(y) \times \mathbf{c}(y, \mathbf{v}_x) \ . \tag{48}$$

With these definitions, velocity $\mathbf{v}_x$ at x has the following partition at y

$$\mathbf{v}_x(y) \equiv \mathbf{D}(y) + u(y, \mathbf{v}_x)\mathbf{b}(y) + w(y, \mathbf{v}_x)\mathbf{c}(y, \mathbf{v}_x) \ . \tag{49}$$

The zeroth order gyrocenter coordinate transformation

$$g_0 : z = (t, \mathbf{x}, \mathbf{v}) \mapsto \bar{Z} = (\bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta}) \tag{50}$$

is given by

$$\mathbf{x} \equiv \bar{\mathbf{X}} + \rho(\bar{\mathbf{X}}, \mathbf{v}) \ , \ \bar{u} \equiv u(\bar{\mathbf{X}}, \mathbf{v}) \ , \ \bar{w} \equiv w(\bar{\mathbf{X}}, \mathbf{v}) \ , \ \sin \bar{\theta} \equiv -\mathbf{c}(\bar{\mathbf{X}}) \cdot \mathbf{e}_1(\bar{\mathbf{X}}) \ , \ t \equiv t \ , \tag{51}$$

where $\mathbf{e}_1(\bar{\mathbf{X}})$ is an arbitrary unit vector field in the perpendicular direction, and it can depend on t as well. Using the $\bar{Z}$ coordinate, $\mathbf{v}_x$ can be expressed as

$$\mathbf{v}_x = \mathbf{D}(\bar{\mathbf{X}}) + \bar{u}\mathbf{b}(\bar{\mathbf{X}}) + \bar{w}\mathbf{c}(\bar{\mathbf{X}}) \ . \tag{52}$$

Substituting Eqs. (51) and (52) into Eq. (26), and expanding terms using the ordering in Eqs. (34)-(37), we have

$$\gamma = \bar{\gamma}_0 + \bar{\gamma}_1 + \ O(\varepsilon^2) \ , \tag{53}$$

$$\bar{\gamma}_0 = (\mathbf{A}_0 + \bar{u}\mathbf{b} + \mathbf{D}) \cdot d\bar{\mathbf{X}} + \bar{\mu} d\bar{\theta} - \left(\frac{\bar{u}^2 + \bar{w}^2 + D^2}{2} + \phi_0 \right) dt \ , \tag{54}$$

$$\begin{aligned} \bar{\gamma}_1 = & \left[\frac{\bar{w}}{B_0} \nabla \mathbf{a} \cdot \left(\bar{u}\mathbf{b} + \frac{\bar{w}\mathbf{c}}{2} \right) + \frac{1}{2} \rho \cdot \nabla \mathbf{B}_0 \times \rho - \frac{\bar{w}}{B_0} \nabla \mathbf{D} \cdot \mathbf{a} + \mathbf{A}_1(\bar{\mathbf{X}} + \rho) \right] \cdot d\bar{\mathbf{X}} \\ & + \left[-\frac{\bar{w}^3}{2B_0^3} \mathbf{a} \cdot \nabla \mathbf{B}_0 \cdot \mathbf{b} + \frac{\bar{w}}{B_0} \mathbf{A}_1(\bar{\mathbf{X}} + \rho) \cdot \mathbf{c} \right] d\bar{\theta} + \left[\frac{1}{\bar{w}} \mathbf{A}_1(\bar{\mathbf{X}} + \rho) \cdot \mathbf{a} \right] d\bar{\mu} \\ & - \left[\phi_1(\bar{\mathbf{X}} + \rho) + \rho \cdot \frac{\partial \mathbf{D}}{\partial t} - \frac{1}{2} \rho \cdot \nabla \mathbf{E}_0 \cdot \rho - \left(\bar{u}\mathbf{b} + \frac{\bar{w}\mathbf{c}}{2} \right) \cdot \frac{\bar{w}}{B_0} \frac{\partial \mathbf{a}}{\partial t} \right] dt \ . \end{aligned} \tag{55}$$

Here, $\mathbf{A}_0$ and ϕ_0 are the leading order vector and scalar potentials corresponding to the leading order $\mathbf{E}_0$ and $\mathbf{B}_0$, $\bar{\mu} = \bar{w}^2/2B_0$, and every field is evaluated at $\bar{Z}$ and can depend on t. Exact terms of the form $d\alpha$ have been discarded because of their insignificance in Hamilton's equation (4). It is obvious that $\partial\bar{\gamma}_0/\partial\bar{\theta} = 0$, but $\partial\bar{\gamma}_1/\partial\bar{\theta} \neq 0$. We now introduce a coordinate perturbation to the zeroth order gyrocenter coordinates $\bar{Z}$,

$$Z = g_1(\bar{Z}, \varepsilon), \quad \frac{dg_1}{d\varepsilon}|_{\varepsilon=0} = G_1(\bar{Z}) \ , \tag{56}$$

such that $\partial\gamma_1/\partial\theta = 0$ in the first order gyrocenter coordinates $Z = (\mathbf{X}, u, \mu, \theta)$. Considering the fact that an arbitrary exact term can be added to γ_1, we write

$$\gamma_1(Z) = \bar{\gamma}_1(Z) - i_{G_1(Z)} d\gamma_0(Z) + dS_1(Z) \ , \tag{57}$$

