Princeton Plasma Physics Laboratory

PPPL-4222 PPPL-4222

Bishop-Taylor Equilibrium and Equilibrium Reconstruction Codes

Joshua B. Kallman and Leonid E. Zakharov

Preprint
(February 2007)

Prepared for the U.S. Department of Energy under Contract DE-AC02-76CH03073.

Bishop-Taylor equilibria for calibration equilibrium and equilibrium reconstruction codes

Joshua B. Kallman, Leonid E. Zakharov
Princeton University, Princeton Plasma Physics Laboratory, Princeton NJ

February 08, 2007

For easy navigation the enumeration in the Table of Contents, and the "(to ToC)" right after the section names are the forward and backward hyperlinks between Table of Contents and the beginning of sections.

Contents

Abstract

The properties of Bishop-Taylor equilibria and the algorithm of their calculations, implemented into numerical code `Cbbsh`, are described. These equilibria are unique in having the shape of magnetic surfaces not dependent on the poloidal flux distribution, which, in its turn, determines both the current and pressure profiles in this case. These equilibria can be calculated with any precision, using only 3 ordinary differential equations. Possessing a free profile, they can be used for calibrating equilibrium and stability codes. They are also especially valuable for equilibrium reconstruction as a precise example of a configuration where the external magnetic field does not depend on the current density.

1 Introduction *(to ToC)*

In a 1985 paper, Bishop and Taylor (Ref.[1]) showed that there exists a configuration of nested toroidal magnetic surfaces that can be created from an infinite number of plasma equilibria. Since then, with the exception of the following paper by the same authors (Ref.[2]), these equilibria have been left essentially untouched by the plasma physics community.

In fact, there are several reasons for interest in the Bishop-Taylor equilibria, whose calculation can be done essentially with any precision. First, they can be used for benchmarking two-dimensional equilibrium code. Second, the precise equilibrium data for different profiles of the current density or the safety factor can be used for testing stability codes. Third, these equilibria are unique in having the magnetic field outside the plasma independent on the current density distribution. In this regard, Bishop-Taylor equilibria are indispensable for testing the equilibrium reconstruction codes, which produce the information on the current and pressure distribution inside the plasma based on external magnetic measurements. At present, the equilibrium reconstruction is widely used for fusion devices and its possible improvement can make a significant impact on progress in tokamak diagnostics.

This paper follows the Bishop and Taylor derivation and introduces a code that can be used to recreate this equilibria for different current distributions and aspect ratios of the plasma. The code also generate interface files for other magnetic confinement codes.

2 Derivation of Bishop-Taylor equilibria *(to ToC)*

We use the cylindrical coordinate system r, φ, z and the following rationalized notations for physics profiles

$$\bar{\Psi} \equiv \frac{\Psi}{2\pi}, \quad \bar{\Phi} \equiv \frac{\Phi}{2\pi}, \quad \mathbf{B} = (\nabla\bar{\Psi} \times \nabla\varphi) + \bar{F}\nabla\varphi, \quad \bar{p} \equiv \mu_0 p, \quad \bar{F} \equiv rB_\varphi, \quad \mu_0 \equiv 0.4\pi, \tag{2.1}$$

where Ψ is the poloidal and Φ toroidal fluxes [V·sec], p is the plasma pressure [MPa], and B_φ is the toroidal component of magnetic field $\mathbf{B}$ [T]. "Barred" notations are used for the variables in rationalized units, typical for equilibrium theory. Our $\bar{\Psi}$, originated from the poloidal flux, has an oppsite sign to that used in Ref.[1].

