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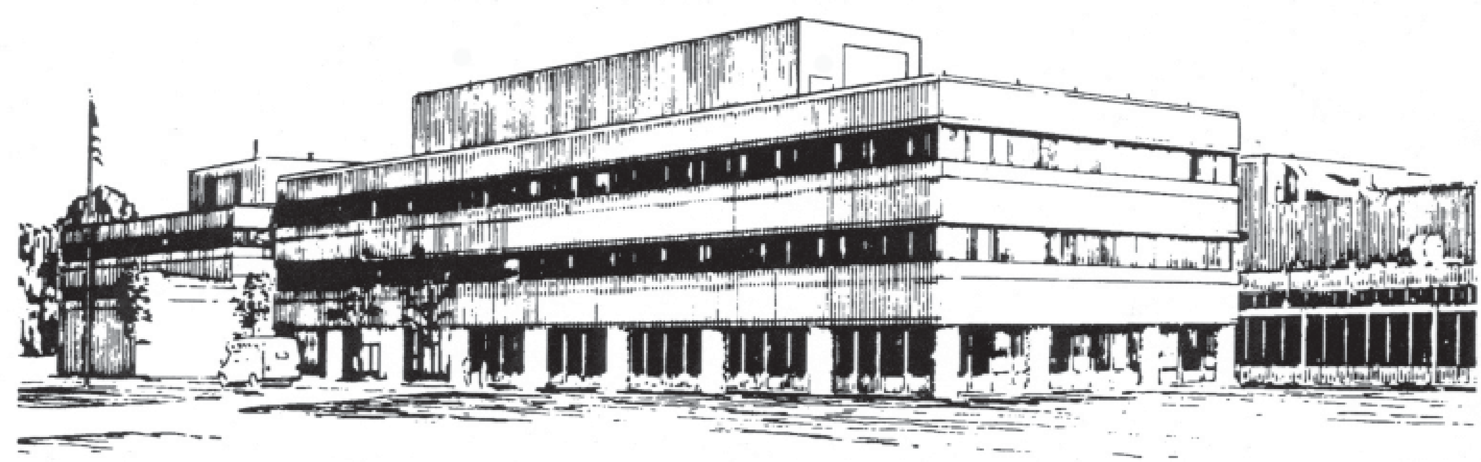

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MHD Field Line Resonances and Global Modes in Three-Dimensional Magnetic Fields

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Abstract.

By assuming a general isotropic pressure distribution $P = P(\psi, \alpha)$, where ψ and α are three-dimensional scalar functions labeling the field lines with $\mathbf{B} = \nabla\psi \times \nabla\alpha$, we have derived a set of MHD eigenmode equations for both global MHD modes and field line resonances (FLR). Past MHD theories are restricted to isotropic pressures with $P = P(\psi)$ only. The present formulation also allows the plasma mass density to vary along the field line. The linearized ideal MHD equations are cast into a set of global differential equations from which the field line resonance equations of the shear Alfvén waves and slow magnetosonic modes are naturally obtained for general three-dimensional magnetic field geometries with flux surfaces. Several new terms associated with $\partial P/\partial\alpha$ are obtained. In the FLR equations a new term is found in the shear Alfvén FLR equation due to the geodesic curvature and the pressure gradient pressure gradient in the poloidal flux surface. The coupling between the shear Alfvén waves and the magnetosonic waves is through the combined effects of geodesic magnetic field curvature and plasma pressure as previously derived. The properties of the FLR eigenfunctions at the resonance field lines are investigated, and the behavior of the FLR wave solutions near the FLR surface are derived. Numerical solutions of the FLR equations for three-dimensional magnetospheric fields in equilibrium with high plasma pressure will be presented in a future publication.

1. Introduction

Field line resonances (FLRs) with multiharmonic frequencies have been observed extensively, for example, for $2 \leq L \leq 9$ by AMPTE/CCE [*Takahashi et al.*, 1990, 2002]. It was firmly established that they are the most commonly excited low- to mid- frequency Pc 3-5 waves in the dayside magnetosphere from the plasmapause to the magnetopause [*Engebretson et al.*, 1986]. The theory of local field-line resonances of standing shear Alfvén waves in response to the propagation of external disturbances [*Radoski*, 1966; *Cummings et al.*, 1969; *Tataronis and Grossmann*, 1973; *Chen and Hasegawa*, 1974; *Southwood*, 1974; *Cheng and Chance*, 1986; *Cheng et al.*, 1993] seemed to be able to explain the basic features of these Pc 3-5 transverse waves. The corresponding eigenfrequencies for the transverse shear Alfvén waves standing along the field lines vary spatially and constitutes the so-called shear Alfvén continuum. For an excitation frequency matching an eigenfrequency inside the shear Alfvén continuum, the wave resonance generates perturbations that are radially localized at the particular resonant magnetic field line. However, most theories of the field line resonance have been limited to the cold plasma model in simple one-dimensional straight but nonuniform magnetic field intensity [*Southwood and Kivelson*, 1986], or a dipole field geometry [*Chen and Cowley*, 1989; *Lee and Lysak*, 1990], or stretched magnetic fields based on empirical magnetic field models such as the Tsyganenko's T96 model [*Rankin et al.*, 2000]. In a realistic magnetosphere, besides being nonuniform in the radial direction the Alfvén velocity is also nonuniform in the azimuthal direction as well as in the direction along the ambient magnetic field. Moreover, the plasma pressure is larger than the magnetic pressure in the plasma sheet and thus the pressure effect must be included. With the magnetospheric magnetic field represented by two Euler potentials as $\mathbf{B} = \nabla\psi \times \nabla\alpha$, where ψ is the poloidal flux and α is a toroidal angle-like variable, the plasma pressure can be written as $P = P(\psi, \alpha)$. By further assuming that the pressure is only a function of the poloidal flux, $P = P(\psi)$, the continuous and discrete spectra of shear Alfvén wave and slow magnetosonic wave for two-dimensional axisymmetric equilibrium magnetic field models [*Cheng*, 1992], where the magnetic field is nonuniform along and across the ambient magnetic field, have been studied [e.g., *Cheng and Chance*, 1986; *Cheng et al.*, 1993]. Moreover, this type of calculation was recently extended for stretched fields obtained from the axisymmetric equilibrium solutions to understand the significant reduction of FLR frequency frequency to $1-4$ mHz at auroral latitudes in the night sector [*Lui and Cheng*, 2001].

Global magnetospheric ULF pulsations with frequencies in the Pc 5 range ($f =$ 1.7 - 6.7 mHz) and below have been observed for decades in space and on the Earth [*Herron*, 1967; *Samson et al.*, 1991; *Nikutowski et al.*, 1995; *Rinnert*, 1996; *Kepko et al.*, 2002]. Observation of discrete frequencies with f = 1.3, 1.9, 2.6, 3.4, and 4.2 mHz [*Samson et al.*, 1991] have been attributed to global wave-guide modes [*Samson et al.*, 1992]. Recently, these global magnetospheric ULF pulsations are explained as driven directly from the fluctuations in the solar wind because of the good correlation between the fluctuation spectrum observed by WIND spacecraft in the upstream solar wind region and the measured spectrum by the geosynchronous satellite GOES 10 [*Kepko et al.*, 2002]. In order to study these global modes, numerical solutions of global MHD eigenmode equations must be pursued. Moreover, these global mode frequencies must be less than the FLR frequencies in order for compressional MHD waves to propagate to the lower L-shell region.

It is to be noted that the MHD model for isotropic pressure dictates that the plasma pressure is constant along a field line, but can vary from field lines to field lines. In particular, because in the plasma sheet the pressure is in general not just a function of poloidal flux and plasma β is higher than unity, the property of global MHD modes and field line resonances can be greatly modified by the pressure effect. Therefore, it is important to re-formulate the MHD equations for general isotropic pressure distributions. In the paper, we derive the linearized ideal MHD eigenmode equations for arbitrary isotropic pressure distributions in a form to provide for a better physical representation of the MHD continuous spectra in general three-dimensional magnetic field geometries with magnetic flux surfaces. Several new terms are obtained in the global eigenmode equations due to $\partial P/\partial \alpha$. The field line resonance equations of the shear Alfvén waves and slow magnetosonic waves are naturally obtained from the global eigenmode equations. In particular, a new term is found in the shear Alfvén FLR equation due to the geodesic curvature and $\partial P/\partial \alpha$. The coupling between the shear Alfvén waves and the magnetosonic waves is through the combined effects of geodesic magnetic field curvature and plasma pressure as previously derived. In 3D fields the FLR surface, which consists of field lines with the same resonance frequency, usually differs from the constant ψ (poloidal flux) surface (or L-shell) in the magnetosphere. The FLR eigenfunctions will be singular at the FLR surface and the singularity can be removed by introducing non-ideal MHD physics. However, the behavior of the FLR wave solutions near the FLR surface determines the strength of the coupling of the fast wave perturbations into the shear Alfvén wave and the slow mode field line resonance eigenfunctions at the local resonance surface.

In the following, the ideal MHD eigenmode equations are given in Section 2. The plasma pressure is constrained to be constant along the field line, but the mass density is allowed to vary along the field lines. The detailed derivation of the MHD eigenmode equations is given in the Appendix. In Section 3 the field line resonances that correspond to two branches (shear Alfvén waves and slow magnetosonic waves) of the MHD continuous spectra are naturally defined from the global MHD eigenmode equations, and the properties of the FLR eigenfunctions at the resonance field lines are presented. In Section 4 we present the asymptotic behavior of solutions of the MHD equations perpendicular to the field line near the field line resonance surface. Finally, in Section 5 a summary of the major results is given, and the implications of physical effects that are absent in the MHD model and future efforts involving global computation of wave propagation are discussed.

2. MHD Eigenmode Equations

We consider static magnetospheric equilibria described by the system of equations

$$\begin{aligned} \mathbf{J} \times \mathbf{B} &= \boldsymbol{\nabla} P, \\ \boldsymbol{\nabla} \times \mathbf{B} &= \mathbf{J}, \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \end{aligned} \tag{1}$$

where $\mathbf{J}$, $\mathbf{B}$, and P are the equilibrium current, magnetic field, and plasma pressure, respectively. The above equilibrium equations can be cast into the following form:

$$\boldsymbol{\nabla}(P + B^2/2) = \boldsymbol{\kappa} B^2, \tag{2}$$

where $\boldsymbol{\kappa} = (\mathbf{B}/B) \cdot \boldsymbol{\nabla}(\mathbf{B}/B)$ is the magnetic field curvature. For a general three dimensional magnetospheric equilibrium with nested flux surfaces, the magnetic field can be expressed as

$$\mathbf{B} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\alpha, \tag{3}$$

where ψ and α are three-dimensional functions of configuration space variable $\mathbf{x}$. We choose ψ to be the magnetic flux function labeling the nested flux surfaces and α to be an angle-like variable. The lines where surfaces of constant ψ and surfaces of constant α intersect represent magnetic field lines along which both ψ and α are constant. Because $\mathbf{B} \cdot \boldsymbol{\nabla} P = 0$, the pressure is constant along field lines and has the general form $P = P(\psi, \alpha)$. However, we note that the plasma density is allowed to vary along the field lines. It is to be noted that previous works on MHD eigenmode equations were based on the assumption that $P = P(\psi)$ only. By allowing a more general two-dimensional pressure distribution function, several new terms are obtained in the eigenmode equations. We also note that for magnetospheric magnetic fields α is a periodic function of toroidal angle ϕ in the cylindrical (R, ϕ, Z) coordinate system to ensure periodicity constraint on each flux surface.

With the time dependence of perturbed quantities as $e^{-i\omega t}$ and with the application of the Laplace transform, the linearized ideal MHD equations governing the asymptotic behaviors of the perturbed quantities are the momentum equation

$$\rho\omega^2\boldsymbol{\xi} = \boldsymbol{\nabla}\delta p + \delta\mathbf{B}\times\mathbf{J} + \mathbf{B}\times(\boldsymbol{\nabla}\times\delta\mathbf{B}), \tag{4}$$

the equation of state

$$\delta p + \boldsymbol{\xi}\cdot\boldsymbol{\nabla}P + \Gamma_s P\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0, \tag{5}$$

the Faraday's law

$$i\omega\delta\mathbf{B} = \boldsymbol{\nabla}\times(\delta\mathbf{E}), \tag{6}$$

and the Ohm's law

$$\delta\mathbf{E} = \boldsymbol{\xi}\times\mathbf{B}, \tag{7}$$

where $\boldsymbol{\xi}$ is the usual fluid displacement vector, $\delta\mathbf{B}$ is the perturbed magnetic field, δp is the perturbed plasma pressure, ρ is the total plasma mass density, $\delta\mathbf{E}$ is the perturbed electric field, and $\Gamma_s = 5/3$ is the ratio of specific heats.

To derive the MHD eigenmode equations in scalar forms, we decompose the displacement vector and perturbed magnetic field as

$$\boldsymbol{\xi} = \frac{\xi_\psi\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2} + \frac{\xi_s(\mathbf{B}\times\boldsymbol{\nabla}\psi)}{B^2} + \frac{\xi_b\mathbf{B}}{B^2}, \tag{8}$$

and

$$\delta\mathbf{B} = \frac{Q_\psi\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2} + \frac{Q_s(\mathbf{B}\times\boldsymbol{\nabla}\psi)}{|\boldsymbol{\nabla}\psi|^2} + \frac{Q_b\mathbf{B}}{B^2}, \tag{9}$$

where $\xi_\psi = \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\psi$, $\xi_s = \boldsymbol{\xi}\cdot\mathbf{B}\times\boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|^2$, $\xi_b = \boldsymbol{\xi}\cdot\mathbf{B}$, $Q_\psi = \delta\mathbf{B}\cdot\boldsymbol{\nabla}\psi$, $Q_s = \delta\mathbf{B}\cdot\mathbf{B}\times\boldsymbol{\nabla}\psi/B^2$, $Q_b = \delta\mathbf{B}\cdot\mathbf{B}$. We also define $\Delta = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}$, the geodesic curvature

$\kappa_s = 2\boldsymbol{\kappa}\cdot\mathbf{B}\times\boldsymbol{\nabla}\psi/B^2$, the radial curvature $\kappa_\psi = 2\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|^2$, and the local magnetic shear as $S = (\mathbf{B}\times\boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|^2)\cdot\boldsymbol{\nabla}\times(\mathbf{B}\times\boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|^2)$. The detailed derivation of the eigenmode equations is given in the Appendix. We obtain the following four three-dimensional eigenmode equations which are similar to those obtained by *Cheng and Chance* [1986, 1987] for warm plasmas, but with several new terms.

$$\begin{aligned}\left[\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{|\boldsymbol{\nabla}\psi|^2}{B^2}\mathbf{B}\cdot\boldsymbol{\nabla}\xi_s\right)+\frac{\rho\omega^2|\boldsymbol{\nabla}\psi|^2}{B^2}\xi_s+\kappa_s\frac{\partial P}{\partial\alpha}\xi_s\right]+\Gamma_s P\kappa_s\Delta = \\ \left[\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{|\boldsymbol{\nabla}\psi|^2}{B^2}S\xi_\psi\right)-\frac{\mathbf{J}\cdot\mathbf{B}}{B^2}\mathbf{B}\cdot\boldsymbol{\nabla}\xi_\psi-\kappa_s\left(\frac{\boldsymbol{\nabla}P\cdot\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\right)\xi_\psi\right] \\ +\left[\left(\frac{\mathbf{B}\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\delta P}{B^2}\right)-\kappa_s\delta P\right],\end{aligned} \tag{10}$$

and

$$\left[\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\Gamma_s P}{\rho\omega^2B^2}\mathbf{B}\cdot\boldsymbol{\nabla}\Delta\right)+\frac{B^2+\Gamma_s P}{B^2}\Delta\right]+\kappa_s\xi_s = -\kappa_\psi\xi_\psi-\frac{\delta P}{B^2}, \tag{11}$$

and

$$\begin{aligned}\frac{\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\delta P}{|\boldsymbol{\nabla}\psi|^2} &= \kappa_\psi\delta P \\ &+\left[\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\xi_\psi}{|\boldsymbol{\nabla}\psi|^2}\right)+\frac{\rho\omega^2}{|\boldsymbol{\nabla}\psi|^2}\xi_\psi+\kappa_\psi\left(\frac{\boldsymbol{\nabla}P\cdot\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\right)\xi_\psi-(|\boldsymbol{\nabla}\psi|^2S-\mathbf{J}\cdot\mathbf{B})\frac{S}{B^2}\xi_\psi\right] \\ &+\left[\kappa_\psi\frac{\partial P}{\partial\alpha}\xi_s+(|\boldsymbol{\nabla}\psi|^2S-\mathbf{J}\cdot\mathbf{B})\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\xi_s}{B^2}\right]+\Gamma_s P\kappa_\psi\Delta,\end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\xi_\psi}{|\boldsymbol{\nabla}\psi|^2} &= -\left[\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\right)+\kappa_\psi\right]\xi_\psi-\frac{\delta P}{B^2} \\ &-\left[\frac{\mathbf{B}\times\boldsymbol{\nabla}\psi}{B^2}\cdot\boldsymbol{\nabla}\xi_s+\frac{\partial P}{\partial\alpha}\left(\frac{\xi_s}{B^2}\right)\right]-\left(\frac{\Gamma_s P}{B^2}\right)\Delta,\end{aligned} \tag{13}$$

Note that the above four equations are different from those derived previously [*Cheng and Chance*, 1986, 1987; *Cheng et al.*, 1993]; there are three new terms (one each in Eq.(10), Eq.(12), and Eq.(13)) resulting from the longitudinal pressure variation due to $\partial P/\partial\alpha$. In addition, $\boldsymbol{\nabla}P\cdot\boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|^2$ in two terms are different from $\partial P/\partial\psi$ because $\boldsymbol{\nabla}\psi$ is not orthogonal to $\boldsymbol{\nabla}\alpha$.

Equations (10) - (13) represent a set of four eigenmode equations describing the MHD wave propagation and MHD instabilities in a general magnetized plasma system, and they can be written symbolically in the following form

$$E \begin{pmatrix} \xi_s \\ \Delta \end{pmatrix} = F \begin{pmatrix} \delta P \\ \xi_\psi \end{pmatrix} \tag{14}$$

and

$$\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2} \cdot \boldsymbol{\nabla} \begin{pmatrix} \delta P \\ \xi_\psi \end{pmatrix} = C \begin{pmatrix} \delta P \\ \xi_\psi \end{pmatrix} + D \begin{pmatrix} \xi_s \\ \Delta \end{pmatrix} \tag{15}$$

where E and C are 2×2 matrix operators involving only the $\mathbf{B} \cdot \boldsymbol{\nabla}$ operator along field lines, and D and F are 2×2 matrix operators involving both $\mathbf{B} \cdot \boldsymbol{\nabla}$ and $(\mathbf{B} \times \boldsymbol{\nabla}\psi) \cdot \boldsymbol{\nabla}$ operators. We emphasize that this set of scalar MHD equations is derived for general 3D magnetic fields in equilibrium with scalar pressure distributions through $\mathbf{J} \times \mathbf{B} = \boldsymbol{\nabla} P$.

3. Field Line Resonance Frequency Spectrum

For a given magnetospheric equilibrium, we first solve ξ_s and Δ in terms of δP and ξ_ψ from Eq.(14) by inverting the matrix operator E that contains only derivatives along field lines. Eq.(15) then reduces to an equation for δP and ξ_ψ, i.e.,

$$\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2} \cdot \boldsymbol{\nabla} \begin{pmatrix} \delta P \\ \xi_\psi \end{pmatrix} = \left(C + DE^{-1}F\right) \begin{pmatrix} \delta P \\ \xi_\psi \end{pmatrix} \tag{16}$$

Admissible regular solutions of Eq.(16) must satisfy the proper boundary conditions. This procedure fails if the inversion of the operator E does not exist (i.e., $det|E| = 0$ subject to the appropriate boundary conditions) for a given frequency at certain field lines. Then Eq.(16) has singular solutions at these field line locations. Because of this singular nature of the MHD solutions, these field lines are in resonance with the external driver with the same frequency. Therefore, at each field line the eigenvalues ω with non-trivial single-valued eigenfunctions ξ_s and Δ of the equation

$$E\begin{pmatrix} \xi_s \\ \Delta \end{pmatrix} = 0 \tag{17}$$

subject to appropriate boundary conditions exists and they are called the field line resonance frequencies. Eq.(17) is explicitly expressed as

$$\left[\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{|\boldsymbol{\nabla}\psi|^2}{B^2}\mathbf{B}\cdot\boldsymbol{\nabla}\xi_s\right)+\frac{\rho\omega^2|\boldsymbol{\nabla}\psi|^2}{B^2}\xi_s+\kappa_s\frac{\partial P}{\partial\alpha}\xi_s\right]+\Gamma_s P\kappa_s\Delta=0, \tag{18}$$

and

$$\left[\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\Gamma_s P}{\rho\omega^2 B^2}\mathbf{B}\cdot\boldsymbol{\nabla}\Delta\right)+\frac{B^2+\Gamma_s P}{B^2}\Delta\right]+\kappa_s\xi_s=0. \tag{19}$$

Since Eqs. (18) and (19) can be combined to form a fourth order ordinary differential equation along the field line with the coefficients being all non-singular, the eigenvalues ω must be discrete for closed field lines. Thus, at each field line a discrete set of eigenvalues ω_n, where the index $n = 0, 1, 2, ...$ can be found with the corresponding eigenfunctions ξ_{sn} and Δ_n satisfying appropriate boundary conditions along closed field lines. Note that ξ_{sn} and Δ_n are linearly dependent through Eqs.(18) and (19). Because the field lines are continuous in space, each ω_n takes a continuous range of values for different field lines and form a continuous spectrum.

It is clear that there are only two branches of MHD field line resonances - the shear Alfvén branch (Eq.(18)) and the slow magnetosonic branch (Eq.(19)), and the coupling of these two branches of MHD field line resonances is through the geodesic magnetic field curvature κ_s and plasma pressure. We also note that there is an additional term in the shear Alfvén equation due to pressure gradient in the $\mathbf{B}\times\boldsymbol{\nabla}\psi$ direction. In the cold plasma limit ($P = 0$), the slow magnetosonic wave no longer exists. Eq.(18) then reduces to a second order ordinary differential equation for ξ_s and describes the shear Alfvén resonance (toroidal magnetic field resonance) in the cold plasma limit previously investigated by *Cummings et al.* [1969]. However, from Eq.(19) $\Delta = -\kappa_s\xi_s$ and thus the shear Alfvén waves retain a finite plasma compressibility if the geodesic magnetic field curvature κ_s is non-vanishing.

A variational principle can be obtained from Eqs.(18)-(19). Multiplying Eq.(18) by ξ_s^* (the complex conjugate of ξ_s) and integrating along the field line with respect to ds/B, we obtain

$$\int_{s_1}^{s_2} \frac{ds}{B} \left\{ \left(\rho\omega^2 \frac{|\nabla\psi|^2}{B^2} + \kappa_s \frac{\partial P}{\partial\alpha} \right) |\xi_s|^2 - \frac{|\nabla\psi|^2}{B^2} |\mathbf{B}\cdot\nabla\xi_s|^2 + \Gamma_s P \kappa_s \xi_s^* \Delta \right\} = 0 \tag{20}$$

where s denotes the distance along the field line so that $\mathbf{B}\cdot\nabla = B(d/ds)$, s_1 and s_2 are the two end points of the field line, and the boundary condition at the field line end points is assumed to be $\xi_s^* \mathbf{B}\cdot\nabla\xi_s = 0$. Multiplying the complex conjugate of Eq.(19) by $\Gamma_s P \Delta$ and integrating along the field line with respect to ds/B, we obtain

$$\int_{s_1}^{s_2} \frac{ds}{B} \left\{ \frac{\Gamma_s P \left(\Gamma_s P + B^2\right)}{B^2} |\Delta|^2 - \rho\omega^2 B^2 |Z|^2 + \Gamma_s P \kappa_s \xi_s^* \Delta \right\} = 0 \tag{21}$$

where $Z = \Gamma_s P(\mathbf{B}\cdot\nabla\Delta)/\rho\omega^2 B^2$, and the boundary condition $\Delta Z^* = 0$ at the field line end points is assumed. Subtracting Eq.(21) from Eq.(20) we obtain a Lagrangian functional δL given by

$$\begin{aligned} \delta L = \int_{s_1}^{s_2} \frac{ds}{B} &\left\{ \rho\omega^2 \left(\frac{|\nabla\psi|^2}{B^2} |\xi_s|^2 + B^2 |Z|^2 \right) \right. \\ &\left. - \left[\frac{|\nabla\psi|^2}{B^2} |\mathbf{B}\cdot\nabla\xi_s|^2 - \kappa_s \frac{\partial P}{\partial\alpha} |\xi_s|^2 + \frac{\Gamma_s P B^2}{\Gamma_s P + B^2} |\kappa_s \xi_s + \mathbf{B}\cdot\nabla Z|^2 \right] \right\} = 0 \end{aligned} \tag{22}$$

where we have also made use of Eq.(19) to substitute Δ in terms of ξ_s and $\mathbf{B}\cdot\nabla Z$. It is straight forward to verify that Eqs.(18) and (19) are a consequence of the requirement that the functional δL is stationary. Since $\delta L = 0$, it is clear that the eigenvalues ω^2 and the corresponding eigenfunctions ξ_s and Δ must be real. The determination of the field line resonance frequency spectrum reduces to that of finding the eigenvalues ω^2 and eigenfunctions so that the Lagrangian functional δL is stationary with respect to variations of ξ_s and Δ. The admissible variational functions must be square-integrable and satisfy the standing wave boundary condition for closed field lines. It should be noted from Eq.(22) that ω^2 is not necessarily positive definite and thus there is a possibility of $\omega^2 < 0$ if $\kappa_s(\partial P/\partial\alpha) > 0$, and if $\omega^2 < 0$ the plasma is unstable at these field lines. Although numerical solutions show that $\omega^2 \geq 0$ for some three-dimensional quasi-static magnetospheric equilibrium fields, a proof for it has not been performed.

One can also show that at each field line the discrete set of eigenfunctions $\{\xi_{sn}; \Delta_n\}$ with corresponding eigenvalues ω_n are complete and orthogonal. Multiplying Eq.(18) by ξ_{sn} and integrating along the field line with respect to ds/B, we obtain

$$\begin{aligned} \int_{s_1}^{s_2} \frac{ds}{B} \Bigg\{ \left(\rho\omega^2 \frac{|\nabla\psi|^2}{B^2} + \kappa_s \frac{\partial P}{\partial\alpha} \right) \xi_s \xi_{sn} \\ - \frac{|\nabla\psi|^2}{B^2} \left(\mathbf{B}\cdot\nabla\xi_s \right) \left(\mathbf{B}\cdot\nabla\xi_{sn} \right) + \Gamma_s P \kappa_s \xi_{sn} \Delta \Bigg\} = 0. \end{aligned} \tag{23}$$

Multiplying Eq.(19) for $\{\omega_n; \xi_{sn}; \Delta_n\}$ by $\Gamma_s P\Delta$ and integrating along the field line with respect to ds/B, we obtain

$$\int_{s_1}^{s_2} \frac{ds}{B} \left\{ \frac{\Gamma_s P \left(\Gamma_s P + B^2 \right)}{B^2} \Delta\Delta_n - \rho\omega^2 B^2 Z Z_n + \Gamma_s P \kappa_s \xi_{sn} \Delta \right\} = 0 \tag{24}$$

where $Z_n = \Gamma_s P \mathbf{B}\cdot\nabla\Delta_n/\rho\omega_n^2 B^2$. Subtracting Eq.(23) by Eq.(24) we obtain

$$\rho\omega^2 \int_{s_1}^{s_2} \frac{ds}{B} \left(\frac{|\nabla\psi|^2}{B^2} \xi_s \xi_{sn} + B^2 Z Z_n \right) = A \tag{25}$$

where

$$\begin{aligned} A = \int_{s_1}^{s_2} \frac{ds}{B} \Bigg\{ \frac{|\nabla\psi|^2}{B^2} \left(\mathbf{B}\cdot\nabla\xi_s \right) \left(\mathbf{B}\cdot\nabla\xi_{sn} \right) - \kappa_s \frac{\partial P}{\partial\alpha} \xi_s \xi_{sn} \\ + \frac{\Gamma_s P \left(\Gamma_s P + B^2 \right)}{B^2} \Delta\Delta_n \Bigg\}. \end{aligned} \tag{26}$$

Similarly, multiplying the complex conjugate of Eq.(18) for $\{\omega_n; \xi_{sn}; \Delta_n\}$ by ξ_s, multiplying Eq.(19) for $\{\omega; \xi_s; \Delta\}$ by $\Gamma_s P\Delta_n$, subtracting these two equations and integrating along the field line with respect to ds/B, we also obtain

$$\rho\omega_n^2 \int_{s_1}^{s_2} \frac{ds}{B} \left(\frac{|\nabla\psi|^2}{B^2} \xi_s \xi_{sn} + B^2 Z Z_n \right) = A \tag{27}$$

Thus,

$$\rho\left(\omega^2-\omega_n^2\right)\int_{s_1}^{s_2}\frac{ds}{B}\left(\frac{|\boldsymbol{\nabla}\psi|^2}{B^2}\xi_s\xi_{sn}+B^2ZZ_n\right)=0. \tag{28}$$

Then, for $\omega^2 \neq \omega_n^2$, we have

$$\int_{s_1}^{s_2}\frac{ds}{B}\left(\frac{|\boldsymbol{\nabla}\psi|^2}{B^2}\xi_s\xi_{sn}+B^2ZZ_n\right)=0, \tag{29}$$

and

$$\int_{s_1}^{s_2}\frac{ds}{B}\left[\frac{|\boldsymbol{\nabla}\psi|^2}{B^2}\left(\mathbf{B}\cdot\boldsymbol{\nabla}\xi_s\right)\left(\mathbf{B}\cdot\boldsymbol{\nabla}\xi_{sn}\right)-\kappa_s\frac{\partial P}{\partial\alpha}\xi_s\xi_{sn}\right.$$
$$\left.+\frac{\Gamma_s P\left(\Gamma_s P+B^2\right)}{B^2}\Delta\Delta_n\right]=0. \tag{30}$$

Therefore, the eigenfunctions for different eigenvalues are orthogonal in the sense defined by Eqs.(28) - (30).

Finally, let us suppose that the field line resonance equations, Eq.(17), have eigenfunctions ξ_{sn} and Δ_n. Then, Eq.(14) can have solutions if and only if the right hand side of Eq.(14), which is regarded as inhomogeneous terms, satisfies some compatibility condition. To obtain this compatibility condition, we multiply Eq.(14) by the matrix $(\xi_{sn} \quad \Gamma_s P\Delta_n)$, integrate Eq.(14) along the field line with respect to ds/B, and apply the orthogonality condition Eq.(28). After some algebraic manipulation, we find

$$\int_{s_1}^{s_2}\frac{ds}{B}\left(\xi_{sn}\quad \Gamma_s P\Delta_n\right)F\begin{pmatrix}\delta P\\ \xi_\psi\end{pmatrix}=0 \tag{31}$$

which is the desired compatibility condition. The compatibility condition will be useful for understanding the wave structure near field line resonance locations.

4. Solutions Near Field Line Resonance Surface

To obtain the wave structure near the field line resonance locations, we will follow our previous approach [*Pao*, 1975; *Cheng et al.*, 1993] in the investigations of the continuous field line resonance frequency spectrum. We first note that the eigenvalues $\omega_n(\psi,\alpha)$ are smooth functions of (ψ,α) and form continuous spectra. For a given excitation frequency ω_{res}, these field line resonance locations with the same $\omega_n = \omega_{res}$ form a line (or more) in the (ψ,α) space and thus the resonance field lines form a surface. We can transform the old (ψ,α) coordinate to a new (ψ,α) coordinate such that $\psi = \psi_{0n}$ labels the field line resonance surface of an external disturbance with the excitation frequency $\omega_{res}^2 = \omega_n^2(\psi_{0n})$. Because the equilibrium magnetic field still has the same form as Eq.(3), the equations and conclusions derived so are still correct in this new (ψ,α) coordinate.

To study the behavior of the solutions of Eqs.(14)-(15) near the field line resonance surface ψ_{0n}, we introduce the variable $y = \psi - \psi_{0n}$ across the field line resonance surface and introduce a smallness parameter $\epsilon = |y/\psi_{0n}| << 1$. We note the following orderings: $|\psi_{0n}(\partial/\partial\psi)| = |\psi_{0n}(\partial/\partial y)| \sim O(1/\epsilon) >> 1$, and the operators $E, C, D, F \sim O(1)$. Near the resonance surface ψ_{0n} the general solutions of Eqs.(14)-(15) are a linear superposition of singular solutions and regular solutions with the coefficients determined by boundary conditions in the ψ-direction. The singular solutions can be expanded asymptotically as

$$\begin{aligned}
\delta P &= \lambda(y)[\delta P^{(0)}(s,\alpha) + y\delta P^{(1)}(s,\alpha)] + ... \\
\xi_\psi &= \lambda(y)[\xi_\psi^{(0)}(s,\alpha) + y\xi_\psi^{(1)}(s,\alpha)] + ... \\
\xi_s &= \lambda'(y)[\xi_s^{(0)}(s,\alpha) + y\xi_s^{(1)}(s,\alpha)] + ... \\
\Delta &= \lambda'(y)[\Delta^{(0)}(s,\alpha) + y\Delta^{(1)}(s,\alpha)] + ...
\end{aligned}$$

where the functions with superscripts are defined at the resonance surface $\psi = \psi_{0n}$ and are functions of (s,α), and the superscripts denote orderings in ϵ. Note that $\lambda'(y)$ is the derivative of $\lambda(y)$ with respect to y and the choice of $\lambda'(y)$ for ξ_s and Δ is suggested by Eq.(15). It is also assumed that $\lambda(y)$ is singular as $y \to 0$ and therefore, $\lambda/\lambda' \to 0$ as $y \to 0$. We also note that all functions in s satisfy the boundary conditions at the end points of the field lines. We also expand the operators E, C, D, F around the field line resonance surface ψ_{0n} as $E = E^{(0)}(s,\alpha) + yE^{(1)}(s,\alpha) + ...$, etc.

Substituting the above expansion expressions into Eqs.(14)-(15), we obtain in the lowest order that the eigenfunctions $\xi_s^{(0)}(s,\alpha)$ and $\Delta^{(0)}(s,\alpha)$ must satisfy the field line resonance equation, Eq.(17), with the operator $E^{(0)}(s,\alpha)$ and eigenvalue ω_{res}^2. To the next order we obtain

$$y\lambda' \left[E^{(0)} \begin{pmatrix} \xi_s^{(1)} \\ \Delta^{(1)} \end{pmatrix} + E^{(1)} \begin{pmatrix} \xi_s^{(0)} \\ \Delta^{(0)} \end{pmatrix} \right] = \lambda F \begin{pmatrix} \delta P^{(0)} + y\delta P^{(1)} \\ \xi_\psi^{(0)} + y\xi_\psi^{(1)} \end{pmatrix} \tag{32}$$

We now differentiate Eq.(32) with respect to y, multiply the differentiated equation with the matrix $(\xi_{sn} \quad \Gamma_s P \Delta_n)$, integrate along the field line with respect to ds/B, and obtain

$$[y\lambda']' \, G(\psi_{0n},\alpha) = O\left(\lambda, y\lambda'\right) \tag{33}$$

where

$$G(\psi_{0n},\alpha) = \int_{s_1}^{s_2} \frac{ds}{B} \left(\xi_{sn} \quad \Gamma_s P \Delta_n\right) \left[E^{(0)} \begin{pmatrix} \xi_s^{(1)} \\ \Delta^{(1)} \end{pmatrix} + E^{(1)} \begin{pmatrix} \xi_s^{(0)} \\ \Delta^{(0)} \end{pmatrix} \right], \tag{34}$$

and the right hand side of Eq.(33) is on the order of λ or $y\lambda'$ because the contribution to the order of $\lambda'(y)$ vanishes due to the compatibility condition, Eq.(31). In general $G \neq 0$ and Eq.(33) has the leading order solution given by

$$\lambda(y) = c_1 \ln y + c_2 \tag{35}$$

where c_1 and c_2 are integration constants, and $\lambda' = c_1/y$. Once $\lambda(y)$ is determined we can go to higher order equations to compute higher order eigenfunctions $\xi_s^{(1)}$, etc. This yields the behavior of the singular solution of Eqs.(14)-(15) in the neighborhood of the resonant surface ψ_{0n}. Finally, the regular solutions of Eqs.(14)-(15) near the resonant surface ψ_{0n} can be obtained by expanding the solutions and operators around ψ_{0n} in terms of power series in y and by solving equations in each order in y. The detailed construction of the regular solutions will not be presented here.

From the analytical procedure presented in this section, we conclude that δP and ξ_ψ have a logarithmic singularity as $\psi \to \psi_0$, and ξ_s and Δ diverge as $(\psi - \psi_0)^{-1}$. This result is similar to the axisymmetric equilibrium magnetic field cases [*Cheng et al.*, 1993] where ψ represents the poloidal flux labeling the flux surfaces which coincide with field line resonance surfaces. We emphasize that for general three-dimensional ψ represents the field line resonance surface corresponding to a constant excitation frequency. Thus, the dominant field components of the shear Alfvén branch near the resonance surface are mainly the radial electric field $\delta\mathbf{E}\cdot\boldsymbol{\nabla}\psi$ and shear magnetic field $\delta\mathbf{B}\cdot\mathbf{B}\times\boldsymbol{\nabla}\psi/B^2$ as seen from Eqs.(7). The dominant field component of the slow magnetosonic branch near the resonance is the compressional magnetic field $\delta\mathbf{B}\cdot\mathbf{B}$ contributed by $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ and ξ_s as seen from Eqs.(A5) and (A14). The other electromagnetic field components are smaller. The strength of the fast wave coupling to field line resonances depends on the integration constants c_1 and c_2, which represent the projection of the fast wave perturbations δP and ξ_ψ into the shear Alfvén wave field line resonance eigenfunction ξ_{sn} and the slow wave field line resonance eigenfunction Δ_n at the local resonance surface ψ_0. Finally, we also note that if there is more than one resonance surface for a given excitation frequency, the behavior of the wave solutions is similar near each of these resonance surfaces.

5. Summary and Discussions

By assuming a pressure distribution $P = P(\psi, \alpha)$ that is constant along the field line and by allowing the mass density to vary along the field line, we have derived a set of ideal MHD global eigenmode equations in general magnetic field geometries with flux surfaces. There are new terms in the MHD equations, which are not included in the previous formulation [e.g., *Cheng and Chance*, 1986; *Cheng et al.*, 1993] with the assumption that $P = P(\psi)$ only, due to pressure gradient in the $\mathbf{B}\times\boldsymbol{\nabla}\psi$ (mainly azimuthal) direction. From the global MHD equations the field line resonance equations for standing shear Alfvén and slow magnetosonic waves are naturally defined. In particular, a new term is found in the shear Alfvén FLR equation due to the geodesic curvature and $\partial P/\partial\alpha$. These two branches of continuous spectra are represented by the field line resonance eigenfunctions $\{\xi_{sn}; \Delta_n\}$ with corresponding eigenfrequencies ω_n, and they couple through the combined effects of geodesic magnetic field curvature and plasma pressure as previously derived. Thus, we expect the coupling to be strong for high β plasma in the central plasma sheet in the magnetosphere. In 3D fields the FLR resonance surface, which consists of field lines with the same resonance frequency, usually differs from the constant ψ (poloidal flux) surface (or L-shell in the magnetosphere). The solutions of the FLR eigenfunctions are singular at the FLR location and the singularity can be removed by introducing non-ideal MHD physics. The behavior of the FLR wave solutions near the FLR surface determines the strength of the coupling of the fast wave perturbations into the shear Alfvén wave and the slow mode field line resonance eigenfunctions at the resonance surface. The theoretically predicted wave structures across the field line resonance surface are similar to those of two-dimensional axisymmetric magnetosphere model.

In order to have a better theoretical understanding of how global fast waves couple to the shear Alfvén and slow mode field line resonances, we have to solve the global MHD equations, Eqs.(14)-(15), to obtain the global wave propagation property. By imposing a source disturbance at the plasma sheet boundary layer or the magnetopause boundary as a boundary condition, one can obtain the spatial distribution of the field line resonance power spectrum. Thus, a global MHD solution will not only provide the information of radial wave structures, but also improve our understanding of the azimuthal variation of the field line resonances. The field line resonances and the global radial wave structures can be studied numerically by employing self-consistent 3D magnetospheric equilibria in force balance with the isotropic plasma pressure *Cheng* [1995]. Numerical results of field line resonances in 3D fields have been obtained and will be presented in the future publication [*Cheng and Zaharia*, 2002].

Finally, the plasma pressure is in general anisotropic, and to understand the pressure anisotropy effects, we need to develop a proper formulation for both equilibrium and wave equations. Furthermore, implications due to kinetic effects [*Cheng*, 1991; *Cheng and Johnson*, 1999] such as finite particle gyroradii and particle trapping in nonuniform magnetic field need to be studied in the future.

Appendix:

In this Appendix we derive the final four MHD eigenmode equations, Eqs.(10)-(13). The equation of state, Eq.(5), can be written as

$$\delta P - Q_b + \frac{\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla} P}{|\boldsymbol{\nabla}\psi|^2}\xi_\psi + \frac{\mathbf{B} \times \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla} P}{B^2}\xi_s + \Gamma_s P \Delta = 0, \tag{A1}$$

where δP is the total perturbed pressure given by $\delta P = \delta p + Q_b$. The scalar product of the induction equation, Eq.(6), with $\boldsymbol{\nabla}\psi$ leads to

$$Q_\psi = \boldsymbol{\nabla} \cdot [(\boldsymbol{\xi} \times \mathbf{B}) \times \boldsymbol{\nabla}\psi] = \mathbf{B} \cdot \boldsymbol{\nabla}\xi_\psi, \tag{A2}$$

where we have made use of $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and $\mathbf{B} \cdot \boldsymbol{\nabla}\psi = 0$. The scalar product of the induction equation, Eq.(6), with $\mathbf{B} \times \boldsymbol{\nabla}\psi$ gives

$$Q_s = \boldsymbol{\nabla} \cdot \left[(\boldsymbol{\xi} \times \mathbf{B}) \times \frac{\mathbf{B} \times \boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\right] + (\boldsymbol{\xi} \times \mathbf{B}) \cdot \boldsymbol{\nabla} \times \left(\frac{\mathbf{B} \times \boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\right) = \mathbf{B} \cdot \boldsymbol{\nabla}\xi_s - S\xi_\psi \tag{A3}$$

where $S = (\mathbf{B} \times \boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|^2) \cdot \boldsymbol{\nabla} \times (\mathbf{B} \times \boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|^2)$ is the local magnetic shear, and we have made use of the relationship that $\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla} \times (\mathbf{B} \times \boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|^2) = 0$ which is due to $\mathbf{B} \cdot \boldsymbol{\nabla}\psi = 0$. The scalar product of the induction equation, Eq.(6), with $\mathbf{B}$ is given by

$$\begin{aligned} Q_b &= \boldsymbol{\nabla} \cdot [(\boldsymbol{\xi} \times \mathbf{B}) \times \mathbf{B}] - (\boldsymbol{\xi} \times \mathbf{B}) \cdot \boldsymbol{\nabla} \times \mathbf{B} \\ &= \mathbf{B} \cdot \boldsymbol{\nabla}\xi_b - B^2\Delta - \boldsymbol{\xi} \cdot \boldsymbol{\nabla}(P + B^2) \end{aligned} \tag{A4}$$

where we have made use of the equilibrium relation $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B}$ and $\mathbf{J} \times \mathbf{B} = \boldsymbol{\nabla} P$. Then, making use of the relation $\boldsymbol{\nabla}_\perp(P + B^2/2) = \boldsymbol{\kappa} B^2$ and $\boldsymbol{\xi} \cdot \boldsymbol{\kappa} = \xi_\psi \boldsymbol{\kappa} \cdot \boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|^2 + \xi_s \boldsymbol{\kappa} \cdot \mathbf{B} \times \boldsymbol{\nabla}\psi/B^2$, the parallel component of the induction equation becomes

$$\begin{aligned} Q_b = \mathbf{B} \cdot \boldsymbol{\nabla}\xi_b - (\mathbf{B} \cdot \boldsymbol{\nabla} B^2)\xi_b - B^2\Delta + \frac{\boldsymbol{\nabla}\psi \cdot (\boldsymbol{\nabla} P - 2\boldsymbol{\kappa} B^2)}{|\boldsymbol{\nabla}\psi|^2}\xi_\psi \\ + \frac{\mathbf{B} \times \boldsymbol{\nabla}\psi \cdot (\boldsymbol{\nabla} P - 2\boldsymbol{\kappa})}{B^2}\xi_s \end{aligned} \tag{A5}$$

The scalar product of the momentum equation, Eq.(4), with $\boldsymbol{\nabla}\psi$ gives

$$\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\delta p = \rho\omega^2\xi_\psi + \mathbf{J}\cdot(\delta\mathbf{B}\times\boldsymbol{\nabla}\psi) + (\mathbf{B}\times\boldsymbol{\nabla}\psi)\cdot\boldsymbol{\nabla}\times\delta\mathbf{B} \tag{A6}$$

Now

$$\mathbf{J}\cdot(\delta\mathbf{B}\times\boldsymbol{\nabla}\psi) = \frac{\mathbf{J}\cdot(\mathbf{B}\times\boldsymbol{\nabla}\psi)}{B^2}Q_b - (\mathbf{J}\cdot\mathbf{B})Q_s \tag{A7}$$

and

$$\begin{aligned}\boldsymbol{\nabla}\times\delta\mathbf{B} = \boldsymbol{\nabla}\left(\frac{Q_\psi}{|\boldsymbol{\nabla}\psi|^2}\right)\times\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}Q_s\times\left(\frac{\mathbf{B}\times\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\right) + \boldsymbol{\nabla}\times\left(\frac{\mathbf{B}\times\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\right)Q_s \\ +\boldsymbol{\nabla}\times\left(\frac{\mathbf{B}}{B^2}\right)Q_b + \boldsymbol{\nabla}Q_b\times\left(\frac{\mathbf{B}}{B^2}\right)\end{aligned} \tag{A8}$$

Then,

$$\begin{aligned}(\mathbf{B}\times\boldsymbol{\nabla}\psi)\cdot\boldsymbol{\nabla}\times\delta\mathbf{B} = |\boldsymbol{\nabla}\psi|^2\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{Q_\psi}{|\boldsymbol{\nabla}\psi|^2}\right) + |\boldsymbol{\nabla}\psi|^2 S Q_s - \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}Q_b \\ +\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}(P+B^2)\frac{Q_b}{B^2}\end{aligned} \tag{A9}$$

where we have made use of the identity $\mathbf{J}\times\mathbf{B} = \boldsymbol{\nabla}P$ and $(\mathbf{B}\times\boldsymbol{\nabla}\psi)\cdot\boldsymbol{\nabla}\times(\mathbf{B}/B^2) = \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}(P+B^2)/B^2$. Then, making use of the identity $\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}(2P+B^2) = 2\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\psi B^2$, Eq.(A6) becomes

$$\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\delta P = \rho\omega^2\xi_\psi + |\boldsymbol{\nabla}\psi|^2\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{Q_\psi}{|\boldsymbol{\nabla}\psi|^2}\right) + (|\boldsymbol{\nabla}\psi|^2 S - \mathbf{J}\cdot\mathbf{B})Q_s + 2\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\psi Q_b. \tag{A10}$$

The scalar product of the momentum equation, Eq.(4), with $\mathbf{B}\times\boldsymbol{\nabla}\psi$ leads to

$$(\mathbf{B}\times\boldsymbol{\nabla}\psi)\cdot\boldsymbol{\nabla}\delta p = \rho\omega^2|\boldsymbol{\nabla}\psi|^2\xi_s + (\mathbf{J}\cdot\mathbf{B})Q_\psi - (\mathbf{J}\cdot\boldsymbol{\nabla}\psi)Q_b - B^2\boldsymbol{\nabla}\cdot(\delta\mathbf{B}\times\boldsymbol{\nabla}\psi). \tag{A11}$$

Making use of $\boldsymbol{\nabla}\cdot(\delta\mathbf{B}\times\boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\cdot[(\mathbf{B}\times\boldsymbol{\nabla}\psi)Q_b/B^2 - \mathbf{B}Q_s]$, Eq.(A11) becomes

$$(\mathbf{B}\times\boldsymbol{\nabla}\psi)\cdot\boldsymbol{\nabla}\delta P = \rho\omega^2|\boldsymbol{\nabla}\psi|^2\xi_s + (\mathbf{J}\cdot\mathbf{B})Q_\psi + B^2\mathbf{B}\cdot\boldsymbol{\nabla}Q_s + 2\boldsymbol{\kappa}\cdot\mathbf{B}\times\boldsymbol{\nabla}\psi Q_b. \tag{A12}$$

The scalar product of the momentum equation, Eq.(4), with $\mathbf{B}$ leads to

$$\rho\omega^2\xi_b = \mathbf{B}\cdot\boldsymbol{\nabla}\delta p + \delta\mathbf{B}\cdot\boldsymbol{\nabla}P \tag{A13}$$

Making use of $\delta\mathbf{B}\cdot\boldsymbol{\nabla}P = \boldsymbol{\nabla}\cdot[(\boldsymbol{\xi}\times\mathbf{B})\times\boldsymbol{\nabla}P] = \mathbf{B}\cdot\boldsymbol{\nabla}(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P)$ and the equation of state, Eq.(5), Eq.(A13) becomes

$$\rho\omega^2\xi_b = -\Gamma_s P\mathbf{B}\cdot\boldsymbol{\nabla}\Delta \tag{A14}$$

We can also express Δ explicitly as

$$\Delta = \frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\cdot\boldsymbol{\nabla}\xi_\psi + \boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\right)\xi_\psi + \frac{\mathbf{B}\times\boldsymbol{\nabla}\psi}{B^2}\cdot\boldsymbol{\nabla}\xi_s + \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\boldsymbol{\nabla}\psi}{B^2}\right)\xi_s + \mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\xi_b}{B^2}\right). \tag{A15}$$

Next, we eliminate Q_s, Q_ψ, Q_b, and ξ_b by substituting Eqs. (A1), (A2), (A3), and (A14) into Eqs. (A5), (A10), (A12) and (A15). Explicitly, from Eq.(A12) we obtain Eq.(10); from Eq.(A5) we obtain Eq.(11); from Eq.(A10) we obtain Eq.(12); and finally making use of Eq.(11), Eq.(A15) reduces to Eq.(13). The final MHD eigenmode equations, Eqs.(10)-(13), are in terms of the dependent variables ξ_s, Δ, δP, and ξ_ψ.

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