which expands into

$$\gamma_1(Z) = \left[\mathbf{G}_{1\mathbf{X}} \times \mathbf{B}^\dagger - G_{1u}\mathbf{b} + \nabla S_1 + \frac{w}{B_0}\nabla\mathbf{a}\cdot\left(u\mathbf{b} + \frac{w\mathbf{c}}{2}\right) + \frac{1}{2}\rho\cdot\nabla\mathbf{B}_0\times\rho\right.$$
$$\left. -\frac{w}{B_0}\nabla\mathbf{D}\cdot\mathbf{a} + \mathbf{A}_1(\mathbf{X}+\rho)\right]\cdot d\mathbf{X} + \left[\mathbf{G}_{1\mathbf{X}}\cdot\mathbf{b} + \frac{\partial S_1}{\partial u}\right] du + \left[G_{1\theta} + \frac{\partial S_1}{\partial \mu} + \right.$$
$$\left. +\frac{1}{w}\mathbf{A}_1(\mathbf{X}+\rho)\cdot\mathbf{a}\right] dw + \left[-G_{1\mu} + \frac{\partial S_1}{\partial\theta} - \frac{w^3}{2B_0^3}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b}\right.$$
$$\left. +\frac{w}{B_0}\mathbf{A}_1(\mathbf{X}+\rho)\cdot\mathbf{c}\right] d\theta + \left[-\mathbf{E}^\dagger\cdot\mathbf{G}_{1\mathbf{X}} + uG_{1u} + B_0G_{1\mu} + \frac{\partial S_1}{\partial t} - \phi_1(\mathbf{X}+\rho)\right.$$
$$\left. -\rho\cdot\frac{\partial\mathbf{D}}{\partial t} + \frac{1}{2}\rho\cdot\nabla\mathbf{E}_0\cdot\rho + \left(u\mathbf{b} + \frac{w\mathbf{c}}{2}\right)\cdot\frac{w}{B_0}\frac{\partial\mathbf{a}}{\partial t}\right] dt \ . \tag{58}$$

In Eq. (58), we have chosen not to transform the time, *i.e.*, $G_t = 0$. All the other components of G_1 and S_1 are determined from the requirement that $\partial\gamma_1/\partial\theta = 0$. The results are listed as follows without giving the details of the derivation,

$$\mathbf{G}_{1\mathbf{X}} = -\frac{\partial S_1}{\partial u}\left(\mathbf{b} + \frac{\mathbf{B}_\perp^\dagger}{B_\parallel^\dagger}\right) + \frac{w^2}{2B_0^2B_\parallel^\dagger}\mathbf{aa}\cdot\nabla\mathbf{B}_0 + \frac{wu}{B_0B_\parallel^\dagger}\left(\nabla\mathbf{a}\cdot\mathbf{b}\right)\times\mathbf{b}$$
$$-\frac{w}{B_0B_\parallel^\dagger}(\nabla\mathbf{D}\cdot\mathbf{a})\times\mathbf{b} \ + \frac{\nabla S_1 + \mathbf{A}_1(\mathbf{X}+\rho)}{B_\parallel^\dagger}\times\mathbf{b} \tag{59}$$

$$G_{1u} = \left(\mathbf{B}_\perp^\dagger\times\mathbf{b}\right)\cdot\mathbf{G}_{1\mathbf{X}}\frac{w^2}{2B_0^2}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{c} \ + \frac{wu}{B_0}\mathbf{b}\cdot\nabla\mathbf{a}\cdot\mathbf{b}$$
$$-\frac{w}{B_0}\mathbf{b}\cdot\nabla\mathbf{D}\cdot\mathbf{a} - \mathbf{b}\cdot\left[\nabla S_1 + \mathbf{A}_1(\mathbf{X}+\rho)\right] \ , \tag{60}$$

$$G_{1\mu} = \frac{\partial S_1}{\partial\theta} - \frac{w^2}{2B_0^3}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b} \ + \frac{w}{B_0}\mathbf{c}\cdot\mathbf{A}_1(\mathbf{X}+\rho) \ , \tag{61}$$

$$G_{1\theta} = -\frac{\partial S_1}{\partial\mu} \ - \frac{1}{w}\mathbf{a}\cdot\mathbf{A}_1(\mathbf{X}+\rho) \ . \tag{62}$$

The determining equation for S_1 is

$$
\begin{aligned}
&\frac{\partial S_1}{\partial t}+\left(\frac{\mathbf{E}_0^\dagger\times\mathbf{b}+\mathbf{B}_\perp^\dagger u}{B_\parallel^\dagger}+u\mathbf{b}\right)\cdot\nabla S_1+\left(E_{0\parallel}^\dagger+\frac{E_{0\parallel}^\dagger\cdot\mathbf{B}_\perp^\dagger}{B_\parallel^\dagger}\right)\frac{\partial S_1}{\partial u}+B_0\frac{\partial S_1}{\partial\theta}=\\
&\left(\mathbf{E}_{0\perp}^\dagger-\mathbf{B}_\perp^\dagger\times u\mathbf{b}\right)\cdot\left[\frac{w^2}{2B_0^3}\widetilde{\mathbf{aa}}\cdot\nabla B_0+\frac{wu}{B_0^2}\left(\nabla\mathbf{a}\cdot\mathbf{b}\right)\times\mathbf{b}-\frac{w}{B_0^2}(\nabla\mathbf{D}\cdot\mathbf{a})\times\mathbf{b}\right]\\
&\quad-\frac{w^2u}{2B_0^2}\nabla\mathbf{B}_0\!:\!\widetilde{\mathbf{ca}}-\frac{wu^2}{B_0}\mathbf{b}\cdot\nabla\mathbf{a}\cdot\mathbf{b}\;+\frac{wu}{B_0}\mathbf{b}\cdot\nabla\mathbf{D}\cdot\mathbf{a}+\frac{w^3}{2B_0^2}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b}\\
&\qquad+\frac{w}{B_0}\mathbf{a}\cdot\frac{\partial\mathbf{D}}{\partial t}-\frac{w^2}{2B_0^2}\nabla\mathbf{E}_0\!:\!\widetilde{\mathbf{aa}}+\frac{uw}{B_0}\mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial t}+\widetilde{\psi_1}\;. \qquad (63)
\end{aligned}
$$