In a magneto-static equilibrium the poloidal flux $\bar{\Psi}$ satisfies the Grad-Shafranov equation

$$\Delta^*\bar{\Psi} = r^2(\frac{\nabla\bar{\Psi}}{r^2}) = -r^2\bar{p}'(\bar{\Psi}) - \bar{F}\bar{F}'(\bar{\Psi}), \tag{2.2}$$

where prime stands for derivative. For another equilibrium with magnetic flux functions $\chi(r,z)$ and $\bar{\Psi}(r,z)$ to have the same shape magnetic surfaces $\chi = \chi(\bar{\Psi})$ the following equation should be fulfilled

$$\Delta^*\chi = \frac{d\chi}{d\bar{\Psi}}\Delta^*\bar{\Psi} + \frac{d^2\chi}{d\bar{\Psi}^2}|\nabla\bar{\Psi}|^2 \tag{2.3}$$

Then the magneto-static equilibrium requires

$$|\nabla\bar{\Psi}|^2 = \alpha(\bar{\Psi}) + r^2\beta(\bar{\Psi}). \tag{2.4}$$

Following Bishop and Taylor we represent the derivatives in the following way

$$|\nabla\bar{\Psi}| \equiv g(\bar{\Psi}, r), \quad \bar{\Psi}'_r \equiv rh(\bar{\Psi}, r), \quad \bar{\Psi}'_z = \sqrt{g^2 - r^2h^2}, \quad g^2 = \alpha + r^2\beta \tag{2.5}$$

Putting this into the Grad-Shafranov equation yields

$$rh'_r - \frac{1}{2}(\alpha'_{\bar{\Psi}} + r^2\beta'_{\bar{\Psi}}) = r^2\bar{p}'_{\bar{\Psi}} + \bar{F}\bar{F}'_{\bar{\Psi}} \tag{2.6}$$

and leads to the following form of h

$$\begin{aligned} &L(\bar{\Psi}) \equiv \bar{p}'_{\bar{\Psi}} + \frac{1}{2}\beta'_{\bar{\Psi}}, \quad M(\bar{\Psi}) \equiv \bar{F}\bar{F}'_{\bar{\Psi}} + \frac{1}{2}\alpha'_{\bar{\Psi}}, \quad rh'_r = r^2L + M \\ &h = L\frac{r^2 - X}{2} + \frac{M}{2}\log\frac{r^2}{X}, \quad X = X(\bar{\Psi}). \end{aligned} \tag{2.7}$$

The radius $r^2 = X$, where $h = 0$ and $\bar{\Psi}'_r = 0$, corresponds to the top and bottom points, $z = z_{max}, z = z_{min}$, of the magnetic surface.

The relationship $\bar{\Psi}''_{rz} = \bar{\Psi}''_{zr}$ gives the following important equation, specifying the Bishop-Taylor equilibria

$$(\alpha + r^2\beta)'_{\bar{\Psi}}h - (\alpha + r^2\beta)2h'_{\bar{\Psi}} + 2\beta - r(h^2)'_r - 2h^2 = 0. \tag{2.8}$$

The h^2 term here contains

$$M^2 \left(\log \frac{r^2}{X}\right)^2 \tag{2.9}$$

and cannot be balanced, unless $M = 0$. This gives the expression for $\bar{F}^2$ and $|\mathbf{B}|^2$

$$\bar{F}^2(\bar{\Psi}) = \bar{F}_0^2 - \alpha(\bar{\Psi}), \quad |\mathbf{B}|^2 = \frac{|\nabla\bar{\Psi}|^2 + \bar{F}^2}{r^2} = \beta(\bar{\Psi}) + \frac{\bar{F}_0^2}{r^2}, \quad \bar{F}_0 = \text{const.} \tag{2.10}$$

The equation for h is now reduced to

$$\left(\frac{\alpha + r^2\beta}{h^2}\right)'_{\bar{\Psi}} + \frac{2\beta}{h^3} - L\frac{3r^2 - X}{h^2} = 0. \tag{2.11}$$

Balancing different powers of r, which is a parameter in this equation, allows to introduce characteristics of Bishop-Taylor equilibria and the ordinary differential equations for them.

3 Equations for characteristics and poloidal angle *(to ToC)*

In addition to X two new functions $S(\bar{\Psi}), Q(\bar{\Psi})$ can be introduced at this stage

$$\alpha \equiv L^2 S, \quad \beta \equiv L^2 Q \tag{3.1}$$

together with a new radial coordinate λ, defined by

$$\frac{d\bar{\Psi}}{d\lambda} = -L. \tag{3.2}$$

It will be proven later on that λ represents simply the volume V of the magnetic surfaces in Bishop-Taylor equilibria

$$V = 4\pi^2\lambda. \tag{3.3}$$

Subsituting $2h = L(r^2 - X)$ into Eq.(2.11) gives

$$(r^2 - X)(S'_\lambda + r^2 Q'_\lambda) + 2(S + r^2 Q)X'_\lambda + 4Q - (3r^2 - X)(r^2 - X) = 0. \tag{3.4}$$

Equating terms of the same order in r gives an expression for Q and two differential equations for characteristics $X(\lambda), S(\lambda)$

$$Q = 3\lambda, \quad S'_\lambda = -6\lambda X'_\lambda - X, \quad X'_\lambda = -\frac{6\lambda}{S + 3\lambda X}. \tag{3.5}$$

The function λ is 0 at magnetic axis. This gives the initial conditions for Eqs.(3.5)

$$S = -\lambda, \quad X = 1 - 3\lambda. \tag{3.6}$$

The shape of the magnetic surfaces is now determined by

$$dr = -\frac{\bar{\Psi}'_z}{|\nabla\bar{\Psi}|}dl, \quad dz = \frac{\bar{\Psi}'_r}{|\nabla\bar{\Psi}|}dl, \quad \frac{dz}{dr^2} = \mp\frac{r^2 - X}{2\sqrt{4S + 12r^2\lambda - r^2(r^2 - X)^2}}. \tag{3.7}$$

Two intersection of them r_1, r_2 (innermost and outermost) with the middle plane $z = 0$ is determined by the condition

$$4S + 12Y_{1,2}\lambda - Y_{1,2}(Y_{1,2} - X)^2 = 0, \quad Y_{1,2} \equiv r_{1,2}^2. \tag{3.8}$$

In terms of Y_1, the radial position of the outermost intersection Y_2 can be calculated as

$$Y_2 = X + \frac{2Y_1(X - Y_1) + 24\lambda}{Y_1 + \sqrt{Y_1(4X - 3Y_1) + 48\lambda}}. \tag{3.9}$$

Now, by introducing two new definitions

$$a \equiv \frac{Y_2 - Y_1}{2}, \quad d \equiv \frac{Y_2 + Y_1}{2} - X, \quad a'_\lambda = \frac{4a}{a^2 - d^2}, \quad d'_\lambda = \frac{2}{X + 2d}, \quad X'_\lambda + d'_\lambda = -\frac{4d}{a^2 - d^2} \tag{3.10}$$

the shape of magnetic surfaces can be described in terms of poloidal angle θ

$$r = \sqrt{X^2 + d + a\cos\theta}, \quad z = -\int_0^\theta \frac{d + a\cos\theta}{2\sqrt{X + 3d + a\cos\theta}} d\theta. \tag{3.11}$$

4 The radial variable *(to ToC)*

In advance, the range of the independent variable λ in Eqs.(3.5) is unknown. For this reason it is better to define a new radial variable $s = 1 - r_1$, representing the distance between the magnetic axis and the innermost point r_1 of the cross-section. The variable s has a predetermined range

$$0 \le s \le 1, \tag{4.1}$$

and its derivative with respect to λ can be found as

$$\frac{dY_1}{d\lambda} = -\frac{4}{X - Y_1}. \tag{4.2}$$

In terms of s as a radial coordinate the equations for the three characteristics obtain the form

$$r_1 = 1 - s, \quad Y_1 = (1 - s)^2, \quad \frac{d\lambda}{ds} = r_1 \frac{X - Y_1}{2}, \quad S'_s = -(6\lambda X'_\lambda + X)\lambda'_s, \quad X'_s = -\frac{6\lambda\lambda'_s}{S + 3X\lambda}. \tag{4.3}$$

Near the magnetic axis $\lambda \simeq s^2/2$, which allows to start the intregration.

Fig.1 shows the solution to Eqs.(4.3) as functions of s in a full range $0 \le s \le 1$. The fact that function $S_{s=1} = 0$ can be seen explicitly from Eq.(3.8), when $Y_1 = 0$ is substituted into it for $s = 1$.

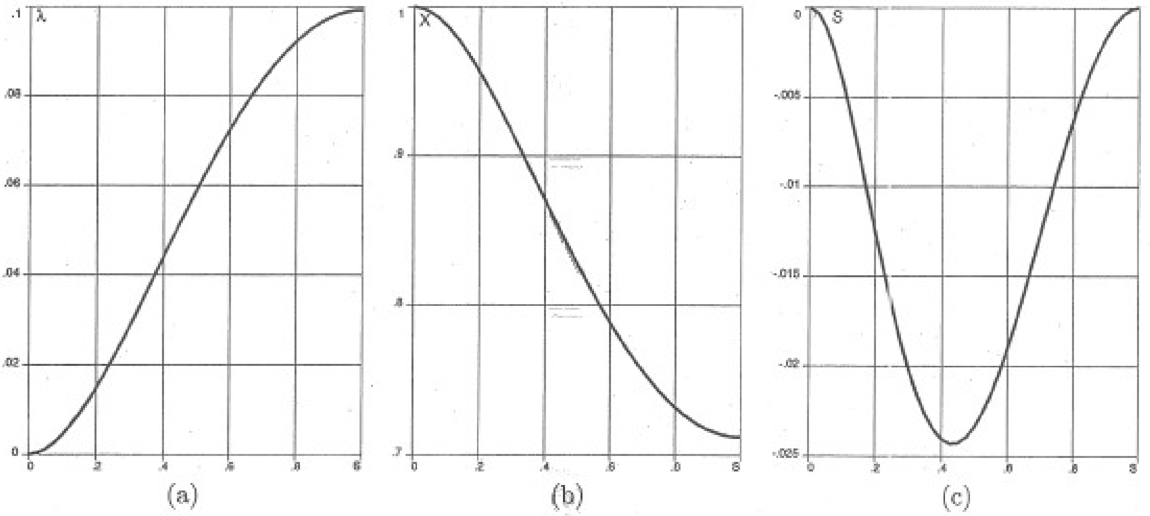


Fig.1. Characteristics of the Bishop-Taylor equilibria: (a) $0 \le \lambda \le 0.0993155012$*, (b)* $0.717017 \le X \le 1$*, (c)* $-.024329 \le S \le 0$.

With 6 valuable digits, the functions $X(\lambda)$ and $S(\lambda)$ can be approximated by

$$X = 1 - \lambda[3 \cdot \hat{\lambda}^2(3-2\hat{\lambda}) + 2.899527 \cdot \bar{\lambda}^2(3-2\bar{\lambda}) - \hat{\lambda}\bar{\lambda}(0.1117299 \cdot \hat{\lambda} - 0.09066381 \cdot \bar{\lambda})],$$
$$S = \bar{\lambda}\hat{\lambda}[-0.09931550 \cdot \hat{\lambda}^2(3-2\hat{\lambda}) - 0.09551659 \cdot \bar{\lambda}^2(3-2\bar{\lambda}) + \hat{\lambda}\bar{\lambda}(0.00425197 \cdot \hat{\lambda} - 0.00340596 \cdot \bar{\lambda})], \quad (4.4)$$
$$0 \le \bar{\lambda} \equiv \frac{\lambda}{\lambda_0} \le 1, \quad \hat{\lambda} \equiv 1 - \bar{\lambda},$$

where $\lambda_0 = 0.0993155012$ is the largest value of λ in the Bishop-Taylor equilibria

The shape of the magnetic surfaces for equidistant s coordinate is showm on Fig.2.

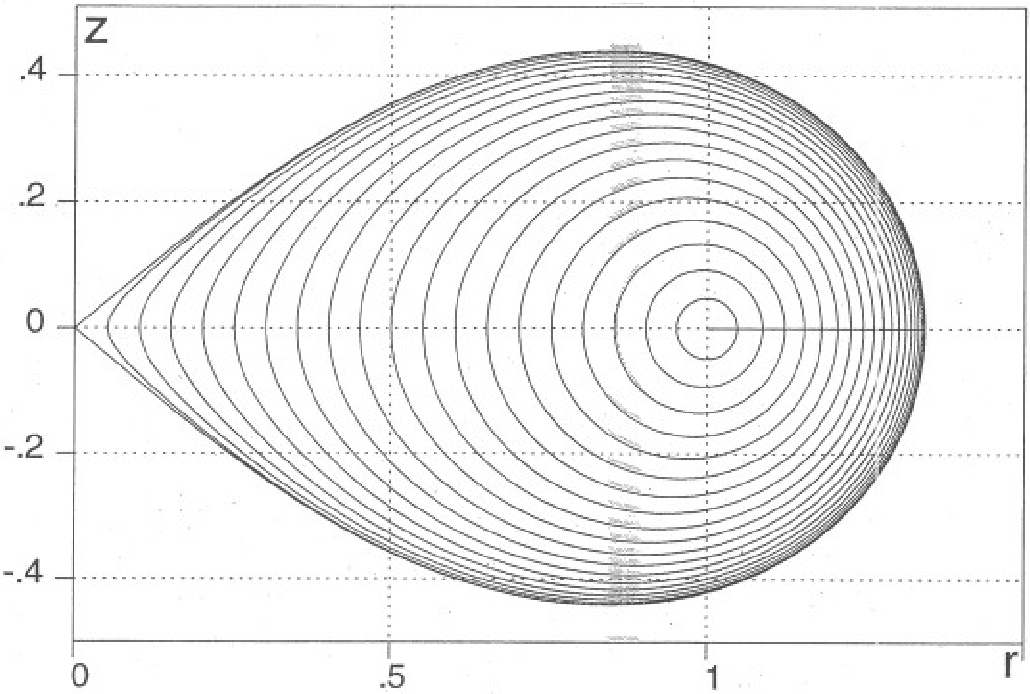


Fig.2. Cross-section of the magnetic configuration with equidistant intervals between magnetic axis and $r = 0$.

5 Basis functions of Bishop-Taylor equilibria *(to ToC)*

The basis functions of this section are necessary and sufficient for calculating quantities necessary for stability, transport, gyro-motion and gyro-kinetic codes.

The differential properties of the flux coordinates λ, θ, φ are given by the following derivatives of r, z

$$r^2 = X + d + a\cos\theta, \quad 2rr'_\theta = -a\sin\theta, \quad 2rr'_\lambda = 4\frac{a\cos\theta - d}{a^2 - d^2}, \quad rr''_{\lambda\theta} = -r'_\theta r'_\lambda - \frac{2a\sin\theta}{a^2-d^2},$$
$$z = -\int_0^\theta \frac{(a\cos\theta + d)d\theta}{2\sqrt{r^2+2d}}, \quad z'_\theta = -\frac{a\cos\theta + d}{2\sqrt{r^2+2d}}, \quad z'_\lambda = -\frac{1}{a^2-d^2}\frac{2a\sin\theta}{\sqrt{r^2+2d}}, \quad (5.1)$$
$$z''_{\lambda\theta} = -\frac{2(r^2+2d)a\cos\theta + a^2\sin^2\theta}{(a^2-d^2)(r^2+2d)^{3/2}}.$$

Using these derivatives the metric tensor can be calculated in a straighforward manner. The Jacobian $\sqrt{g}$ of these coordinates has the form

$$\sqrt{g} = \frac{1}{\sqrt{r^2+2d}}. \tag{5.2}$$

Its λ-derivative can be calculated as

$$(\sqrt{g})'_\lambda = \frac{4}{(a^2-d^2)(X+2d)}\left[-\frac{a\sin\theta}{\sqrt{r^2+2d}} - z\right]'_\theta. \tag{5.3}$$

From the expressions for the derivatives z'_λ and $(\sqrt{g})'_\lambda$ it follows that

$$\begin{aligned}
&(z_{\theta=2\pi} - z_{\theta=0})'_\lambda = \int_0^{2\pi} z'_\lambda d\theta = 0, \quad z_{\theta=2\pi} = z_{\theta=0},\\
&V''_{\lambda\lambda} = 4\pi^2\int_0^{2\pi}(\sqrt{g})'_\lambda d\theta = 0, \quad V'_\lambda = 4\pi^2\oint\frac{d\theta}{\sqrt{r^2+2d}} = 4\pi^2, \quad V = 4\pi^2\lambda.
\end{aligned} \tag{5.4}$$

This proves the closure of all magnetic surfaces, which are determined by an expression for z, and specifies the meaning of the parameter λ.

Four basis profiles and their first derivatives, which are necessary, e.g., for stability codes, are given by the following expressions, containing an arbitrary function L

$$\begin{aligned}
&\bar\Psi'_\lambda = -L, \quad \bar\Psi''_{\lambda\lambda} = -L'_\lambda, \quad \bar F^2 = \bar F_0^2 - SL^2,\\
&H \equiv \frac{1}{r^2\sqrt{r^2+2d}}, \quad H_0 \equiv \frac{1}{2\pi}\oint H d\theta, \quad \bar\Phi'_\lambda = \bar F H_0, \quad \bar\Phi''_{\lambda\lambda} = \bar F'_\lambda H_0 + \bar F H'_{0\lambda},\\
&P \equiv \bar p'_{\bar\Psi} = \frac{5}{2}L + 3\lambda L'_\lambda, \quad P'_\lambda = \frac{11}{2}L'_\lambda + 3\lambda L''_{\lambda\lambda},\\
&T \equiv \bar F\bar F'_{\bar\Psi} = \frac{1}{2}LS'_\lambda + SL'_\lambda, \quad T'_\lambda = \frac{1}{2}LS''_{\lambda\lambda} + \frac{3}{2}S'_\lambda L'_\lambda + SL''_{\lambda\lambda}.
\end{aligned} \tag{5.5}$$

In the code `Cbbsh`, the derivatives of H, H_0 are calculated using the derivatives of r^2 and evaluating the integral numerically.

In order to complete the set of basis functions, two more 2-dimensional functions and their derivatives should be provided, i.e.,

$$\begin{aligned}
&|\mathbf{B}| = \sqrt{3\lambda L^2 + \frac{\bar F_0^2}{r^2}}, \quad |\mathbf{B}|'_\lambda = \frac{3L^2 + 6\lambda LL'_\lambda - \bar F_0^2\frac{(r^2)'_\lambda}{r^4}}{2|\mathbf{B}|}, \quad |\mathbf{B}|'_\theta = \bar F_0^2\frac{a\sin\theta}{2r^4|\mathbf{B}|},\\
&\eta'_\theta \equiv \frac{H}{H_0} - 1, \quad \eta''_{\lambda\theta} = \frac{H'_\lambda}{H_0} - \frac{HH'_{0\lambda}}{H_0^2}, \quad \eta''_{\theta\theta} = \frac{H'_\theta}{H_0}, \quad \eta'''_{\lambda\theta\theta} = \frac{H''_{\lambda\theta}}{H_0} - \frac{H'_\theta H'_{0\lambda}}{H_0^2}.
\end{aligned} \tag{5.6}$$

Module $|\mathbf{B}|$ is necessary for particle orbit codes and calculating curvature of the field lines in stability codes. The function η is the radial covariant component of the vector potential of the magnetic field and is necessary for generating straight field line coordinates.

Although the shape of the magnetic surfaces does not depend on the choice of function $L(\lambda)$, the profiles of the current density, pressure, and magnetic fluxes do. Thus, a unique set of flux surfaces in Bishop-Taylor configuration corresponds to infinitely many equilibria.

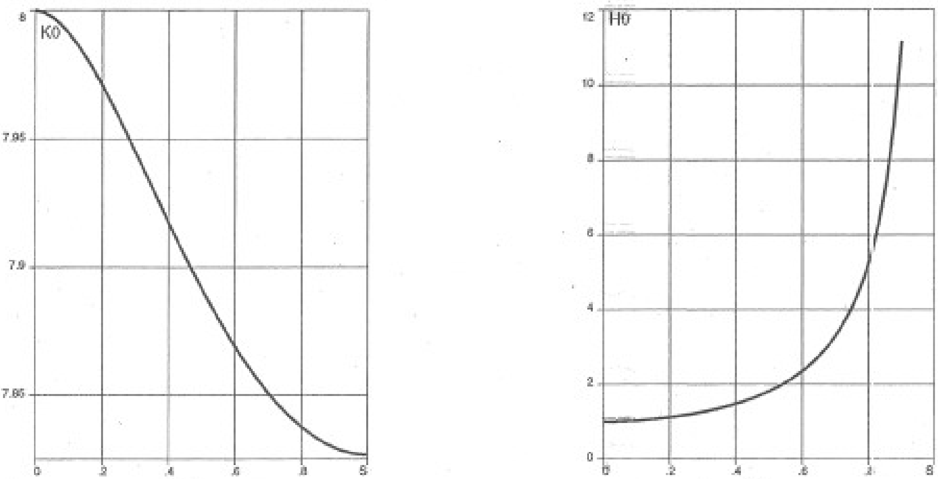
The plasma current $I(\lambda)$ through the magnetic surface is determined by the following universal relationship

$$\mu_0 I = -2\pi\bar K_0\lambda\bar\Psi'_\lambda = 2\pi\bar K_0\lambda L, \quad \bar K_0 \equiv \frac{r'^2_\theta + z'^2_\theta}{\sqrt{g}\lambda} = \frac{1}{2\pi\lambda}\oint\frac{a^2+2ad\cos\theta+d^2}{\sqrt{r^2+2d}}d\theta, \quad q \equiv -\frac{\bar\Phi'_\lambda}{\bar\Psi'_\lambda} = \frac{\bar F H_0}{L}. \tag{5.7}$$

The safety factor q-profile is given by the following expression

$$q \equiv -\frac{\bar{\Phi}'_\lambda}{\bar{\Psi}'_\lambda} = \frac{\bar{F}}{L} H_0 = \frac{\sqrt{F_0^2 - SL^2}}{L} H_0. \tag{5.8}$$

The functions $H_0(\lambda)$ and $K_0(\lambda)$, which are the geometrical characteristics and do not depend on the current profile, are shown in Fig.3.



(a) (b)

Fig.3. Two additional characteristics of the Bishop-Taylor equilibria: (a) $\bar{K}_0$, (b) H_0 (shown on a truncated interval because of a singularity in H_0 at $s \to 1$).

6 Summary. Code Implementation *(to ToC)*

The above equations have been implemented in the `Cbbsh`[3] numerical code, which will create a set of nested magnetic surfaces and comprehensive information on the Bishop-Taylor equilibrium, once the choice of the radial coordinate (e.g., s, λ, $\sqrt{(V/V_0)}$, $\sqrt{(\bar{\Phi}/\bar{\Phi}_0)}$) has been made, and its boundary value, F_0, and the function L have been specified. The code produces an output in the standard Equilibrium Spline Interface (ESI) format[4] of an equilibrium code solver. Specific documentation on usage of the code can be found in the documentation and help files which accompany it. In particular, the present paper is a part of the help file for the main section of the code.

References

[1] C. M. Bishop, J. B. Taylor. *"A special class of toroidal MHD equilibria including minimum-B"*. In Proceedings 12 European Conference on Controlled Fusion and Plasma Physics, Budapest, pp. 401-404 (1985)

[2] Bishop, C. M. and J. B. Taylor. "Degenerate toroidal MHD equilibria and minimum B". Physics of Fluids **29**, 1444 (1986).

[3] L.E.Zakharov, J.B.Kallman. *"Cbbsh-code for calculation of the Bishop-Taylor equilibria"*, http://w3.pppl.gov/~zakharov (Numerical codes and the theory)

[4] L.E.Zakharov, R.B.White *et al* *"Equilibrium Spline Interface (ESI) for magneic confinement codes"*, http://w3.pppl.gov/~zakharov (Numerical codes and the theory)

The Princeton Plasma Physics Laboratory is operated
by Princeton University under contract
with the U.S. Department of Energy.

Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2750
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov