A Kinetic-Fluid Model

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Received ______________________; accepted ______________________

submitted to *Journal of Geophysical Research*, 1998

Short title: KINETIC-FLUID MODEL

Abstract. A nonlinear kinetic-fluid model for high β plasmas with multiple ion species which can be applied to multiscale phenomena is presented. The model embeds important kinetic effects due to finite ion Larmor radius (FLR), wave-particle resonances, magnetic particle trapping, etc. in the framework of simple fluid descriptions. When further restricting to low frequency phenomena with frequencies less than the ion cyclotron frequency the kinetic-fluid model takes a simpler form in which the fluid equations of multiple ion species collapse into single-fluid density and momentum equations and a low frequency generalized Ohm's law. The kinetic effects are introduced via plasma pressure tensors for ions and electrons which are computed from particle distribution functions that are governed by the Vlasov equation or simplified plasma dynamics equations such as the gyrokinetic equation. The ion FLR effects provide a finite parallel electric field, a perpendicular velocity that modifies the $\mathbf{E} \times \mathbf{B}$ drift, and a gyroviscosity tensor, all of which are neglected in the usual one-fluid MHD description. Eigenmode equations are derived which include magnetosphere-ionosphere coupling effects for low frequency waves (*e.g.,* kinetic/inertial Alfvén waves and ballooning-mirror instabilities).

1. Introduction

A grand challenge in space plasma physics is to study low frequency multiscale phenomena in which kinetic physics involving small spatial and fast temporal scales can strongly affect the global structure and long time behavior of plasmas. Dominant global magnetospheric dynamical behavior such as magnetospheric substorms; reconnection and plasma transport processes at the magnetopause; and storm time plasma transport in the ring current region involve complex multiscale low-frequency phenomena with time scales much longer than the ion cyclotron period.

Coupling between multiple spatial and temporal scales is an inherently difficult process to model. The difficulty stems from the disparate scales which traditionally are analyzed separately. Global-scale phenomena are generally studied using the MHD framework, while microscale phenomena are best described with kinetic theories. The most fundamental kinetic description of a collisionless plasma system is to employ the Vlasov equation to obtain particle distribution functions for all particle species and compute the electric and magnetic fields from the particle density and plasma current by the Maxwell's equations. Such a treatment would involve plasma temporal and spatial scales over many orders of magnitude, and consequently, it is extremely difficult to perform analytical analysis or numerical simulations for low frequency global phenomena. To effectively model kinetic effects on low frequency MHD phenomena, we need not only eliminate high frequency (higher than ion cyclotron frequency) and small scale (smaller than ion gyroradius) phenomena from the governing dynamical and field equations, but also retain essential coupling between fast temporal and small spatial scale kinetic physics with slow temporal and large spatial scale MHD physics.

We have previously developed a kinetic-MHD model [*Cheng*, 1991] to study particle kinetic effects on MHD phenomena by taking advantage of the simplicity of the MHD model while properly including major kinetic effects of energetic particles. The kinetic-MHD model assumes that the plasma consists of two components: (1) a low energy core component which has major density fraction and (2) an energetic component which has low density, high energy and high β, and does not satisfy the MHD description. Each component can consist of more than one particle species. Instead of employing full kinetic approach for all particle species, the kinetic-MHD model treats the low energy core plasma by the MHD description and energetic particles by kinetic approach such as the gyrokinetic equation [*Frieman and Chen*, 1982; *Hahm et al.*, 1988; *Brizard*, 1989] or Vlasov equation, and the coupling between the dynamics of these two components of plasma is through the plasma pressure in the momentum equation. Because kinetic effects of the core component are neglected and the energetic particle density is low, the parallel electric field is negligible. The kinetic-MHD model optimizes both the physics content and the theoretical (analytical as well as numerical) effort, and properly accounts for the energetic particle dynamics of high-β plasma with pressure anisotropy in general magnetic field geometries. The kinetic-MHD model has been successfully employed to study the stability of ballooning-mirror instability which has improved understanding of the compressional Pc 5 waves in the ring current region [*Cheng and Qian*, 1994].

The major weakness of the previously developed kinetic-MHD model [*Cheng*, 1991] is that the kinetic effects associated with core plasma component are neglected. The core plasma kinetic effects modify the Ohm's law, introduce gyroviscosity stress tensor to the momentum equation, and modify the adiabatic pressure law.

In this paper we present a new kinetic-fluid model that includes important kinetic effects of all particle species with a minimum of modification to the one-fluid MHD equations: the mass density continuity equation, momentum equation, and a generalized Ohm's law. Kinetic effects are included in the particle pressure tensors which are obtained from the moments of the particle distribution functions. Specifically, important global effects such as background density, temperature and magnetic field gradients; magnetic field curvature; large plasma β; and pressure anisotropy are retained, while important kinetic effects, such as finite Larmor radius; resonant wave-particle interactions; and bounce resonance are added. These kinetic effects are essential when describing multiscale coupling processes; for example, we have demonstrated that wave-particle resonance and background plasma gradients are important in determining the wave structure and stability of global mirror modes in the magnetosheath [*Johnson and Cheng*, 1997a]. In the presence of background gradients, finite Larmor radius effects couple global disturbances with kinetic Alfvén waves which can strongly interact with ions because the perpendicular wavelength is the order of the ion gyroradius. Wave-particle interaction leads to anomalous particle transport and dissipation which can significantly alter the background equilibrium on the transport time scale as demonstrated at the magnetopause for kinetic Alfvén waves [*Johnson and Cheng*, 1997b]. Energetic trapped particles can strongly affect the stability of ballooning-mirror modes in the ring current region. The importance of these effects is exemplified in observed Pc 4-5 waves [*Takahashi et al.*, 1987] which generally exhibit antisymmetric mode field-aligned structure in the parallel magnetic field component contrary to the MHD theory prediction. Energetic trapped particles which are accounted for in the kinetic-fluid theory, but not in MHD, stabilize the symmetric modes and therefore explain the observations [*Cheng and Qian*, 1994; *Cheng et al.*, 1994].

In the following sections we will first motivate the need for a new kinetic-fluid model by discussing the advantage and shortcomings of the ideal MHD model and our previously developed kinetic-MHD model. Then, we present a kinetic-multifluid model for plasmas with multiple ion species that includes kinetic effects of finite ion gyroradii and wave-particle resonances for all particle species. The kinetic-multifluid mode is appropriate for studying phenomena with frequencies up to the order of ion cyclotron frequencies. If we further restrict the temporal scales to frequencies below the ion cyclotron frequency, the kinetic-multifluid mode is greatly simplified to a low frequency kinetic-fluid model that consists of one-fluid equations. Then, we demonstrate that our low frequency kinetic-fluid model properly describes all major particle kinetic effects by obtaining a dispersion relation for low frequency waves and instabilities which properly accounts for the well known kinetic Alfvén waves. Finally we summarize the paper.

2. Ideal MHD Model

The most commonly employed model used to study global plasma dynamics is the one-fluid ideal MHD model. The global dynamics of the ideal MHD plasma is governed by the mass density continuity equation, momentum equation, adiabatic pressure law and Ohm's law. We shall employ the rationalized MKSA unit system in the paper. The center-of-mass density continuity equation is given by

$$\frac{d}{dt}\rho + \rho\nabla\cdot\mathbf{V} = 0, \tag{1}$$

where $d/dt = \partial/\partial t + \mathbf{V}\cdot\nabla$ is the total time derivative, $\rho = \sum_j n_j m_j$ is the center-of-mass density with the summation over all particle species, n_j is the particle density of each particle species, m_j is the particle mass, $\mathbf{V} = \sum_j n_j m_j \mathbf{V}_j/\rho$ is the bulk fluid velocity, and $\mathbf{V}_j$ is the fluid velocity of each particle species. The momentum equation is given by

$$\rho\frac{d}{dt}\mathbf{V} = -\nabla P + \mathbf{J}\times\mathbf{B}, \tag{2}$$

where $\mathbf{J}$ is the plasma current, $\mathbf{B}$ is the magnetic field, P is the isotropic plasma pressure due to all particle species in the center-of-mass reference frame. Both the center-of-mass density continuity equation and the momentum equation are exact. The Maxwell's equations in the magnetostatic limit hold: the Faraday's law, $\partial\mathbf{B}/\partial t = -\nabla\times\mathbf{E}$, where $\mathbf{E}$ is the electric field; the Ampere's law, $\mathbf{J} = \nabla\times\mathbf{B}$; and $\nabla\cdot\mathbf{B} = 0$.

To close the above equations, the ideal MHD model prescribes the relation between the electric field and the fluid velocity by the Ohm's law,

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = 0, \tag{3}$$

so that the perpendicular fluid motion is determined by the $\mathbf{E}\times\mathbf{B}$ motion, and the parallel electric field is zero. In the resistive MHD limit the parallel electric field is proportional to the parallel current density through plasma resistivity. The dynamics of plasma pressure is described by the adiabatic pressure law which relates the plasma pressures to the plasma density, plasma convection as well as compression by

$$\frac{d}{dt}\left(P\rho^{-5/3}\right) = 0. \tag{4}$$

The major advantage of the one-fluid MHD model is that the governing equations are much simpler than the kinetic equations and properly describe the global profile and geometrical effects.

The fundamental shortcomings of the MHD model are in the Ohm's law and adiabatic pressure law: (a) the plasma is frozen in the field lines and moves across the field with a $\mathbf{E} \times \mathbf{B}$ drift velocity, (b) the plasma pressure changes adiabatically. The Ohm's law and adiabatic pressure law are appropriate only if the frequency, ω, and perturbation wavenumber, $\mathbf{k}$, of MHD phenomena satisfy the frequency and spatial scale ordering assumptions that $\omega_{ci} > \omega > \omega_b, \omega_d$ and $L > k^{-1} > \rho_i$, where ω_{ci} is the ion cyclotron frequency, ω_b is the particle bounce frequency, ω_d is the particle magnetic (∇B and curvature) drift frequency, L is the background plasma and magnetic field scale length, and ρ_i is the ion Larmor radius. These assumptions break down if the particle magnetic drift velocity is not small in comparison with the $\mathbf{E} \times \mathbf{B}$ drift velocity or if kinetic effects such as finite particle Larmor radii, wave-particle resonances and particle trapping in a nonuniform magnetic field are important. For example, because ω_d is proportional to the particle energy, energetic particles can significantly affect the MHD stability. For low frequency MHD modes with $\omega \ll \omega_d$ the energetic particle dynamics perpendicular to $\mathbf{B}$ are no longer governed by the $\mathbf{E} \times \mathbf{B}$ drift, but rather by the magnetic drift. For MHD modes with $\omega \simeq \omega_d$, the energetic particle can resonate with MHD waves and excite new type of kinetic-fluid modes. When the Alfvén speed is approximately equal to the particle velocity, the MHD shear Alfvén waves can be driven unstable by free energy of particle pressure gradient via wave-particle bounce resonance processes if the wave frequency is less than the particle diamagnetic drift frequency. In addition, because ion motion across field lines is different from the electron $\mathbf{E} \times \mathbf{B}$ motion, significant charge separation can result if the perpendicular wavelength is on the order of ion gyroradii. The resultant charge separation not only allows the kinetic Alfvén waves to travel across the field lines but also gives rise to a parallel electric field. Because the particle gyroradius is proportional to the particle velocity, energetic ions resonating with the kinetic Alfvén waves can decouple from the magnetic fields and lead to significant diffusion across the magnetic field [*Johnson and Cheng*, 1997b]. In addition, the electric field can effectively accelerate or decelerate resonant ions.

In order to study kinetic effects on MHD phenomena, we have previously developed a hybrid kinetic-MHD model [*Cheng*, 1991] for describing low-frequency phenomena in high β ($\beta \simeq O(1)$) anisotropic plasmas for general magnetic field geometries. However, while the kinetic-MHD model retains full kinetic effects of energetic particles, it neglects kinetic effects of the core plasma component by assuming that the core plasma component is cold. The kinetic-MHD model is applicable to magnetized collisionless plasma systems where the energetic particle density is small in comparison with the cold core plasma component so that parallel electric field effects are negligibly small. However, for problems that require core plasma kinetic effects (such as kinetic Alfvén waves) it is important to modify the kinetic-MHD model so that it properly addresses the relevant core plasma kinetic effects. In the following sections we shall first present a kinetic-multifluid model that is applicable to ion cyclotron wave phenomena for multiple ion species. Then, the kinetic-multifluid model is reduced to a low frequency kinetic-fluid model that is applicable to MHD phenomena with frequencies below ion cyclotron frequencies. The low frequency kinetic-fluid model preserves the one-fluid framework, but retains the kinetic effects of multiple ion species.

3. Kinetic-Multifluid Model

We consider a plasma with multiple ion species which is common in space environment and laboratory fusion devices. First, we shall present a general kinetic-multifluid model that eliminates temporal scales with frequency higher than the ion cyclotron frequency. Because the magnetic field can vary by orders of magnitude in the magnetosphere, it is often necessary for a global model to be valid at the ion cyclotron frequency of each ion species as well as in the MHD regime. For example, consider ion cyclotron waves generated in the central plasma sheet by ion temperature anisotropy which frequently are associated with ground based observations. The magnetic field varies several orders of magnitude between the central plasma sheet and the ionosphere, and in order to describe these waves with a global model, it must be sufficiently general to include ion cyclotron resonances. The kinetic-multifluid model consists of a set of fluid equations which are closed by the solutions of kinetic equations for each species. The fluid equations consist of continuity equations and momentum equations for each particle species and will be closed provided the particle pressure tensors are obtained from solutions of the Vlasov equation. The particle kinetic physics is incorporated through plasma pressure tensors. Because of small electron mass, the electron momentum equation can be replaced by the Ohm's law and the electron density determined by the charge quasi-neutrality condition. In this form, the model is sufficiently general to describe global structures with frequency up to the ion gyrofrequency. Hence, it is appropriate for studying ion-cyclotron waves and other low frequency phenomena.

For each particle species the density continuity and momentum equations are given by

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{V}_j) = 0 \tag{5}$$

and

$$n_j m_j \left(\frac{\partial}{\partial t} + \mathbf{V}_j \cdot \nabla \right) \mathbf{V}_j = n_j q_j (\mathbf{E} + \mathbf{V}_j \times \mathbf{B}) - \nabla \cdot \mathbf{P}_j \tag{6}$$

where q_j is the particle charge, m_j is the particle mass, the pressure tensor for each species is defined in its moving frame by

$$\mathbf{P}_j = m_j \int d^3v(\mathbf{v}-\mathbf{V}_j)(\mathbf{v}-\mathbf{V}_j)f_j, \tag{7}$$

and f_j is the particle distribution function for species j.

A generalized Ohm's law that relates the current to the electric field is obtained by multiplying Eq. (6) by q_j/m_j and by summing over all particle species (conveniently ignoring corrections which are $\mathcal{O}(m_e/m_i)$) and is given by

$$\frac{m_e}{n_e e^2}\left[\frac{\partial \mathbf{J}}{\partial t} + \nabla\cdot(\mathbf{JV}+\mathbf{VJ})\right] + \eta\mathbf{J} =$$

$$\mathbf{E} + \frac{1}{n_e e}\left(\sum_i n_i q_i \mathbf{V}_i - \mathbf{J}\right)\times\mathbf{B} + \frac{1}{n_e e}\nabla\cdot\left(\mathbf{P}_e^{cm} - \sum_i \frac{q_i m_e}{e m_i}\mathbf{P}_i^{cm}\right), \tag{8}$$

where the summation over i is for all ion species, $e = |q_e|$, the center-of-mass pressure tensor for each species is defined relative to the bulk flow velocity $\mathbf{V}$ by

$$\mathbf{P}_j^{cm} = m_j \int d^3v(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})f_j, \tag{9}$$

which is related to $\mathbf{P}_j$ by $\mathbf{P}_j^{cm} = \mathbf{P}_j + n_j m_j(\mathbf{V}-\mathbf{V}_j)(\mathbf{V}-\mathbf{V}_j)$, and the plasma resistivity (η) contribution is conveniently added to model collisional effects. In deriving the generalized Ohm's law we have made use of the charge quasi-neutrality condition, $n_e e = \sum_i n_i q_i$, which relates the electron density to the ion densities. The charge quasi-neutrality condition is equivalent to $\nabla\cdot\mathbf{J} = 0$. Note that the generalized Ohm's law essentially replaces the electron momentum equation. For a single ion species, the generalized Ohm's law reduces to the well known one-fluid form with $\mathbf{V}_i = \mathbf{V}(1+\mathcal{O}(m_e/m_i))$. However, for multiple ion species, the generalized Ohm's law relates the electric field to the density and fluid velocity of all ion species. This complication arises because the diamagnetic and polarization drift velocities are different for each ion species. The one-fluid mass density and momentum can thus not be used to couple with the generalized Ohm's law.

Therefore, in our kinetic-multifluid model, we shall adopt the density and momentum equations for each ion species. The electron momentum equation is replaced by the generalized Ohm's law. The electric and magnetic fields are related to the plasma current and are obtained from Maxwell's equations in the magnetostatic limit: the Faraday's law, $\partial\mathbf{B}/\partial t = -\nabla\times\mathbf{E}$, where $\mathbf{E}$ is the electric field; the Ampere's law, $\mathbf{J} = \nabla\times\mathbf{B}$; and $\nabla\cdot\mathbf{B} = 0$. To close the above multifluid equations, we need to specify the pressure tensor for each particle species. Instead of prescribing a fluid description for electron and ion pressure tensors, we shall compute the pressure tensor from the particle distribution functions. For collisionless plasmas the most fundamental description of particle dynamics is by the Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0. \tag{10}$$

The particle dynamics are correct even for systems with equilibrium profile scale length on the order of ion gyroradii. In many plasma systems the electron dynamics can be adequately simplified by the guiding center particle description with the fast electron gyro-motion ignored. If possible, the electron pressure tensor will be expressed in terms of an appropriate equation of state or in an analytic form to minimize the computational requirements so that computational resources can be devoted to wave-ion interactions and complicated dynamics which occur when the gyroradius is on the order of the background field gradients.

This kinetic-multifluid model has eliminated high frequency wave phenomena with frequency the order of the electron cyclotron frequency. It is appropriate for studying Alfvén , ion cyclotron, and MHD waves. The difference between the kinetic-multifluid model and full kinetic Vlasov model is that the ion densities and velocities are governed by the continuity and momentum equations instead of computing from the particle distribution functions, so that high frequency phenomena involving electron dynamics are basically eliminated. This kinetic-multifluid model is different from the previously developed kinetic-MHD model [*Cheng*, 1991] in that (1) the dynamics of multiple ion species are properly treated, (2) a generalized Ohm's law is adopted instead of the simplified Ohm's law, (3) the particle pressure tensor is computed from appropriate particle distribution functions for all particle species, (4) the Vlasov equation, instead of the low frequency gyrokinetic equation, is employed for particle dynamics if $\rho_i/L \sim O(1)$ and/or $\omega/\omega_{ci} \sim O(1)$, (5) effects of equilibrium flow are included.

This kinetic-multifluid model is appropriate for nonlinear simulations of multiscale phenomena for general magnetic field geometry. However, it still contains the fast ion cyclotron time scale and is therefore difficult for performing simulations of phenomena which have a longer, MHD time scale. Moreover, it is also difficult for analytical analysis because analytical solutions of the Vlasov equation for systems with complex magnetic field geometries are, in general, difficult to obtain. To improve the kinetic-multifluid model for longer time scale simulations of global MHD behavior, we will further reduce the multifluid equations by employing one-fluid equations and a low frequency Ohm's law as well as simplify particle dynamics by employing the gyrokinetic equations.

4. Low Frequency Kinetic-Fluid Model

A major purpose of this paper is to further restrict the temporal scales to frequency below the ion cyclotron frequency and to obtain a low frequency kinetic-fluid model with one-fluid equations. The model eliminates ion cyclotron waves, but is appropriate for studying kinetic effects on MHD phenomena. It is in this regime that the multi-ion fluid model reduces to an one-fluid model. The basic reason for this is that to lowest order in ω/ω_{ci}, all ion velocities are dominated by the $\mathbf{E} \times \mathbf{B}$ and diamagnetic drift motion.

Hence an one-fluid description of a low frequency generalized Ohm's law can be obtained by keeping the information on the ion diamagnetic drift motion which depends on ion mass, charge and pressure gradient. The information that is lost in the one-fluid model is the individual ion cyclotron waves.

To obtain an one-fluid description of the generalized Ohm's law, we first rewrite the ion momentum equation, Eq. (6), as

$$\frac{\partial}{\partial t}(n_i m_i \mathbf{V}_i) = n_i q_i \left(\mathbf{E} + \mathbf{V}_i \times \mathbf{B}\right) - \nabla \cdot \mathbf{P}_i^0, \tag{11}$$

where

$$\mathbf{P}_i^0 = m_i \int d^3 v \mathbf{v}\mathbf{v} f_i. \tag{12}$$

The perpendicular ion velocity, $\mathbf{V}_{i\perp} = \mathbf{B} \times (\mathbf{V} \times \mathbf{B})/B^2$, can be written as

$$\mathbf{V}_{i\perp} = \mathbf{V}_E + \mathbf{V}_{di} + \mathbf{V}_{pi}, \tag{13}$$

where

$$\mathbf{V}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \tag{14}$$

$$\mathbf{V}_{di} = \frac{\mathbf{B} \times \nabla \cdot \mathbf{P}_i^0}{n_i q_i B^2}, \tag{15}$$

and

$$\mathbf{V}_{pi} = \frac{1}{n_i q_i} \frac{\mathbf{B}}{B^2} \times \frac{\partial (n_i m_i \mathbf{V}_i)}{\partial t} \tag{16}$$

are the $\mathbf{E} \times \mathbf{B}$, diamagnetic and polarization drift velocities, respectively, for each species. Multiplying $n_i m_i$ to the ion perpendicular velocity and summing over all ion species we obtain

$$\rho \mathbf{V}_\perp = \rho \mathbf{V}_E + \sum_i n_i m_i (\mathbf{V}_{di} + \mathbf{V}_{pi}). \tag{17}$$

Finally, multiplying $n_i q_i$ to the ion perpendicular velocity and summing over all ion species we have

$$\sum_i n_i q_i \mathbf{V}_{i\perp} = n_e e \mathbf{V}_\perp + \sum_i n_i q_i (\mathbf{V}_{di} + \mathbf{V}_{pi}) - n_e e \sum_i \frac{n_i m_i}{\rho} (\mathbf{V}_{di} + \mathbf{V}_{pi}). \tag{18}$$

We note that the difference

$$\left(\frac{n_i q_i}{n_e e} - \frac{n_i m_i}{\rho} \right) \mathbf{V}_{pi} << \mathbf{V}_{pi} \sim \mathcal{O}(\omega/\omega_{ci})(\mathbf{V}_E + \mathbf{V}_{di}), \tag{19}$$

which can be neglected in Eq. (18). For a single ion species the difference is zero. Then we have

$$\sum_i \frac{n_i q_i}{n_e e} \mathbf{V}_{i\perp} \simeq \mathbf{V}_\perp + \sum_i \left(\frac{m_i}{\rho q_i} - \frac{1}{n_e e} \right) \nabla \cdot \mathbf{P}_i^0 \times \frac{\mathbf{B}}{B^2}, \tag{20}$$

and from Eq. (8) a low frequency generalized Ohm's law can be obtained and is given by

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{1}{n_e e}\left[\mathbf{J} \times \mathbf{B} - \nabla \cdot \left(\mathbf{P}_e^{cm} - \sum_i \frac{q_i m_e}{e m_i}\mathbf{P}_i^{cm}\right)\right]$$
$$+ \sum_i \left(\frac{m_i}{\rho q_i} - \frac{1}{n_e e}\right)\frac{\mathbf{B}}{B} \times \left(\nabla \cdot \mathbf{P}_i^0 \times \frac{\mathbf{B}}{B}\right) + \frac{m_e}{n_e e^2}\left[\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{J}\mathbf{V} + \mathbf{V}\mathbf{J})\right] + \eta \mathbf{J}, \qquad (21)$$

which eliminates the ion density and fluid velocities for all ion species in favor of the center-of-mass density, bulk fluid velocity and the ion pressure tensors. Note that for a single ion species case, the $\mathbf{P}_i^0$ term is on the order of m_e/m_i and can be ignored.

In addition to the low frequency generalized Ohm's law we need to obtain one-fluid mass density and momentum equations. Multiplying Eq. (5) by m_j and summing the equations over all particle species, the equation for mass density transport in the one-fluid form is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \qquad (22)$$

where the summation is over all particle species including both electron and ions. Summing Eq. (6) over all particle species and assuming the charge quasi-neutrality condition gives the equation for momentum transport in the one-fluid form

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right)\mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla \cdot \sum_j \mathbf{P}_j^{cm}. \qquad (23)$$

Thus, our low frequency kinetic-fluid model consists of the set of one-fluid equations, Eqs. (21–23), which is closed by coupling with kinetic descriptions for particle distribution such as the gyrokinetic equation from which pressure tensors for all particle species can be determined. The electron density can be determined either via ion densities by using the charge quasi-neutrality condition or from the electron distribution function. Consistent with the low frequency generalized Ohm's law, the electron density and pressure tensors for all particle species must be valid to $\mathcal{O}(\omega/\omega_{ci})$. Note that there are only three one-fluid equations in our one-fluid model which is much simpler than the multifluid model when there are more than one ion species. Moreover, we have the flexibility of dealing with particle pressure tensors for hot ion species by kinetic analysis and for thermal or cold ion species by fluid descriptions such as adiabatic pressure law or truncated pressure equations derived from the kinetic equations.

5. Gyrokinetic Formulation

If the particle magnetic moment is an adiabatic invariant (which is satisfied if the gyroradius is smaller than the equilibrium scale length perpendicular to the equilibrium magnetic field, $L_\perp$), the gyrokinetic formulation, instead of the full Vlasov equation, can be employed to describe the particle dynamics in our kinetic-fluid model for low-frequency phenomena. The gyrokinetic formulation is also limited by the assumptions that $k_{||}L_{||} > 1$ and $k_\perp L_\perp > 1$, where $L_{||}$ is the parallel background equilibrium scale length. These conditions can be satisfied for most space and laboratory plasma conditions.

To simplify the analytical treatment we shall consider collisionless plasmas without equilibrium flow. The case with equilibrium flow will be presented in the future. In a general three-dimensional equilibrium with nested flux surfaces, the magnetic field can be expressed as $\mathbf{B} = \nabla\psi \times \nabla\alpha$, where ψ is chosen as the magnetic flux function. Both ψ and α are three-dimensional functions and are constant along magnetic field lines. The lines where surfaces of constant ψ and α intersect represent magnetic field lines. For a collisionless plasma the particle energy per particle mass ($\mathcal{E} = v^2/2$) and the adiabatic invariants, magnetic moment per particle mass ($\mu = v_\perp^2/2B$) and the longitudinal invariant ($J_{||} = \int \mathbf{v}\cdot d\mathbf{x}$), are constant during the drift motions, where $v_{||}$ and $v_\perp$ are the components of the velocity parallel and perpendicular to $\mathbf{B}$ in the guiding center coordinate, respectively. The guiding-center equilibrium particle distribution function must have the form $F = F(\mathcal{E}, \mu, J_{||})$. In general, $J_{||} = J_{||}(\mathcal{E}, \mu, \psi, \alpha)$ and $F = F(\mathcal{E}, \mu, \psi, \alpha)$. If all particles on each field line share the same drift surface, where ψ labels the drift surface, then $J_{||} = J_{||}(\mathcal{E}, \mu, \psi)$ and $F = F(\mathcal{E}, \mu, \psi)$. The guiding-center particle distributions $F = F(\mathcal{E}, \mu, \psi)$ can be either prescribed by an analytical form or obtained from the satellite measurements of the particle flux. The equilibrium parallel and perpendicular pressures for each particle species are given by

$$\begin{pmatrix} P_{||} \\ P_\perp \end{pmatrix} = \sum_{\sigma_{||}} 2\pi m \int_0^\infty d\mathcal{E} \int_0^{\mathcal{E}/B} d\mu \frac{BF}{|v_{||}|} \begin{pmatrix} v_{||}^2 \\ \mu B \end{pmatrix}, \tag{24}$$

where the summation is over the particle species j and $\sigma_{||}$ which represents the direction of particle velocity parallel to $\mathbf{B}$, and m_j is the particle mass. The parallel velocity $v_{||}$ has the form $v_{||} = \sigma_{||}\sqrt{2(\mathcal{E} - \mu B)}$. By inspection, $P_\perp$ and $P_{||}$ are functions of ψ and B only.

We consider perturbations with $k_\perp > k_{||}$ and assume a WKB eikonal representation for perturbed quantities, *i.e.*, $\delta f(\mathbf{x}, \mathbf{v}, t) = \delta f(s, \mathbf{k}_\perp, \mathbf{v}, t) \exp\left(i \int d\mathbf{x}_\perp \cdot \mathbf{k}_\perp\right)$, where s is the distance along the equilibrium magnetic field. Following the gyrokinetic formulation the perturbed particle distribution function can be expressed as

$$\delta f = -\frac{q}{m}\frac{\partial F}{\partial \mathcal{E}}\Phi + \frac{q}{mB}\frac{\partial F}{\partial \mu}(\Phi - v_{||}A_{||}) + \sum_l \left\{ g_l - \frac{q}{mB}\frac{\partial F}{\partial \mu}\langle \delta L_l \rangle \right\} e^{iL_l}, \tag{25}$$

where the summation over l is over all integers,

$$\langle \delta L_l \rangle = \left[(\Phi - v_{||}A_{||})J_l(\lambda) - \frac{v_\perp \delta B_{||}}{k_\perp} J_l'(\lambda) \right], \tag{26}$$

Φ is the perturbed electrostatic potential, $A_{\|}$, and $\delta B_{\|}$ are the vector potential and perturbed magnetic field parallel to the equilibrium magnetic field $\mathbf{B}$, respectively, J_l is the Bessel function of order l with argument $\lambda = k_\perp v_\perp/\omega_c$, $L_l = \mathbf{k}_\perp \times \mathbf{v}_\perp \cdot \hat{\mathbf{b}}/\omega_c - l\theta$, θ is the particle gyrophase angle between $\mathbf{k}_\perp$ and $\mathbf{v}_\perp$, $\hat{\mathbf{b}} = \mathbf{B}/B$, and g_l, is the l-th nonadiabatic contribution to the perturbed particle distribution function. For $l = 0$, g_l represents the main low frequency ($\omega < \omega_c$) contribution and is governed by the nonlinear gyrokinetic equation for high β, anisotropic pressure plasmas in a general magnetic field geometry [*Frieman and Chen*, 1982; *Hahm et al.*, 1988; *Brizard*, 1989] given by

$$\left[\frac{\partial}{\partial t} + (\mathbf{v}_{\|} + \mathbf{v}_d)\cdot\nabla\right] g_0 = -\left[\frac{q}{m}\frac{\partial F}{\partial \mathcal{E}}\frac{\partial}{\partial t} - \frac{\mathbf{B}\times\nabla(F+g_0)}{B^2}\cdot\nabla\right]\langle\delta L_0\rangle, \tag{27}$$

where $\mathbf{v}_d = (\mathbf{B}/B\omega_c)\times[\nabla(\mu B) + \boldsymbol{\kappa} v_{\|}^2]$ is the particle magnetic drift velocity in the equilibrium magnetic field, and $\boldsymbol{\kappa} = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$ is the equilibrium magnetic field curvature. Equation (27) shows that the nonlinearity arises from a gyrophase-averaged effective $\mathbf{E}_{\text{eff}}\times\mathbf{B}\cdot\nabla$ coupling, where $\mathbf{E}_{\text{eff}} = -\nabla(\Phi - \mathbf{v}\cdot\mathbf{A})$. Note that the nonlinear polarization drift is contained within the finite Larmor radius corrections in J_0 and J_1. The electric field is related to the electrostatic potential and the vector potential $\mathbf{A}$ by $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$, where $\delta\mathbf{B} = \nabla\times\mathbf{A}$.

For $l \neq 0$, g_l represents the (ω/ω_c) correction to the particle distribution function, which contributes to the leading order finite ion Larmor radius effects in the gyroviscosity and the fluid velocity. Because we are interested in low frequency phenomena, the nonlinear effects in g_l can be ignored. Then, g_l is governed by the linearized general frequency gyrokinetic equation [*Lee et al.*, 1983; *Chen and Tsai*, 1983; *Berk et al.*, 1983]

$$\left(\omega + iv_{\|}\frac{\partial}{\partial s} - \omega_d - l\omega_c\right) g_l = \frac{qF}{T}(\tilde{\omega}_l - \omega_*^T)\langle\delta L_l\rangle, \tag{28}$$

where $\tilde{\omega}_l = -T/m[\omega\partial/\partial\mathcal{E} + (l\omega_c/B)\partial/\partial\mu]\ln F$, $\omega_*^T = (T/qB)\mathbf{k}_\perp\cdot\hat{\mathbf{b}}\times\nabla\ln F$, $\omega_d = \omega_b m\mu B/T + \omega_\kappa m v_{\|}^2/T$, $\omega_b = (T/qB^2)\mathbf{k}_\perp\cdot\hat{\mathbf{b}}\times\nabla B$, and $\omega_\kappa = (T/qB)\mathbf{k}_\perp\cdot\hat{\mathbf{b}}\times\boldsymbol{\kappa}$. We emphasize that the important effects of background gradients, magnetic curvature, and wave-particle resonances are all retained in the gyrokinetic equations through the various drift effects. Finite Larmor radius effects are included through the Bessel functions, and the theoretical framework is valid even for $k_\perp\rho_i \sim O(1)$. For low frequency phenomena with $\omega/\omega_c << 1$, the contribution to the pressure tensor from g_l with $l \neq 0$ can be obtained to the leading order in ω/ω_c. With the frequency ordering: $\omega_c > \omega, k_{\|}v_{\|}, \omega_d$, the solution of g_l is given by

$$g_l - \frac{q}{mB}\frac{\partial F}{\partial \mu}\langle\delta L_l\rangle = \left[\frac{q}{mB}\frac{\partial F}{\partial\mu}\frac{\hat{\omega}}{l\omega_c} - \frac{qF}{T}\frac{\tilde{\omega}_0 - \omega_*^T}{l\omega_c} + O\left(\frac{\hat{\omega}}{\omega_c}\right)^2\right]\langle\delta L_l\rangle, \tag{29}$$

where $\hat{\omega} = (\omega - k_{\|}v_{\|} - \omega_d)$.

The pressure tensor $\mathbf{P}$ of each particle species in the guiding center coordinate can be expressed in terms of the diagonal elements and off-diagonal elements:

$$\mathbf{P} = P_\perp(\mathbf{I} - \mathbf{bb}) + P_{||}\mathbf{bb} + \mathbf{\Pi}, \tag{30}$$

where $\mathbf{I}$ is the unit dyadic, , $\mathbf{I} = \hat{\mathbf{k}}_\perp\hat{\mathbf{k}}_\perp + \hat{\mathbf{b}} \times \hat{\mathbf{k}}_\perp\hat{\mathbf{b}} \times \hat{\mathbf{k}}_\perp + \hat{\mathbf{b}}\hat{\mathbf{b}}$, $\mathbf{\Pi}$ is the off-diagonal tensor element and $\mathbf{b} = \mathbf{B}/B$ is the unit vector along the magnetic field. Usually the diagonal (parallel and perpendicular) elements is more important than the off-diagonal elements in the momentum equation. The diagonal pressure tensor elements are computed from the particle distribution function f by

$$\begin{aligned} P_{||} &= m\int d^3v v_{||}^2 f, \\ P_\perp &= \frac{1}{2}m\int d^3v v_\perp^2 f, \end{aligned} \tag{31}$$

where $v_{||}$ is the particle velocity along the $\mathbf{b}$ direction, and $v_\perp$ is the particle velocity perpendicular to the $\mathbf{b}$ direction in the guiding center coordinate. Employing Eq.(25) and carrying out the gyrophase angle average and the summation over l, the perturbed diagonal pressure elements are given by

$$\delta P_\perp = \int d^3v \frac{mv_\perp^2}{2}\left(g_0 J_0 + \frac{q}{m}\frac{\partial F}{\partial \mathcal{E}}\Phi + \frac{q}{mB}\frac{\partial F}{\partial \mu}\left[(\Phi - v_{||}A_{||})(1 - J_0^2) - \frac{v_\perp \delta B_{||}}{k_\perp}J_0 J_1\right]\right), \tag{32}$$

and

$$\delta P_{||} = \int d^3v m v_{||}^2\left(g_0 J_0 + \frac{q}{m}\frac{\partial F}{\partial \mathcal{E}}\Phi + \frac{q}{mB}\frac{\partial F}{\partial \mu}\left[(\Phi - v_{||}A_{||})(1 - J_0^2) - \frac{v_\perp \delta B_{||}}{k_\perp}J_0 J_1\right]\right). \tag{33}$$

The off-diagonal pressure tensor element is contributed by the perturbed particle distribution and is given by

$$\mathbf{\Pi} = \widetilde{\mathbf{\Pi}} + \frac{1}{2}m\int d^3v(\mathbf{v}_{||}\mathbf{v}_\perp + \mathbf{v}_\perp\mathbf{v}_{||})\delta f \tag{34}$$

where the gyroviscosity tensor is defined as

$$\widetilde{\mathbf{\Pi}} = \frac{1}{2}m\int d^3v\left[\mathbf{v}_\perp\mathbf{v}_\perp - \frac{v_\perp^2}{2}(\mathbf{I} - \mathbf{bb})\right]\delta f \tag{35}$$

If $k_\perp \gg k_{||}$, the contribution from $\widetilde{\mathbf{\Pi}}$ need only be considered in the off-diagonal pressure tensor element. Furthermore, because $\widetilde{\mathbf{\Pi}}$ is on the order of ω/ω_c in comparison with the diagonal pressure tensor elements, we shall retain only linearized perturbed particle distribution function in obtaining $\widetilde{\mathbf{\Pi}}$. To evaluate the off-diagonal terms in the pressure tensor, first we obtain

$$\mathbf{v}_\perp\mathbf{v}_\perp - \frac{v_\perp^2}{2}(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) = \frac{v_\perp^2}{2}\left[(\cos 2\theta\mathbf{C} + \sin 2\theta\mathbf{S})\right], \tag{36}$$

where θ is the particle gyrophase angle between $\hat{\mathbf{b}}$ and $\hat{\mathbf{k}}_\perp$, $\mathbf{C} = 2\hat{\mathbf{k}}_\perp\hat{\mathbf{k}}_\perp - (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}})$, $\mathbf{S} = \hat{\mathbf{k}}_\perp\hat{\mathbf{b}} \times \hat{\mathbf{k}}_\perp + \hat{\mathbf{b}} \times \hat{\mathbf{k}}_\perp\hat{\mathbf{k}}_\perp$, and $\hat{\mathbf{b}}$ and $\hat{\mathbf{k}}_\perp$ are the unit vector along the magnetic field and perpendicular wave vector directions, respectively. Then,

$$\nabla \cdot \widetilde{\mathbf{\Pi}} = \hat{\mathbf{b}} \times (\nabla \delta P_c \times \hat{\mathbf{b}}) + \hat{\mathbf{b}} \times \nabla_\perp \delta P_s, \tag{37}$$

$$\hat{\mathbf{b}} \cdot \nabla \times \nabla \cdot \widetilde{\mathbf{\Pi}} = \nabla_\perp^2 \delta P_s, \tag{38}$$

and

$$\hat{\mathbf{b}} \cdot \nabla \times (\hat{\mathbf{b}} \times \nabla \cdot \widetilde{\mathbf{\Pi}}) = \nabla_\perp^2 \delta P_c, \tag{39}$$

where

$$\delta P_c = \int d^3v \frac{mv_\perp^2}{2} \cos 2\theta \delta f = \int d^3v \frac{mv_\perp^2}{2} \sum_l g_l(J_l(\lambda) + 2J_l''(\lambda)), \tag{40}$$

and

$$\delta P_s = \int d^3v \frac{mv_\perp^2}{2} \sin 2\theta \delta f = \int d^3v \frac{mv_\perp^2}{2} \sum_l 2ilg_l \left(\frac{J_l(\lambda)}{\lambda^2} - \frac{J_l'(\lambda)}{\lambda} \right). \tag{41}$$

In deriving the above equations, the following identities and definitions are used: $e^{i\lambda \sin\theta} = \sum_l J_l(\lambda) e^{il\theta}$, $\langle e^{iL_l} \cos n\theta \rangle = \frac{1}{2}[J_{l-n}(\lambda) + J_{l+n}(\lambda)]$, $\langle A(\theta) \rangle = 1/2\pi \int_0^{2\pi} d\theta A(\theta)$ is the gyrophase average of $A(\theta)$, n is an integer, $\langle e^{iL_l} \sin n\theta \rangle = \frac{1}{2i}[J_{l-n}(\lambda) - J_{l+n}(\lambda)]$, $\sum_l J_l^2 = 1$, $\sum_l J_l'^2 = 1/2$, $\sum_l J_l J_l'' = -1/2$, and $\sum_l J_l J_l' = \sum_l J_l' J_l'' = 0$.

Making use of the solution of g_l given in Eq. (29), the summation over l in δP_c and δP_s can be carried out, and to leading order in (ω/ω_c) we obtain

$$\delta P_c = \delta P_{c1} + \delta P_{c2}, \tag{42}$$

where

$$\delta P_{c1} = \int d^3v \frac{mv_\perp^2}{2} g_0 (J_0 - 2J_1') \tag{43}$$

is due to g_0, $J_1' = J_0 - J_1/\lambda$, and

$$\delta P_{c2} = \int d^3v \frac{mv_\perp^2}{2} \frac{q}{mB} \frac{\partial F}{\partial \mu} \left[(\Phi - v_\parallel A_\parallel)(2J_0 J_1' - J_0^2) - \frac{v_\perp \delta B_\parallel}{k_\perp}(J_0 J_1 - 2J_1 J_1') \right] \tag{44}$$

has no contribution due to g_0. Note that δP_c contain the finite gyroradius contribution and those corrections come in at the same order as $\delta P_\perp$. There is no contribution due to g_0 in δP_s which is given by

$$\delta P_s = \int d^3v \frac{imv_\perp^2}{\lambda^2} \left[\frac{qF}{T} \frac{\tilde{\omega}_0 - \omega_*^T}{\omega_c} - \frac{q}{mB} \frac{\partial F}{\partial \mu} \frac{\hat{\omega}}{\omega_c} \right] \tag{45}$$
$$\times \left[(\Phi - v_\parallel A_\parallel)(\lambda J_0 J_1 + J_0^2 - 1) - \frac{v_\perp \delta B_\parallel}{2k_\perp}(\lambda(1 - 2J_1^2) - 2J_0 J_1) \right].$$

Note that δP_s is smaller than δP_c by ω/ω_c. Because of the small electron mass the gyroviscosity tensor is mainly from ion contribution. We also note the gyroviscosity tensor elements are smaller than the diagonal pressure elements by $(k_\perp \rho_i)^2$ in the small ion gyroradius limit. From the particle equilibrium guiding center distribution functions δP_{c2} and δP_s can be straightforwardly computed.

In order to completely determine the plasma pressure tensor, we need to obtain g_0 by integrating the low frequency gyrokinetic equation, Eq.(27), which is a nonlinear equation and in general can be solved by simulation techniques. However, for the purpose of studying linear stability, progress can be made by integrating the gyrokinetic equation, which has been obtained previously [*Cheng*, 1991]. Employing the gyrokinetic formulation in the kinetic-fluid model is in principle much simpler than that using the Vlasov formulation because particle and wave dynamics involving time scales faster than ion cyclotron period are eliminated so that large numerical time steps can be used in the numerical simulation. It is also important to point out that employing the gyrokinetic formulation in the kinetic-fluid model primarily requires the frequency ordering $\omega/\omega_c << 1$, which is fully justified for low frequency MHD phenomena and associated plasma transport.

Finally we summarize this section with the expression for each particle species:

$$\nabla \cdot \mathbf{P} = \nabla P_\perp + \mathbf{BB} \cdot \nabla \left(\frac{P_\| - P_\perp}{B^2} \right) + \left(\frac{P_\| - P_\perp}{B^2} \right) \mathbf{B} \cdot \nabla \mathbf{B}$$
$$+ \frac{\mathbf{B}}{B} \times \left(\nabla \delta P_c \times \frac{\mathbf{B}}{B} \right) + \frac{\mathbf{B}}{B} \times \nabla \delta P_s. \quad (46)$$

The divergence of the perturbed pressure tensor is approximately given by

$$\nabla \cdot \delta \mathbf{P} \approx \nabla_\perp \delta \bar{P}_\perp + \mathbf{b} \times \nabla_\perp \delta P_s + \mathbf{bb} \cdot \nabla \delta P_\|, \quad (47)$$

where $\delta \bar{P}_\perp = \delta P_\perp + \delta P_c$. Note that the gyroviscosity contribution, δP_c and δP_s, are mainly due to ions because the electron gyroviscosity is much smaller and can be neglected due to small mass.

6. Low Frequency Kinetic-Fluid Eigenmode Equations

In order to show that our low frequency kinetic-fluid model provides proper kinetic and global effects, we will derive the linear eigenmode equations for describing low frequency phenomena with $\omega < \omega_{ci}$. In particular, the kinetic-fluid eigenmode equations should properly take into account the effects of FLR for each ion species as well as electron inertia for kinetic Alfvén waves. To derive the kinetic-fluid eigenmode equations, we will first decompose the nonlinear one-fluid momentum equation into three scalar components by following the paper by *Cheng* [1991]. First we rewrite the one-fluid momentum equation as

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla(P_\perp + B^2/2) + \sigma\mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{B}(\mathbf{B}\cdot\nabla\sigma) - \nabla\cdot\mathbf{\Pi}, \tag{48}$$

where $\sigma = 1 + (P_\perp - P_\|)/B^2$. Note that $\nabla(P_\perp + B^2/2)$ is the dominant term for perturbations with $k_\| < k_\perp$ and $k_\perp\rho_i < 1$. The one-fluid momentum equation can be decomposed into three components. First, the component parallel to $\mathbf{B}$ is given by

$$\mathbf{B}\cdot\left(\rho\frac{d\mathbf{V}}{dt} + \nabla\cdot\mathbf{\Pi}\right) = -\mathbf{B}\cdot\nabla P_\| + \frac{P_\| - P_\perp}{B}\mathbf{B}\cdot\nabla B. \tag{49}$$

To eliminate the $\nabla(P_\perp + B^2/2)$ term we apply $\mathbf{B}\cdot\nabla\times$ to the one-fluid momentum equation, Eq. (48). Also making use of the quasi-neutrality condition, $\nabla\cdot\mathbf{J} = 0$, we obtain the parallel current (or vorticity) equation

$$\begin{aligned} &B^2\mathbf{B}\cdot\nabla\left[\frac{\sigma\mathbf{J}\cdot\mathbf{B}}{B^2}\right] - \mathbf{B}\times\boldsymbol{\kappa}\cdot\nabla(\sigma B^2) \\ &+\nabla\cdot\left(\rho\mathbf{B}\times\frac{d\mathbf{V}}{dt}\right) - \mathbf{B}\cdot\nabla\times(\nabla\cdot\mathbf{\Pi}) - \rho\mathbf{J}\cdot\frac{d\mathbf{V}}{dt} = 0, \end{aligned} \tag{50}$$

where $\boldsymbol{\kappa} = (\mathbf{B}/B)\cdot\nabla(\mathbf{B}/B)$. To retain the $\nabla(P_\perp + B^2/2)$ term we apply the $\mathbf{B}\cdot\nabla\times\mathbf{B}\times$ operator to the one-fluid momentum equation, Eq. (48), and we obtain the perpendicular current equation

$$\begin{aligned} &\mathbf{B}\cdot\nabla\times\left[\mathbf{B}\times\left(\rho\frac{d\mathbf{V}}{dt} + \nabla\cdot\mathbf{\Pi}\right)\right] + B^2\nabla^2\left(P_\perp + \frac{B^2}{2}\right) - \mathbf{B}\cdot\nabla\left[\mathbf{B}\cdot\nabla\left(P_\perp + \frac{B^2}{2}\right)\right] \\ &-\sigma B^2\left(\nabla^2\frac{B^2}{2} - \mathbf{B}\cdot\nabla^2\mathbf{B}\right) + \mathbf{B}\cdot\nabla\left[\sigma\mathbf{B}\cdot\nabla\frac{B^2}{2}\right] - \left(\mathbf{J}\times\mathbf{B} + \nabla\frac{B^2}{2}\right)\cdot\nabla\left(\sigma B^2\right) \\ &+\sigma(\mathbf{B}\cdot\mathbf{J})^2 + \left(\mathbf{J}\times\mathbf{B} + \nabla B^2\right)\cdot\nabla\left(P_\perp + \frac{B^2}{2}\right) - \sigma\mathbf{J}\times\mathbf{B}\cdot\nabla\frac{B^2}{2} = 0. \end{aligned} \tag{51}$$

To obtain the linearized one-fluid momentum equation for low frequency perturbations we consider perturbed quantities $\delta\mathbf{B} \sim e^{-i\omega t}$ with $\omega \ll \omega_{ci}$ and $k_\| \ll k_\perp$. For simplicity quantities such as B, ρ, $\mathbf{B}$, etc., are denoted as equilibrium values. The linearized parallel momentum equation, which describes slow magnetosonic waves and maintains parallel force balance, becomes

$$-i\omega\rho\mathbf{B}\cdot\delta\mathbf{V} + \mathbf{B}\cdot\nabla\delta P_\| + \delta\mathbf{B}\cdot\nabla P_\| - \frac{P_\| - P_\perp}{B^2}\mathbf{B}\cdot\nabla\mathbf{B}\cdot\delta\mathbf{B} \simeq 0. \tag{52}$$

The linearized parallel current equation, which mainly describes the transverse Alfvén type waves and instabilities, reduces to

$$\begin{aligned} &B^2\mathbf{B}\cdot\nabla\left[\frac{\sigma\delta\mathbf{J}\cdot\mathbf{B}}{B^2}\right] - \mathbf{B}\times\boldsymbol{\kappa}\cdot\nabla(\mathbf{B}\cdot\delta\mathbf{B} - \delta P_\|) \\ &\qquad -\nabla\cdot(i\omega\rho\mathbf{B}\times\delta\mathbf{V}) - \mathbf{B}\cdot\nabla\times(\nabla\cdot\delta\mathbf{\Pi}) \simeq 0, \end{aligned} \tag{53}$$

We need to obtain the perturbed center-of-mass velocity $\delta\mathbf{V}$ and parallel electric field potential Ψ from the low frequency generalized Ohm's law, Eq. (21). Substituting the momentum equation, Eq. (23), into the low frequency generalized Ohm's law and making use of Eq. (46), we have

$$\delta\mathbf{V} \times \mathbf{B} \simeq -\delta\mathbf{E}_\perp + \sum_i \left(\frac{m_i}{\rho q_i}\right)\left[\nabla_\perp(\delta P_{\perp i} + \delta P_{ic}) + \frac{\mathbf{B}}{B} \times \nabla_\perp \delta P_{is}\right] - \frac{i\omega\rho}{n_e e}\delta\mathbf{V}_\perp. \quad (54)$$

Combining this equation with the gyroviscous force expression, Eq. (38), and neglecting terms on the order of (ω/ω_{ci}) we have

$$\begin{aligned} &\nabla \cdot (i\omega\rho\mathbf{B} \times \delta\mathbf{V}) + \mathbf{B} \cdot \nabla \times (\nabla \cdot \delta\mathbf{\Pi}) \\ &\simeq i\omega\rho\nabla \cdot \delta\mathbf{E}_\perp - i\omega\nabla_\perp^2 \sum_i \frac{m_i}{q_i}(\delta P_{i\perp} + \delta P_{ic}) - B\nabla_\perp^2 \sum_i \delta P_{is} \\ &\simeq -i\omega \sum_i n_i m_i \nabla_\perp^2 \left[\Phi + \frac{1}{n_i q_i}\left(\delta P_{i\perp} + \delta P_{ic} - \frac{\omega_{ci}}{i\omega}\delta P_{is}\right)\right]. \end{aligned} \quad (55)$$

Note that the parallel current is related to the parallel vector potential through the parallel Ampere's law, $\nabla_\perp^2 A_\| = -\delta J_\|$ and the parallel vector potential is related to the parallel electric field by $\delta\mathbf{E}_\| = -\nabla_\|\Psi = -\nabla_\|\Phi + i\omega\mathbf{A}_\|$. Then, Eq. (53) becomes

$$\begin{aligned} &\mathbf{B} \cdot \nabla \left[\frac{\sigma\nabla_\perp^2}{B^2}\mathbf{B} \cdot \nabla(\Phi - \Psi)\right] + \frac{i\omega\mathbf{B} \times \boldsymbol{\kappa}}{B^2} \cdot \nabla\left(\mathbf{B} \cdot \delta\mathbf{B} - \delta P_\|\right) \\ &+ \sum_i \frac{n_i m_i \omega^2}{B^2}\nabla_\perp^2 \left[\Phi + \frac{1}{n_i q_i}\left(\delta P_{i\perp} + \delta P_{ic} - \frac{\omega_{ci}}{i\omega}\delta P_{is}\right)\right] \simeq 0. \end{aligned} \quad (56)$$

We can give a physical interpretation to Eq. (56). This equation is the charge quasi-neutrality condition $\nabla \cdot \mathbf{J} = 0$. The Larmor radius corrections that appear are due to $\nabla_\perp \cdot \mathbf{J}_\perp$ which is carried by the ion diamagnetic and polarization drift currents. The ion diamagnetic drift contributes only through the gyroviscosity while the ion polarization drift results from the time variation of the $\delta\mathbf{E} \times \mathbf{B}$ and diamagnetic drift. The Larmor radius corrections also contribute to the parallel electric field which influences the parallel current.

The linearized perpendicular current equation, which mainly describes the compressional Alfvén type waves and instabilities, reduces to

$$\begin{aligned} &\mathbf{B} \cdot \nabla \left[\frac{\sigma}{B^2}\mathbf{B} \cdot \nabla(\mathbf{B} \cdot \delta\mathbf{B})\right] + \mathbf{B} \cdot \nabla \times [\mathbf{B} \times (-i\omega\rho\delta\mathbf{V} + \nabla \cdot \delta\mathbf{\Pi})] \\ &+\nabla_\perp^2(\delta P_\perp + \mathbf{B} \cdot \delta\mathbf{B}) \simeq 0. \end{aligned} \quad (57)$$

Because the $\mathbf{B}\times$ operator removes the parallel dynamics information, Eq. (57) describes mainly waves associated with the compressional magnetic field. For $\omega \ll k_\perp V_A$, the perturbed perpendicular pressure is out of phase with the perturbed magnetic pressure. From Eq. (57) $\delta P_\perp$ is related to the compressional magnetic field, and from Eq. (53) $\delta P_{||}$ enters with the magnetic field curvature. From Eq. (54) and the gyroviscous force expression, Eq. (39) and making use of the Faraday's law, we have

$$\mathbf{B}\cdot\nabla\times[\mathbf{B}\times(-i\omega\rho\delta\mathbf{V}+\nabla\cdot\delta\mathbf{\Pi})]\simeq\rho\omega^2\mathbf{B}\cdot\delta\mathbf{B}+B^2\nabla_\perp^2\sum_i\delta P_{ic}, \tag{58}$$

where terms on the order of (ω/ω_{ci}) have been neglected. Then, Eq. (57) becomes

$$\mathbf{B}\cdot\nabla\left[\frac{\sigma}{B^2}\mathbf{B}\cdot\nabla(\mathbf{B}\cdot\delta\mathbf{B})\right]+\frac{\rho\omega^2}{B^2}\mathbf{B}\cdot\delta\mathbf{B}$$
$$+\nabla_\perp^2\left(\mathbf{B}\cdot\delta\mathbf{B}+\delta P_\perp+\sum_i\delta P_{ic}\right)\simeq 0, \tag{59}$$

Note that the gyroviscosity enters this equation through δP_{ic}, which gives a finite ion gyroradius correction. Eq. (59) describes the fast magnetosonic (compressional Alfvén) waves and mirror instabilities.

To close Eqs. (56) and (59) we need to obtain another relationship between Ψ, Φ and $\delta B_{||}$. This relationship may be obtained using the charge quasi-neutrality condition or the parallel component of the low frequency generalized Ohm's law which is given by

$$\mathbf{B}\cdot\delta\mathbf{E}=-\frac{1}{n_e e}\left[\mathbf{B}\cdot\nabla\left(\delta P_{||e}-\sum_i\frac{q_i m_e}{e m_i}\delta P_{||i}\right)+\delta\mathbf{B}\cdot\nabla P_{||e}\right]-\left(\frac{i\omega m_e}{n_e e^2}-\eta\right)\mathbf{B}\cdot\delta\mathbf{J}. \tag{60}$$

Thus, Eqs. (56), (59) and (60) form the kinetic-fluid eigenmode equations for low frequency waves such as kinetic Alfvén waves [*Hasegawa*, 1976], inertial Alfvén waves [*Goertz and Smith*, 1989] as well as the kinetic ballooning and mirror modes [*Cheng and Qian*, 1994]. In the next section we verify that the correct dispersion relations are obtained for well-known kinetic effects on MHD waves.

The perturbed pressures, $\delta P_\perp$ and $\delta P_{||}$, for each particle species must be obtained from the perturbed particle distribution functions. In the paper by *Cheng* [1991] the perturbed particle distribution functions were derived and the perturbed pressures are written as

$$\begin{pmatrix}\delta P_{||}\\ \delta P_\perp\end{pmatrix}=-\frac{i\mathbf{B}\times\nabla\Phi}{\omega B^2}\cdot\tilde{\nabla}\begin{pmatrix}P_{||}\\ P_\perp\end{pmatrix}$$
$$+\frac{\mathbf{B}\cdot\delta\mathbf{B}}{B}\left(\frac{\partial}{\partial B}\right)_\psi\begin{pmatrix}P_{||}\\ P_\perp\end{pmatrix}+\begin{pmatrix}\delta\hat{P}_{||}\\ \delta\hat{P}_\perp\end{pmatrix}, \tag{61}$$

where $\tilde{\nabla}=\nabla-\nabla B(\partial/\partial B)_\psi$, the finite Larmor radius and other kinetic effects such as wave-particle resonances and trapped particle dynamics are included in the nonadiabatic perturbed pressures $\delta\hat{P}_\perp$ and $\delta\hat{P}_{||}$, which can be properly derived by obtaining the solutions of the gyrokinetic equation. Note that on the right-hand side of Eq. (61) the first term represents the convective derivative of plasma pressure, and the second term represents the compressional magnetic field effect associated with pressure non-uniformity along the field line resulting from pressure anisotropy.

7. Alfvén Waves and Instabilities

We have at this point obtained a closed set of kinetic-fluid eigenmode equations for low frequency ($\omega < \omega_{ci}$) waves and instabilities. Our strategy is to solve the gyrokinetic equation in various limits and obtain particle kinetic corrections to the low-frequency eigenmode equations. We will obtain explicit expressions of the perturbed particle pressures and ion gyroviscosity tensors by neglecting the magnetic drift frequency.

We consider a Bi-Maxwellian ion equilibrium distribution with $F(\mathcal{E},\mu,\psi) = N(\psi)(2\pi T_{\|}(\psi)/m)^{-3/2}\exp[-m\mathcal{E}/T_{\|} + m\mu B_0(\psi)/T_0(\psi)] = n(\psi,B)(2\pi T_{\perp}/m)^{-1} (2\pi T_{\|}/m)^{-1/2}\exp[-mv_{\perp}^2/2T_{\perp} - mv_{\|}^2/2T_{\|}]$, where $n(\psi,B) = N(\psi)T_{\perp}/T_{\|}$ is the particle density, $T_{\perp}/T_{\|} = (1 - B_0T_{\|}/BT_0)^{-1}$, the parallel pressure is $P_{\|} = nT_{\|}$, and the perpendicular pressure is $P_{\perp} = nT_{\perp}$. Then, $(q/m)\partial F/\partial\mathcal{E} = -qF/T_{\|}$ and $(q/m)\partial F/\partial\mathcal{E} + (q/mB)\partial F/\partial\mu = -qF/T_{\perp}$. We now proceed to evaluate the perturbed ion pressures and gyroviscosity in the limit $\omega, k_{\|}v_i > \omega_{di}$, where v_i is the ion thermal velocity and ω_{di} is the ion magnetic drift frequency. The solution of the linearized ion gyrokinetic equations is obtained from Eq. (27) and is given by

$$g_{i0} \simeq -\frac{q_i}{m_i}\frac{\partial F}{\partial\mathcal{E}}\frac{\omega - \hat{\omega}_*}{\omega - k_{\|}v_{\|}}\left[(\Phi - v_{\|}A_{\|})J_0 + \frac{v_{\perp}\delta B_{\|}}{k_{\perp}}J_1\right], \tag{62}$$

where $\hat{\omega}_* = m_i\mathbf{B}\times\mathbf{k}\cdot\nabla F/q_iB(\partial F/\partial\mathcal{E})$. Then, by ignoring the temperature gradient effect we obtain from Eq. (33) the perturbed parallel ion pressure

$$\begin{aligned}\delta P_{\|i} = P_{\|i}&\left[-\frac{q_i\Phi}{T_{\|i}} + \left(1 - \frac{\omega_{*i}}{\omega}\right)\left(\Gamma_0\frac{q_i\Phi}{T_{\|i}} + (\Gamma_0 - \Gamma_1)\frac{T_{\perp i}}{T_{\|i}}\frac{\delta B_{\|}}{B}\right)\right] \\ -P_{\|i}&\left(1 - \frac{\omega_{*i}}{\omega}\right)\left[1 - \zeta_i^2 Z'\right]\left(\Gamma_0\frac{q_i\Psi}{T_{\|i}} + (\Gamma_0 - \Gamma_1)\frac{T_{\perp i}}{T_{\|i}}\frac{\delta B_{\|}}{B}\right),\end{aligned} \tag{63}$$

where $\omega_{*i} = (T_{\|i}/q_iB^2N_i)\mathbf{B}\times\nabla N_i\cdot\mathbf{k}_{\perp}$ is the ion diamagnetic drift frequency, $\zeta_i = \omega/\sqrt{2}k_{\|}v_i$, $v_i^2 = T_{\|i}/m_i$, $Z(\zeta_i)$ is the plasma dispersion function, $Z' = -2(1+\zeta_iZ)$, $\Gamma_n(b_i) = I_n(b_i)e^{-b_i}$, I_n is the modified Bessel function of the first kind of order n, $b_i = k_{\perp}^2T_{i\perp}/m_i\omega_{ci}^2$.

Similarly, from Eqs. (32), (43), (44), and (45) we obtain the perturbed ion perpendicular pressure and ion gyroviscosity tensor contributions:

$$\delta P_{\perp i} + \delta P_{ic} = P_{\perp i}\left[-\frac{q_i\Phi}{T_{\perp i}} + \left(1 - \frac{\omega_{*i}T_{\perp i}}{\omega T_{\|i}}\right)(\Gamma_0 - \Gamma_1)\left(\frac{q_i\Phi}{T_{\perp i}} + 2\frac{\delta B_{\|}}{B}\right)\right]$$

$$-P_{\perp i}\left(\frac{T_{\perp i}}{T_{\parallel i}}\right)\left(1-\frac{\omega_{*i}}{\omega}\right)\left[1+\zeta_i Z(\zeta_i)\right](\Gamma_0-\Gamma_1)\left(\frac{q_i\Psi}{T_{\perp i}}+2\frac{\delta B_\parallel}{B}\right), \tag{64}$$

and

$$\frac{i\omega_{ci}\delta P_{is}}{\omega}=P_{\perp i}\left(1-\frac{\omega_{*i}T_{\perp i}}{\omega T_{\parallel i}}\right)$$
$$\times\left[\left(\frac{1-\Gamma_0}{b_i}\right)\frac{q_i\Phi}{T_{\perp i}}+\left(\frac{1-\Gamma_0+\Gamma_1}{b_i}\right)\frac{\delta B_\parallel}{B}-(\Gamma_0-\Gamma_1)\left(\frac{q_i\Phi}{T_{\perp i}}+2\frac{\delta B_\parallel}{B}\right)\right]$$
$$+P_{\perp i}\left(1-\frac{T_{\perp i}}{T_{\parallel i}}\right)\frac{k_\parallel^2 T_{\parallel i}}{m_i\omega^2}\left(\frac{1-\Gamma_0}{b_i}-(\Gamma_0-\Gamma_1)\right)\frac{q_i(\Phi-\Psi)}{T_{\perp i}}. \tag{65}$$

From Eqs. (64) and (65), the vorticity equation, Eq. (56), reduces to

$$\mathbf{B}\cdot\nabla\left[\frac{\sigma\nabla_\perp^2}{B^2}\mathbf{B}\cdot\nabla(\Phi-\Psi)\right]+\frac{i\omega\mathbf{B}\times\boldsymbol{\kappa}}{B^2}\cdot\nabla\left(\delta B_\parallel B-\delta P_\parallel\right)$$
$$+\sum_i\frac{n_i m_i\omega^2}{B^2}\nabla_\perp^2\left\{\left(1-\frac{\omega_{*i}T_{\perp i}}{\omega T_{\parallel i}}\right)\left[\left(\frac{1-\Gamma_0}{b_i}\right)\Phi+\left(\frac{1-\Gamma_0+\Gamma_1}{b_i}\right)\frac{T_{\perp i}\delta B_\parallel}{q_i B}\right]\right.$$
$$-\frac{T_{\perp i}}{T_{\parallel i}}\left(1-\frac{\omega_{*i}}{\omega}\right)(1+\zeta_i Z)(\Gamma_0-\Gamma_1)\left(\Psi+\frac{2T_{\perp i}\delta B_\parallel}{q_i B}\right)$$
$$\left.+\left(1-\frac{T_{\perp i}}{T_{\parallel i}}\right)\frac{k_\parallel^2 T_{\parallel i}}{m_i\omega^2}\left(\frac{1-\Gamma_0}{b_i}-(\Gamma_0-\Gamma_1)\right)(\Phi-\Psi)\right\}\simeq 0. \tag{66}$$

Eq. (66) describes mainly the transverse/kinetic Alfvén waves and ballooning instabilities in high β plasmas with anisotropic pressure. Note that a Padé approximation can be used to simplify the Bessel functions so that the differential operators can remain valid for general values of $k_\perp\rho_i$. Physically, the Larmor radius corrections modify the zero order polarization current which arises from ion inertia. Also note that for $\beta_i\sim O(1)$ the $\delta B_\parallel$ contribution resulting from the gyroviscous force is at least of the same order as the finite Larmor radius correction to the electrostatic potential.

For electrons, we consider $k_\parallel v_e,\omega\gg\omega_{de}$ so that the transition from kinetic to inertial Alfvén waves which occurs for $k_\parallel v_e\sim\omega$ is retained. Neglecting effects of trapped particle dynamics, gyroradii, and pressure anisotropy, the linear perturbed electron distribution function can be straightforwardly obtained from Eqs. (25) and (27) and is given by

$$\delta f_e\simeq-\frac{e}{m}\frac{\partial F}{\partial\mathcal{E}}\left[\frac{\omega_*^T}{\omega}\Phi+\left(1-\frac{\omega_*^T}{\omega}\right)\Psi+\frac{\omega-\omega_*^T}{k_\parallel v_\parallel-\omega}\left(\Psi+\frac{v_\perp^2}{2\omega_{ce}}\delta B_\parallel\right)\right] \tag{67}$$

Integration over the velocity space and ignoring the temperature gradient we obtain the perturbed electron density

$$\delta n_e=n_e\left[\frac{\omega_{*e}}{\omega}\frac{e\Phi}{T_e}+\left(1-\frac{\omega_{*e}}{\omega}\right)\left((1+\zeta_e Z(\zeta_e))\frac{e\Psi}{T_e}-\zeta_e Z(\zeta_e)\frac{\delta B_\parallel}{B}\right)\right], \tag{68}$$

the perturbed parallel electron pressure is given by

$$\delta P_{\|e} = n_e T_e \left[\frac{\omega_{*e}}{\omega}\frac{e\Phi}{T_e} + \left(1 - \frac{\omega_{*e}}{\omega}\right)\left(\left(1 - \zeta_e^2 Z'(\zeta_e)\right)\frac{e\Psi}{T_e} + \zeta_e^2 Z'(\zeta_e)\frac{\delta B_\|}{B}\right)\right], \qquad (69)$$

and the perturbed perpendicular electron pressure is given by

$$\delta P_{\perp e} = n_e T_e \left[\frac{\omega_{*e}}{\omega}\frac{e\Phi}{T_e} + \left(1 - \frac{\omega_{*e}}{\omega}\right)\left(\left(1 + \zeta_e Z(\zeta_e)\right)\frac{e\Psi}{T_e} - 2\zeta_e Z(\zeta_e)\frac{\delta B_\|}{B}\right)\right], \qquad (70)$$

where $\zeta_e = \omega/\sqrt{2}k_\| v_e$, $v_e^2 = T_e/m_e$, and $\omega_{*e} = -(T_e/eB^2 n_e)\mathbf{B}\times\nabla n_e\cdot\mathbf{k}_\perp$.

We obtain the parallel current and parallel electric field from the parallel component of the low frequency Ohm's law, Eq. (60). Because $\delta\mathbf{B}_\perp = \nabla\times\mathbf{A}_\|$ and the parallel electric field definition, $\delta\mathbf{E}_\| = -\nabla_\|\Psi = -\nabla_\|\Phi + i\omega\mathbf{A}_\|$, we also have

$$\delta\mathbf{B}\cdot\nabla P_e = -ien_e B k_\| \left(\frac{\omega_{*e}}{\omega}\right)(\Phi - \Psi). \qquad (71)$$

Employing the parallel Ampere's law, $\nabla_\perp^2 A_\| = -\delta J_\|$, and ignoring plasma resistivity, the parallel component of the low frequency Ohm's law, Eq. (60) then reduces to

$$\delta J_\| = -\nabla_\perp^2\left[\frac{k_\|}{\omega}(\Phi - \Psi)\right] \simeq \sum_j \frac{n_j q_j^2}{T_{\|j}}\frac{\omega - \omega_{*j}}{k_\|}\frac{Z'(\zeta_j)}{2}\left[\Gamma_0\Psi + (\Gamma_0 - \Gamma_1)\frac{T_{\perp j}}{q_j}\frac{\delta B_\|}{B}\right] \qquad (72)$$

where the summation over j is over all particle species. We have also neglected terms that are on the order of (m_e/m_i). Note that the parallel electric field in the Ohm's law is cancelled by a contribution from the $\mathbf{B}\cdot\nabla\delta P_{\|e}$ term and thus the parallel electric field potential is determined by the balance of other terms in the low frequency Ohm's law.

Equations (59), (66), and (72) together with the expressions for perturbed particle pressures and ion gyroviscosity tensors form a closed set of eigenmode equations describing the dispersion of dispersive kinetic/inertial Alfvén waves as well as kinetic ballooning and mirror modes for high β plasmas. The operator $i\mathbf{k}_\| = \nabla_\|$ acts along field lines, and an integral equation should be used if there are plasma and magnetic field gradients along field lines. Kinetic effects of finite ion Larmor radii and electron-wave particle resonances are included for nonuniform plasmas in a general magnetic field geometry. However, kinetic effects of trapped particles and particle magnetic drifts are neglected.

It is worthwhile to note that for high β plasmas we must retain in Eq. (72) the ion parallel current contribution, $\delta J_{\|i}$, which is due to the $\sum_i (m_e q_i/m_i e)\mathbf{B}\cdot\nabla\delta P_{\|i}$ term in the parallel Ohm's law. This term has been neglected in all previous studies involving the Ohm's law. To assess the importance of $\delta J_{\|i}$ relative to $\delta J_{\|e}$ we consider three different limits. In the small parallel wave phase velocity limit, $\omega/k_\| \ll v_i \ll v_e$, we have $Z' \simeq -2$, and

$$\frac{\delta J_{\|i}}{\delta J_{\|e}} \sim \frac{\omega - \omega_{*i}}{\omega - \omega_{*e}}\frac{T_e}{T_i} \sim \frac{T_e}{T_i}.$$

Thus, the ion parallel current is on the same order as the electron parallel current and must be retained. In the medium parallel wave phase velocity limit, $v_i \ll \omega/k_{||} \ll v_e$, we have $Z'(\zeta_i) \simeq 1/\zeta_i^2$ and thus

$$\frac{\delta J_{||i}}{\delta J_{||e}} \sim -\frac{\omega - \omega_{*i}}{\omega - \omega_{*e}} \frac{T_e}{T_i} \left(\frac{k_{||} v_i}{\omega} \right)^2 .$$

In the high parallel wave phase velocity limit, $v_i \ll v_e \ll \omega/k_{||}$, the ion parallel current can be neglected because

$$\frac{\delta J_{||i}}{\delta J_{||e}} \sim \frac{\omega - \omega_{*i}}{\omega - \omega_{*e}} \frac{T_e}{T_i} \left(\frac{v_i}{v_e} \right)^2 \sim \frac{m_e}{m_i} \ll 1.$$

For transverse Alfvén waves, $\omega \simeq k_{||} V_A$, and thus $2(k_{||} v_e/\omega)^2 = \beta_e m_i/m_e$ and $2(k_{||} v_i/\omega)^2 = \beta_i$. For $\beta_e > m_e/m_i$ and $\beta_i > 1$ the small parallel wave phase velocity limit applies and the ion parallel current is the same order as the electron parallel current. For $\beta_e > m_e/m_i$ and $\beta_i < 1$ the medium parallel wave phase velocity limit applies and $\delta J_{||i}/\delta J_{||e} \sim \beta_e/2$. Obviously, if $\beta_e > 1$ (due to $T_e/T_i \gg 1$) the parallel ion current contribution becomes important. For $\beta_e < m_e/m_i$ and $\beta_i < 1$, the high parallel wave phase velocity limit applies and the ion parallel current can be neglected. Near the ionosphere, $v_e < V_A$ ($\beta_e < m_e/m_i$) and the electron inertia effect is important.

7.1. Dispersive Transverse Alfvén Waves in Low β Limit

We will demonstrate the dispersive aspects of Larmor radius corrections and electron inertia on the shear Alfvén waves. For low β ($\beta < 1$) plasmas the kinetic-fluid eigenmode equations for dispersive transverse Alfvén waves can be further simplified by considering the limit $\omega \gg k_{||} v_i$. We can also neglect the parallel ion current contribution, the $\delta B_{||}$ terms, and the magnetic field curvature term in Eq. (66) which is usually smaller than the ion inertia and field line bending terms. Then, from Eq. (72) the parallel electric field potential is given by

$$\Psi = k_\perp^2 \lambda_p^2 \Phi \left[k_\perp^2 \lambda_p^2 + \left(1 - \frac{\omega_{*e}}{\omega} \right) \zeta_e^2 Z'(\zeta_e) \right]^{-1} , \qquad (73)$$

where $\lambda_p = c/\omega_{pe}$ is the electron skin depth. By neglecting the temperature anisotropy effect we obtain from Eq. (66) the eigenmode equation for dispersive transverse Alfvén waves

$$\mathbf{B} \cdot \nabla \left[\frac{\sigma \nabla_\perp^2}{B^2} \mathbf{B} \cdot \nabla \Upsilon \right] + \sum_i \frac{n_i m_i \omega (\omega - \omega_{*i})}{B^2} \left(\frac{1 - \Gamma_0}{b_i} \right) \left(1 - \frac{\omega \lambda_p^2 \nabla_\perp^2}{(\omega - \omega_{*e}) \zeta_e^2 Z'(\zeta_e)} \right) \nabla_\perp^2 \Upsilon \simeq 0, \qquad (74)$$

where $\Upsilon = \Phi - \Psi$ and the effects of full ion Larmor radii, diamagnetic drifts, and electron-wave resonances are included. Without diamagnetic drift effects the local dispersion relation of this equation was previously derived [*Hasegawa and Chen*, 1976; *Lysak and Lotko*, 1996]. For $b_i \ll 1$ the Bessel function can be expanded with $(1-\Gamma_0)/b_i \simeq 1-3b_i/4$. The Bessel function can also be approximated by the Padé approximation, $(1-\Gamma_0)/b_i \simeq 1/(1+b_i)$, which is correct for both the $b_i \ll 1$ and $b_i \gg 1$ limits. Equation (74) is a three-dimensional eigenmode equation and determines the magnetosphere-ionosphere coupling of transverse Alfvén waves. Near the field line resonance location the equation can be simplified to a one-dimensional field-aligned equation by taking the limit $k_\perp \gg k_\|$ and treating $\nabla_\perp^2 = -k_\perp^2$.

Two limits of Eq. (74) are worth pointing out. For the limit $\zeta_e \ll 1$ (i.e., $V_A \ll v_e$ and $m_e/m_i \ll \beta_e < 1$), $Z'(\zeta_e) = -2 + \mathcal{O}(\zeta_e^2)$, and the parallel electric field potential is given by $\Psi \simeq k_\perp^2\lambda_p^2\Phi/\left(k_\perp^2\lambda_p^2 - m_e\omega(\omega-\omega_{*e})/k_\|^2 T_e\right)$. For $\omega \gg \omega_{*e}$ and $k_\perp^2 T_e/m_i\omega_{ci}^2 \ll 1$, we have the well-known parallel electric field potential $\Psi \simeq -(k_\perp^2 T_e/m_i\omega_{ci}^2)\Phi$ for kinetic Alfvén waves. For one ion species we obtain the dispersion relation for kinetic Alfvén waves

$$\frac{\omega(\omega-\omega_{*i})}{k_\|^2 V_A^2} = \frac{\sigma b_i}{1-\Gamma_0} + \frac{k_\perp^2 T_e}{m_i\omega_{ci}^2}\left(\frac{\omega-\omega_{*i}}{\omega-\omega_{*e}}\right). \tag{75}$$

Considering $b_i < 1$ and ignoring the diamagnetic drift and pressure anisotropy effects, we obtain the well-known dispersion relation for the kinetic Alfvén waves

$$\omega^2 = k_\|^2 V_A^2\left[1+\left(\frac{3}{4}+\frac{T_e}{T_i}\right)\frac{k_\perp^2 T_i}{m_i\omega_{ci}^2}\right]. \tag{76}$$

Equation (76) had been previously derived based on the gyrokinetic theory [*Hasegawa*, 1976; *Hasegawa and Mima*, 1978; *Goertz*, 1984] for low β plasmas. The eigenmode equations including full Larmor radius effects have also been derived for kinetic Alfvén wave and kinetic ballooning modes [*Cheng et al.*, 1995].

Near the ionosphere where $v_e \ll V_A$ we adopt the ordering $1 \ll \zeta_e$ ($\beta_e \ll m_e/m_i$), then $Z'(\zeta_e) \sim 1/\zeta_e^2$, and we find the well known parallel electric field for inertial Alfvén waves with $\Psi \simeq k_\perp^2\lambda_p^2\Phi/(1-\omega_{*e}/\omega + k_\perp^2\lambda_p^2)$. It is worthwhile to note that for $\omega \gg \omega_{*e}$ and $k_\perp^2 T_e/m_i\omega_{ci}^2 \ll 1$ this parallel electric field potential has an opposite sign from the kinetic Alfvén wave case. For one ion species we obtain the dispersion relation for inertial Alfvén waves

$$\frac{\omega(\omega-\omega_{*i})}{k_\|^2 V_A^2} = \frac{\sigma b_i}{1-\Gamma_0}\left(\frac{1-\omega_{*e}/\omega}{1-\omega_{*e}/\omega + k_\perp^2\lambda_p^2}\right). \tag{77}$$

Ignoring the diamagnetic drift and pressure anisotropy effects, the inertial Alfvén wave dispersion relation for $b_i \ll 1$ becomes

$$\omega^2 = \frac{k_\|^2 v_A^2}{1+k_\perp^2\lambda_p^2} \tag{78}$$

as previously obtained [*Goertz and Bosewell*, 1979].

7.2. Reduced Two Fluid Equations for Dispersive Transverse Alfvén Waves

A reduced two-fluid model has been employed in studying shear/kinetic Alfvén waves in low β plasmas [*Streltsov et al.*, 1998] without gradient in density and temperature. The equations involved are the electron parallel momentum equation, electron continuity equation, current continuity equation and ion momentum equation. The perpendicular current is determined by the ion momentum equation, and the parallel current from the electron equation. An important question is whether an isothermal electron pressure, $\delta P_e = \delta n_e T_e$, can be properly used in the electron momentum equation.

The electron density and parallel momentum equations are

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{V}_{||e}) \simeq 0, \tag{79}$$

and

$$m_e n_e \frac{d\mathbf{V}_{||e}}{dt} + e n_e \mathbf{E}_{||} + \nabla_{||} p_e \simeq 0. \tag{80}$$

If we use the representation $\mathbf{E}_{||} = -\nabla_{||}\Psi$, assume $\delta \mathbf{J}_{||} \simeq -n_e e \mathbf{V}_{||e}$, and ignore the diamagnetic drift frequency contribution, then from the parallel Ampere's law and the linear electron continuity equation, Eq. (79), we obtain

$$\frac{\delta n_e}{n_0} = -e \left(\frac{k_{||} v_A}{\omega} \right)^2 \frac{k_\perp^2}{m_i \omega_{ci}^2} \left(\Phi - \Psi \right). \tag{81}$$

Combining this equation with the parallel electron momentum equation, Eq. (80), yields

$$\Psi = \frac{k_\perp^2 \lambda_p^2 \left(1 - (k_{||} v_e/\omega)^2 \right)}{1 + k_\perp^2 \lambda_p^2 \left(1 - (k_{||} v_e/\omega)^2 \right)} \Phi. \tag{82}$$

The result is in agreement with the $(k_{||} v_e/\omega)^2 \gg 1$ and $(k_{||} v_e/\omega)^2 \ll 1$ limits of Eq (73). However, from Eqs. (68) and (69) we can easily obtain

$$\delta P_e = T_e \delta n_e \left[\frac{1 + 2\zeta_e^2 (1 + \zeta_e Z(\zeta_e))}{1 + \zeta_e Z(\zeta_e)} \right]. \tag{83}$$

Near the ionosphere $\zeta_e \gg 1$, $\delta P_e \sim 3 T_e \delta n_e$, which is a factor of 3 different from the isothermal model. But, the pressure does not enter the leading order balance of the parallel electric field and inertia so the pressure law is not important. In the magnetosphere $\zeta_e \ll 1$ and the pressure balances the electric field in the leading order and the inertia is not important. Any attempt to keep the inertia term in the reduced two-fluid model in this limit is, in fact futile because the error introduced by the isothermal electron pressure model is larger than the inertia term and gives the wrong sign. (Note that for $\zeta_e \ll 1$, $\delta P_e \approx T_e \delta n_e (1 + 4\zeta_e^2)$, and the error is the same order as the inertia term.) While the qualitative behavior of dispersive transverse Alfvén waves is recovered by the simple isothermal electron pressure model, quantitative studies will be inaccurate because they neglect important physical effects near the transition where $\zeta_e \sim 1$ where electron Landau damping is important.

8. Summary and Discussion

In this paper we have formulated two nonlinear kinetic-fluid models for high β plasmas with multiple ion species to study multiscale phenomena: one is a kinetic-multifluid model for studying phenomena with frequency on the order of ion cyclotron frequencies; the other is a low frequency kinetic-fluid model for studying phenomena with frequency below the ion cyclotron frequency. These two kinetic-fluid models were developed by taking advantage of the simplicity of the fluid models and by properly taking into account finite ion Larmor radius (FLR) and other major particle kinetic effects. The kinetic-multifluid model treats each particle species by fluid descriptions as well as particle kinetic approach such as the Vlasov or gyrokinetic equation to determine particle distribution functions, and the coupling between the particle kinetic dynamics and multifluid models is through the particle pressure in the fluid momentum equations.

The low-frequency kinetic-fluid model is obtained from the kinetic-multifluid model by further restricting the time scales to be longer than the ion cyclotron time. Instead of the electron and multiple ion fluid equations, the one-fluid density and momentum equations and a newly derived low frequency Ohm's law are employed. The particle kinetic physics are again coupled to the one-fluid equations and the low-frequency Ohm's law through particle pressure tensors. The major advantage of the low-frequency kinetic-fluid model is that important kinetic effects can be accurately described with a minimum of modification to the one-fluid equations. We note that important particle kinetic effects such as finite Larmor radius; resonant wave-particle interactions; and trapped particle dynamics are properly retained. These kinetic effects are essential when describing multiscale coupling processes for long time scale global phenomena.

From the low frequency kinetic-fluid model we have derived the eigenmode equations for low frequency ($\omega < \omega_{ci}$) waves and instabilities in high β plasmas such as dispersive transverse Alfvén waves (kinetic and inertial Alfvén waves) and ballooning-mirror instabilities. The eigenmode equations take into account the magnetosphere-ionosphere coupling. It is demonstrated that for $\beta < 1$ plasmas effects due to ion Larmor radii, electron Landau damping and electron inertia on parallel electric field are properly retained in the dispersion relation of dispersive transverse Alfvén waves. Note that the ion Larmor radius effects on the KAW are not properly included in the popularly employed reduced two-fluid equations without proper gyroviscosity contribution. In the presence of background gradients, finite ion Larmor radius effects couple global MHD disturbances with kinetic Alfvén waves which can strongly interact with ions because the perpendicular wavelength is on the order of ion gyroradii.

Finally, the kinetic-fluid model presented in the paper represents a major advance in laying the theoretical foundation for studying long time behavior of multiscale phenomena in high β plasmas. Besides the study of wave propagation and stability analysis in realistic plasma geometries the natural next step is to develop global simulation codes based on the kinetic-fluid model. Such simulation studies will enhance our understanding of important nonlinear physics of plasma heating and transport, which in turn determines the dynamics and structure of plasma and magnetic field profiles.

Acknowledgments. This work is supported by the NSF grant ATM-9523331 and DoE Contract No. DE-AC02-76-CHO3073.

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