With G_1 and S_1 taking the forms in Eqs. (59)-(63), the θ-dependence in γ_1 is removed,

$$\gamma\left(Z\right)=\gamma_0(Z)+\gamma_1(Z)\ , \qquad (64)$$

$$\gamma_0=\left(\mathbf{A}_0+u\mathbf{b}+\mathbf{D}\right)\cdot d\mathbf{X}+\mu d\theta-\left(\frac{u^2+w^2+D^2}{2}+\phi_0\right)dt\ , \qquad (65)$$

$$\gamma_1(Z)=-\mu\mathbf{R}\cdot d\mathbf{X}-H_1dt\ , \qquad (66)$$

$$
\begin{aligned}
H_1=&\left(\mathbf{E}_{0\perp}^\dagger-\mathbf{B}_\perp^\dagger\times u\mathbf{b}\right)\cdot\frac{w^2}{4B_0^2B_\parallel^\dagger}\nabla\mathbf{B}_0+\frac{w^2u}{4B_0}\mathbf{b}\cdot\nabla\times\mathbf{b}\\
&-\frac{w^2}{4B_0^2}\left(\nabla\cdot\mathbf{E}_0-\mathbf{bb}:\nabla\mathbf{E}_0\right)-\frac{w^2}{2B_0}R_0+\left\langle\psi_1\right\rangle\ , \qquad (67)
\end{aligned}
$$

$$\mathbf{R}\equiv\nabla\mathbf{c}\cdot\mathbf{a}\ ,\quad R_0\equiv-\frac{\partial\mathbf{c}}{\partial t}\cdot\mathbf{a}\ , \qquad (68)$$

$$\psi_1\equiv\phi_1(\mathbf{X}+\rho)-\left(\frac{\mathbf{E}_0^\dagger\times\mathbf{b}+\mathbf{B}_\perp^\dagger u}{B_\parallel^\dagger}+u\mathbf{b}+w\mathbf{c}\right)\cdot\mathbf{A}_1(\mathbf{X}+\rho), \qquad (69)$$

$$\left\langle\alpha\right\rangle\equiv\frac{1}{2\pi}\int_0^{2\pi}\alpha d\theta\ ,\quad\widetilde{\alpha}\equiv\alpha-\left\langle\alpha\right\rangle\ , \qquad (70)$$

$$\phi_0^\dagger\equiv\phi_0+\mu B_0+\frac{D^2}{2}\ ,\ \mathbf{A}^\dagger\equiv\mathbf{A}_0+u\mathbf{b}+\mathbf{D}\ , \qquad (71)$$

$$\mathbf{B}^\dagger\equiv\nabla\times\mathbf{A}^\dagger,\ B_\parallel^\dagger\equiv\mathbf{B}^\dagger\cdot\mathbf{b}\,, \qquad (72)$$

$$\mathbf{E}_0^\dagger\equiv-\nabla\phi_0^\dagger-\frac{\partial\mathbf{A}^\dagger}{\partial t}=\mathbf{E}_0-\nabla\left[\mu B_0+\frac{D^2}{2}\right]-u\frac{\partial\mathbf{b}}{\partial t}-\frac{\partial\mathbf{D}}{\partial t}\ . \qquad (73)$$

The perturbation procedure has been carried out to the second order by introducing another coordinate transformation $g_2 : g_1(\bar{Z}) \to Z = g_2 \circ g_1(\bar{Z})$. The results up to $O(\varepsilon_1^2)$ are

$$\gamma(Z) = \gamma_0(Z) + \gamma_1(Z) + \gamma_1(Z) \ , \tag{74}$$

$$\gamma_2 = -\langle \psi_2 \rangle \, dt \, , \tag{75}$$

$$\psi_2 \equiv \frac{1}{2}\mathbf{E}_{0\perp} \cdot \left[\left(\mathbf{G}_1^\dagger \times \mathbf{B}_1\right) \times \mathbf{b}\right] - \frac{1}{2}\left(u\mathbf{b} + w\mathbf{c}\right) \cdot \left(\mathbf{G}_1^\dagger \times \mathbf{B}_1\right) \tag{76}$$

$$+ \frac{1}{2}\left[\mathbf{G}_{1\mathbf{x}} \cdot \mathbf{E}_1 + \left(\mathbf{E}_1 \cdot \frac{\mathbf{a}}{\sqrt{2\mu B_0}} - \frac{\partial \langle \psi_1 \rangle}{\partial \mu}\right) G_{1\mu} + \mathbf{E}_1 \cdot \mathbf{c}\sqrt{\frac{2\mu}{B_0}} G_{1\theta}\right] ,$$

$$\mathbf{G}_1^\dagger \equiv \mathbf{G}_{1\mathbf{x}} + G_{1\mu}\frac{\mathbf{a}}{\sqrt{2\mu B_0}} + G_{1\theta}\sqrt{\frac{2\mu}{B_0}}\mathbf{c} \, , \tag{77}$$

$$\mathbf{E}_1 \equiv -\nabla\phi_1 - \frac{\partial \mathbf{A}_1}{\partial t} \, . \tag{78}$$

Given γ, a particle's trajectory (worldline) on the phase space is uniquely determined by Eq. (4) through its tangent vector τ. The gyrocenter motion equation in terms of $Z = (\mathbf{X}, u, \mu, \theta)$ can be obtained through

$$\frac{d\mathbf{X}}{dt} = \frac{\tau_{\mathbf{X}}}{\tau_t} \ , \ \frac{du}{dt} = \frac{\tau_u}{\tau_t} \ , \ \frac{dw}{dt} = \frac{\tau_w}{\tau_t} \ , \ \frac{d\theta}{dt} = \frac{\tau_\theta}{\tau_t} \ . \tag{79}$$

The explicit expressions for gyrocenter dynamics are

$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{B}^\dagger}{B_\parallel^\dagger}(u + \frac{\mu}{2}\mathbf{b} \cdot \nabla \times \mathbf{b}) - \frac{\mathbf{b} \times \mathbf{E}^\dagger}{\mathbf{b} \cdot \mathbf{B}^\dagger} \ , \tag{80}$$

$$\frac{du}{dt} = \frac{\mathbf{B}^\dagger \cdot \mathbf{E}^\dagger}{B_\parallel^\dagger} \ , \tag{81}$$

$$\frac{d\theta}{dt} = B_0 + \mathbf{R} \cdot \frac{d\mathbf{X}}{dt} - R_0 + \frac{\mathbf{E}_0 \cdot \nabla B_0}{B_0^2} + \frac{u}{2}\mathbf{b} \cdot \nabla \times \mathbf{b}$$

$$- \frac{1}{2B_0}\left[\nabla \cdot \mathbf{E}_0 - \mathbf{bb} : \nabla \mathbf{E}_0\right] + \frac{\partial}{\partial \mu}\langle \psi_1 + \psi_2 \rangle \ , \tag{82}$$

$$\frac{d\mu}{dt} = 0 \ , \ \mu \equiv \frac{w^2}{2B_0} \ , \tag{83}$$

$$\mathbf{E}^\dagger \equiv \mathbf{E}_0^\dagger - \nabla \langle \psi_1 + \psi_2 \rangle \ . \tag{84}$$

In the right hand sides of Eqs. (80)-(84), all the fields are evaluated at the gyrocenter coordinate Z and can depend on t. All the terms on the right hand sides of Eqs. (80)-(84) are gyrophase independent. The spirit of the general gyrokinetic theory is to decouple the gyrophase dynamics from the rest of particle dynamics by finding the gyro-symmetry, instead of "averaging out" the "fast gyro-motion". This objective was accomplished by asymptotically constructing a good coordinate system using the Lie coordinate perturbation method enabled by the geometric nature of the phase space dynamics. Note that in Eq. (80) the usual curvature drift is hidden in the first term on the right hand side, and the second term is the Banõs drift [53]. The last term is the generalized $\mathbf{E}\times\mathbf{B}$ drift that contains the usual gradient $\mathbf{B}$ drift along with several other "cross-$\mathbf{B}$" drifts, such as that induced by the spacetime inhomogeneities of $\mathbf{E}_0$. Compared with previous gyrokinetic theories, the time-dependence of the background magnetic field B_0 and the spacetime-dependence of the background $\mathbf{E}_0$ field and the associated $\mathbf{E}\times\mathbf{B}$ flow are self-consistently and systematically included in our analysis. These factors are treated on equal footing with the spatial inhomogeneity of the background magnetic field in the perturbative procedure. The spacetime inhomogeneities associated with the background $\mathbf{E}\times\mathbf{B}$ flow have important physical consequences. For example, the large scale length shear flow can effectively suppress the micro-turbulence and result in better transport properties in H-modes. This effect enters into the gyrokinetic equation system through the spatial derivative of $\mathbf{D}\equiv\mathbf{E}_0\times\mathbf{B}_0/B_0^2$ in Eq. (73).

It is necessary to note that there are freedoms in defining the gyrocenter coordinates. For example, in Ref. [12], a different definition of the zeroth order gyrocenter coordinates are used, which results in more terms in the expression for $\bar{\gamma}_1$. In addition, the requirement $\partial\gamma/\partial\theta = 0$ does not uniquely determine the coordinate perturbation G and the gauge function S, and therefore the higher order gyrocenter coordinates. We will call the freedoms in choosing gyrocenter coordinates gyro-center gauges. One special gyro-center gauge is the so-called gyro-gauge, a gauge group associated how the gyrophase θ is measured. This gauge group of transformation is given by

$$R \longrightarrow R' + \nabla\delta(X)\ ,\ \theta \longrightarrow \theta' + \delta(X)\ , \tag{85}$$

where $R = (R_0, \mathbf{R})$, $X = (t, \mathbf{X})$, $\nabla = (-\partial/\partial t, \nabla)$, and $\delta(X)$ is an arbitrary scalar function on M. The γ in Eq. (64) is invariant under this group of transformation

V. GYROKINETIC SYSTEMS

After obtaining the expression of γ in Eq. (74) for gyrocenter, we can apply the geometric field theory developed in Sec. (II) to derive the corresponding geometric gyrokinetic theory. The gyrokinetic equation is given by Eq. (8), which is explicitly

$$\frac{dZ_j}{dt}\frac{\partial F}{\partial Z_j} = 0\ ,\quad (0 \leq j \leq 6)\ . \tag{86}$$

Here, F is the particle distribution function in the gyrocenter coordinates $Z = (t, \mathbf{X}, u, \mu, \theta)$. Let

$$F = \langle F \rangle + \widetilde{F}\ , \tag{87}$$

where $\langle F \rangle$ and $\widetilde{F}$ are the gyrophase independent and dependent parts of F. Because

$$\frac{\partial}{\partial \theta}\left(\frac{dZ}{dt}\right) = 0\ , \tag{88}$$

the gyrokinetic equation can be easily split into gyrophase dependent and independent parts as well

$$\frac{\partial \langle F \rangle}{\partial t} + \frac{d\mathbf{X}}{dt}\cdot\nabla_{\mathbf{X}}\langle F \rangle + \frac{du}{dt}\frac{\partial \langle F \rangle}{\partial u} = 0\ , \tag{89}$$

$$\frac{\partial \widetilde{F}}{\partial t} + \frac{d\mathbf{X}}{dt}\cdot\nabla_{\mathbf{X}}\widetilde{F} + \frac{du}{dt}\frac{\partial \widetilde{F}}{\partial u} + \frac{d\theta}{dt}\frac{\partial \widetilde{F}}{\partial \theta} = 0\ , \tag{90}$$

where $d\mathbf{X}/dt$, du/dt, and $d\theta/dt$ are given by Eqs. (80)-(82). The gyrophase dependent $\widetilde{F}$ can be decoupled from the system by setting $\widetilde{F} = 0$, and Eqs. (89) and (14) form a close system for $\langle F \rangle$ and $A = (-\phi, \mathbf{A})$. However $\widetilde{F} = 0$ does not imply that $\widetilde{f} = 0$. The distribution function f in the laboratory coordinates becomes gyrophase dependent through the pullback transformation [28]. Finally, the gyrokinetic system is completed by the Maxwell's equation and the 4-current given by Eqs. (14) and (17). We emphasize that by using the geometric field theory developed in Sec. II, the Maxwell's equation and the 4-current are uniquely determined by the 1-form γ as well.

The expression of γ in Eq. (74) is rather complicated. It contains all the physical aspects of the gyrocenter dynamics in an inhomogeneous, time-dependent electromagnetic fields with long and short spacetime wavelength. For studies focusing on different physics phenomena, we can selectively adopting different terms in γ to investigate the corresponding physics. The value of the geometric field theory developed here is that it allow arbitrary approximations and simplifications to be made at the level of the 1-form, and the resulting kinetic system still posses the good geometric properties, such as the conservation of phase space volume and energy-momentum if the 1-form does not depend on the spacetime coordinate explicitly. To look at several special gyrokinetic systems, we pick

$$\gamma = \mathbf{A}^{\dagger} \cdot d\mathbf{X} + \mu d\theta - H dt \tag{91}$$

$$H = \frac{u^2 + w^2 + D^2}{2} + \phi_0 + \langle \psi_1 + \psi_2 \rangle \tag{92}$$

as a model for gyrocenter dynamics in the gyrocenter coordinates $Z = (t, \mathbf{X}, u, \mu, \theta)$. From Eq. (6), the Liouville 6-form corresponding to Eq. (91) is

$$\begin{aligned}
\Omega = {} & B_{\parallel}^{\dagger} dx^1 \wedge dx^2 \wedge dx^3 \wedge du \wedge d\mu \wedge d\theta \\
& + \left[A_{j,t}^{\dagger} b_i - A_{i,j}^{\dagger} u - b_j H_{,i} \right] dt \wedge dx^j \wedge dx^i \wedge du \wedge d\mu \wedge d\theta \\
& + \left[A_{i,j}^{\dagger} H_{,l} + A_{i,j}^{\dagger} A_{l,t}^{\dagger} \right] dx^j \wedge dx^i \wedge dx^l \wedge dt \wedge d\mu \wedge d\theta \\
& - A_{i,j}^{\dagger} b_l H_{,\mu} dx^j \wedge dx^i \wedge dx^l \wedge dt \wedge du \wedge d\mu \,.
\end{aligned} \tag{93}$$

The conservative form of the gyrokinetic equation is obtained from the general Vlasov equation in conservative form Eq. (10). It can be explicitly written as

$$\frac{\partial}{\partial t} \left[B_{\parallel}^{\dagger} \langle F \rangle \right] + \nabla_{\mathbf{X}} \cdot \left[B_{\parallel}^{\dagger} \langle F \rangle \dot{\mathbf{X}} \right] + \frac{\partial}{\partial u} \left[B_{\parallel}^{\dagger} \langle F \rangle \dot{u} \right] = 0 \,, \tag{94}$$

where $B_{\parallel}^{\dagger}$ is allowed to depend on t. Because γ in Eq. (91) does not indeed depend on the spacetime explicitly, the energy-momentum of the system is conserved. We study three special cases of the gyrokinetic systems by further simplifying the γ in Eq. (91).

A. Gyrokinetic theory without FLR effect – drift kinetic theory

For physical processes where the finite Lamour radius effect is not important, we can ignore all the first order terms related to $(\phi_1, \mathbf{A}_1)$ in Eq. (91) and identify particle positions with gyrocenters. This is the drift kinetic limit of the gyrokinetic theory. As a model, we will also ignore the background $\mathbf{E}\times\mathbf{B}$ flow in this analysis. Let

$$\gamma = \mathbf{A}^{\dagger}\cdot d\mathbf{X} + \mu d\theta - H dt\,, \tag{95}$$

$$H = \frac{u^2+w^2}{2} + \phi_0\,. \tag{96}$$

$$\mathbf{B}^{\dagger} \equiv \nabla\times(\mathbf{A}_0 + u\mathbf{b})\,,\; B_{\parallel}^{\dagger} \equiv \mathbf{B}^{\dagger}\cdot\mathbf{b}\,, \tag{97}$$

$$\mathbf{E}_0^{\dagger} \equiv \mathbf{E}_0 - \nabla\left[\mu B_0\right] - u\frac{\partial\mathbf{b}}{\partial t}\;. \tag{98}$$

Using the general 4-current expression in Eq. (17) and the Liouville 6-form in Eq. (93) , we have

$$\mathbf{j} = \mathbf{j}_g + \mathbf{j}_M\,, \tag{99}$$

$$\mathbf{j}_g = \int_{\pi^{-1}(x)} qF\Omega\wedge\frac{\partial\gamma(x)}{\partial A_i(x)} = \int qF\left(\frac{\mathbf{E}^{\dagger}\times\mathbf{b}+\mathbf{B}^{\dagger}u}{B_{\parallel}^{\dagger}}\right)B_{\parallel}^{\dagger}du\wedge d\mu\wedge d\theta\,, \tag{100}$$

$$\begin{aligned}\mathbf{j}_M &= -\frac{\partial}{\partial x^j}\left[\int_{\pi^{-1}(x)} qF\Omega\wedge\frac{\partial\gamma(x)}{\partial A_{i,j}(x)}\right]\\ &= -\nabla\times\left[\int qF\left(\mu\mathbf{b} - \frac{u\mathbf{E}^{\dagger}\times\mathbf{b}}{BB_{\parallel}^{\dagger}} - \frac{u^2\mathbf{B}_{\perp}^{\dagger}}{BB_{\parallel}^{\dagger}}\right)B_{\parallel}^{\dagger}du\wedge d\mu\wedge d\theta\right]\,.\end{aligned} \tag{101}$$

Here $\mathbf{j}_g$ is the current associated with the gyrocenter drift and $\mathbf{j}_M$ is the diamagnetic current. It is interesting to note that current $\mathbf{j}_g$ contains all the particle drift motion except for the Banõs drift. The first term in $\mathbf{j}_M$ is the usual diamagnetic current, and the second and third terms are additional diamagnetic current related to inhomogeneities of the electromagnetic field. From

$$\frac{\partial\gamma}{\partial\phi_0} = -dt\,, \tag{102}$$

the density n is simply

$$n(x) = \int FB_{\parallel}^{\dagger}du\wedge d\mu\wedge d\theta\,. \tag{103}$$

We emphasize that the current and density as functions of the gyrocenter distribution function are self-consistently derived from the geometrically generalized Vlasov-Maxwell equations. Previously such expressions are obtained using the pullback transformation [28].

Overall the drift equation and the Maxwell's equation can be explicitly written out as

$$\frac{\partial F}{\partial t} + \frac{d\mathbf{X}}{dt} \cdot \nabla_{\mathbf{X}} F + \frac{du}{dt}\frac{\partial F}{\partial u} = 0 \ , \tag{104}$$

$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{B}^{\dagger}}{B_{\parallel}^{\dagger}}(u + \frac{\mu}{2}\mathbf{b} \cdot \nabla \times \mathbf{b}) - \frac{\mathbf{b} \times \mathbf{E}^{\dagger}}{\mathbf{b} \cdot \mathbf{B}^{\dagger}} \ , \tag{105}$$

$$\frac{du}{dt} = \frac{\mathbf{B}^{\dagger} \cdot \mathbf{E}^{\dagger}}{B_{\parallel}^{\dagger}} \ , \tag{106}$$

$$\mathbf{A}^{\dagger} \equiv \mathbf{A}_0 + u\mathbf{b},\ \mathbf{B}^{\dagger} \equiv \nabla \times \mathbf{A}^{\dagger},\ B_{\parallel}^{\dagger} \equiv \mathbf{B}^{\dagger} \cdot \mathbf{b}\,, \tag{107}$$

$$\mathbf{E}^{\dagger} \equiv -\nabla \phi_0^{\dagger} - \frac{\partial \mathbf{A}^{\dagger}}{\partial t} = \mathbf{E}_0 - \nabla\left[\mu B_0\right] - u\frac{\partial \mathbf{b}}{\partial t}\,, \tag{108}$$

$$\nabla^2 \mathbf{A}_0 = -\sum_{species} \frac{4\pi}{c}\mathbf{j}\,, \tag{109}$$

$$\nabla^2 \phi_0 = -\sum_{species} 4\pi q n\,, \tag{110}$$

where $\mathbf{j}$ and n are given by Eqs. (99) and (103). The $(\mathbf{A}_0, \phi_0)$ in Eqs. (109) and (110) are un-normalized.

B. Gyrokinetic theory in a time-independent background

The next example that we consider is the gyrokinetic system for electrostatic perturbation in a given time-independent, inhomogeneous background ($\mathbf{B}_0$, $\mathbf{E}_0$). We select

$$\gamma = \mathbf{A}^{\dagger} \cdot d\mathbf{X} + \mu d\theta - H dt\,, \tag{111}$$

$$H = \frac{u^2 + w^2 + D^2}{2} + \phi_0 + \langle \psi_1 + \psi_2 \rangle \ , \tag{112}$$

$$\phi_0^{\dagger} \equiv \phi_0 + \mu B_0 + \frac{D^2}{2} \ , \ \mathbf{A}^{\dagger} \equiv \mathbf{A}_0 + u\mathbf{b} + \mathbf{D} \ , \tag{113}$$

$$\langle \psi_1 \rangle = \langle \phi_1(\mathbf{X} + \rho) \rangle \ , \tag{114}$$

$$S_1 = \frac{1}{B_0}\int \tilde{\phi}_1(\mathbf{X} + \rho) d\theta \equiv \frac{1}{B_0}\tilde{\phi}_1^{(1)}\,, \tag{115}$$

$$\langle \psi_2 \rangle = -\frac{1}{2}\left\langle \nabla\tilde{\phi}_1 \cdot \nabla\tilde{\phi}_1^{(1)} \right\rangle \times \frac{\mathbf{b}}{B_{\parallel}^{\dagger}\bar{B}_0} - \frac{1}{2B_0}\left\langle \frac{\partial \tilde{\phi}_1^2}{\partial \mu} \right\rangle \ , \tag{116}$$

where we have kept the second order contribution $\langle \psi_2 \rangle$, and $\bar{B}_0$ is a spatially averaged B_0. The expression for S_1 and $\langle \psi_2 \rangle$ are obtained from Eqs. (63) and (76) under the electrostatic and low-frequency approximations. Since these approximations are introduced at the 1-form level, the resulting kinetic system will still have the desirable exact conservation properties. The density response is calculated from $\delta\gamma/\delta\phi_1$ according to Eqs. (17) and (111),

$$n(x) = \int_X \int_{\pi^{-1}(X)} \delta\left[x - g(X)\right] F\Omega \wedge dt \left[\frac{\delta\left\langle\psi_1\left(g\right)\right\rangle}{\delta\phi_1(g)} + \frac{\delta\left\langle\psi_2\left(g\right)\right\rangle}{\delta\phi_1(g)}\right]$$
$$= \int \left[F + \frac{\partial F}{\partial\mu}\frac{\tilde{\phi}_1}{B_0} + \nabla F \times \frac{\nabla\tilde{\phi}_1^{(1)}}{B_\parallel^\dagger \bar{B}_0}\cdot\mathbf{b} + \frac{\partial F}{\partial u}\frac{\mathbf{B}^\dagger\cdot\nabla\tilde{\phi}_1^{(1)}}{B_\parallel^\dagger\bar{B}_0}\right]\delta\left[\mathbf{x} - \mathbf{X} - \rho(\mathbf{X})\right] B_\parallel^\dagger d^3X du d\mu d\theta \ , \tag{117}$$

where $g(\mathbf{X}) = \mathbf{X} + \rho$. The final gyrokinetic system for this case is

$$\frac{\partial F}{\partial t} + \frac{d\mathbf{X}}{dt}\cdot\nabla_{\mathbf{X}}F + \frac{du}{dt}\frac{\partial F}{\partial u} = 0 \ , \tag{118}$$
$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{B}^\dagger}{B_\parallel^\dagger}(u + \frac{\mu}{2}\mathbf{b}\cdot\nabla\times\mathbf{b}) - \frac{\mathbf{b}\times\mathbf{E}^\dagger}{\mathbf{b}\cdot\mathbf{B}^\dagger} \ , \tag{119}$$
$$\frac{du}{dt} = \frac{\mathbf{B}^\dagger\cdot\mathbf{E}^\dagger}{B_\parallel^\dagger} \ , \tag{120}$$
$$\mathbf{A}^\dagger \equiv \mathbf{A}_0 + u\mathbf{b} + \mathbf{D},\ \mathbf{B}^\dagger \equiv \nabla\times\mathbf{A}^\dagger,\ B_\parallel^\dagger \equiv \mathbf{B}^\dagger\cdot\mathbf{b}\,, \tag{121}$$
$$\mathbf{E}^\dagger \equiv -\nabla\left[\phi_0 + \mu B_0 + \frac{D^2}{2} + \left\langle\psi_1 + \psi_2\right\rangle\right], \tag{122}$$
$$\nabla^2\left(\phi_1 + \phi_0\right) = -\sum_{species} 4\pi q n\,, \tag{123}$$

where n is given by Eq. (117) and ($\mathbf{B}_0$, $\mathbf{E}_0$) are assumed given. The potentials in Eq. (123) are un-normalized. Since γ in Eq. (111) does not depend on t explicitly, the total energy of the kinetic system is conserved, and this is valid for inhomogeneous background. In comparison with previous results in Refs. [13] and [15] we note that in Eq. (116), the denominator of the first term for $\langle\psi_2\rangle$ is different from that in Refs. [13] and [15], and an additional term proportional to $\partial F/\partial u$ is found in Eq. (117) for the density. These differences are of physical importance in that Eqs. (111)–(123) guarantee an exact energy conservation in the general inhomogeneous magnetic field.

C. Gyrokinetic theory in a time-dependent background

The last simplified gyrokinetic system that we investigate is the gyrokinetic system that allows for time-dependent background. To simplify the problem, we only consider the electrostatic case. The 1-form that uniquely determines the equation system is the same as that in Eq. (111), except that ϕ_0 is allowed to depend on time. Since gyrokinetic theory treats the equilibrium field ϕ_0 and the perturbed field ϕ_1 differently, in order to determine both fields self-consistently from the distribution function F, we need two field equations. Let

$$\phi = \phi_0 + \phi_1 \,. \tag{124}$$

It is easy to verify that $\delta S/\delta\phi_0 = 0$ and $\delta S/\delta\phi_1 = 0$ both give the Poisson equation for ϕ, but with different source terms. To calculate the density in terms of the distribution function from $\delta\gamma/\delta\phi_0$, we first calculate

$$-\Omega \wedge \frac{\partial\gamma}{\partial\phi_{0,i}} = -\Omega \wedge \frac{\partial \mathbf{D}}{\partial\phi_{0,i}} \cdot d\mathbf{X} + \Omega \wedge \frac{\partial D^2/2}{\partial\phi_{0,i}} dt \,, \tag{125}$$

which gives

$$\begin{aligned} n^{(0)}(x) = \int F B_\parallel^\dagger du \wedge d\mu \wedge d\theta \, + \nabla \cdot \int \Bigg\{ & \frac{\mathbf{E}_{0\perp}}{B_0}\left(1 - \frac{B_\parallel^\dagger}{B_0}\right) \\ & - \frac{1}{B_0} u\mathbf{b} \times \nabla \times (u\mathbf{b} + \mathbf{D}) - \frac{\mathbf{E}_\perp^\dagger}{B_0} \Bigg\} F du d\mu d\theta \,. \end{aligned} \tag{126}$$

The calculation of $\delta\gamma/\delta\phi_1 = 0$ is the same as that for Eq. (117),

$$n^{(1)}(x) = \int \left[F \, + \frac{\partial F}{\partial \mu}\frac{\tilde{\phi}_1}{B_0} + \nabla F \times \frac{\nabla\tilde{\phi}_1^{(1)}}{B_\parallel^\dagger \bar{B}_0} \cdot \mathbf{b} + \frac{\partial F}{\partial u}\frac{\mathbf{B}^\dagger \cdot \nabla\tilde{\phi}_1^{(1)}}{B_\parallel^\dagger \bar{B}_0} \right] \delta\left[\mathbf{x} - \mathbf{X} - \rho(\mathbf{X})\right] B_\parallel^\dagger d^3X du d\mu d\theta \,. \tag{127}$$

These two Poisson equations and the gyrokinetic equation form a complete equation systems for (F, ϕ_0, ϕ_1),

$$\frac{\partial F}{\partial t} + \frac{d\mathbf{X}}{dt} \cdot \nabla_{\mathbf{X}} F + \frac{du}{dt}\frac{\partial F}{\partial u} = 0 \ , \tag{128}$$

$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{B}^{\dagger}}{B_{\parallel}^{\dagger}}(u + \frac{\mu}{2}\mathbf{b} \cdot \nabla \times \mathbf{b}) - \frac{\mathbf{b} \times \mathbf{E}^{\dagger}}{\mathbf{b} \cdot \mathbf{B}^{\dagger}} \ , \tag{129}$$

$$\frac{du}{dt} = \frac{\mathbf{B}^{\dagger} \cdot \mathbf{E}^{\dagger}}{B_{\parallel}^{\dagger}} \ , \tag{130}$$

$$\mathbf{A}^{\dagger} \equiv \mathbf{A}_0 + u\mathbf{b} + \mathbf{D},\ \mathbf{B}^{\dagger} \equiv \nabla \times \mathbf{A}^{\dagger},\ B_{\parallel}^{\dagger} \equiv \mathbf{B}^{\dagger} \cdot \mathbf{b} \,, \tag{131}$$

$$\mathbf{E}^{\dagger} \equiv -\nabla \left[\phi_0 + \mu B_0 + \frac{D^2}{2} + \langle \psi_1 + \psi_2 \rangle \right] - \frac{\partial \mathbf{D}}{\partial t} \ , \tag{132}$$

$$\nabla^2 \phi = \nabla^2 \left(\phi_0 + \phi_1 \right) = -\sum_{species} 4\pi q n^{(0)} \,, \tag{133}$$

$$\nabla^2 \phi = \nabla^2 \left(\phi_0 + \phi_1 \right) = -\sum_{species} 4\pi q n^{(1)} \,, \tag{134}$$

where $n^{(0)}$ and $n^{(1)}$ are given by Eqs. (126) and (127). Eqs. (133) and (134) are two independent field equations for the two fields of ϕ_0 and ϕ_1, because the source terms dependent on ϕ_0 and ϕ_1 differently. The potentials in Eqs. (133) and (134) are un-normalized.

VI. CONCLUSIONS

During the pedestal cycle of H-mode edge plasmas in tokamak experiments, large-amplitude pedestal build-up and destruction coexist with small-amplitude drift wave turbulence. The pedestal dynamics simultaneously includes fast time-scale electromagnetic instabilities, long time-scale turbulence-induced transport processes, and more interestingly the interaction between them. To numerically simulate the pedestal dynamics from first principles, it is desirable to develop an effective algorithm based on the gyrokinetic theory. However, existing gyrokinetic theories cannot treat fully nonlinear electromagnetic perturbations with multi-scale-length structures in spacetime, and therefore do not apply to edge plasmas. In this paper, we first constructed a geometrically generalized Vlasov-Maxwell system using the Poincaré-Cartan-Einstein 1-form. Geometric gyrokinetic theory is then developed as a special case of the geometrically generalized Vlasov-Maxwell system. Arbitrary approximations based on physical intuition or mathematical simplification can be made at the 1-form level and the resulting gyrokinetic systems still possess exact geometric conservation properties. The construction of the gyrokinetic system is essentially the construction of the gyro-symmetry using the Lie perturbation method. The gyrokinetic system developed allows for time-dependent electromagnetic background coexisting with short wavelength electromagnetic perturbation, and therefore applicable to the edge plasmas in magnetic confinement devices.

Acknowledgment

This research was supported by the U.S. Department of Energy under contract AC02-76CH03073, and by Lawrence Livermore National Laboratory's LDRD Project 04-SI-03, Kinetic Simulation of Boundary Plasma Turbulent Transport. We are grateful to Prof. Ronald C. Davidson and Dr. Janardhan Manickam for their continuous support of this work. We thank Drs. Alian Brizard, Peter J. Catto, Bruce I. Cohen, Andris Dimits, Alex Friedman, Gregory W. Hammet, W. Wei-li Lee, Lynda L. Lodestro, Thomas D. Rognlien, Philip B. Snyder, William M. Tang, Ronald E. Waltz, and Weixing Wang for fruitful discussions.

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Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2750
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov