Stability and Transport Properties of an Intense Ion Beam Propagating Through an Alternating-Gradient Focusing Lattice

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Abstract

Stability and transport properties of an intense ion beam propagating through an alternating-gradient focusing lattice with initial Kapchinskij-Vladimirskij (KV) distribution are studied using newly-developed perturbative (δf) particle simulation techniques. Stability properties are investigated over a wide range of beam current and focusing field strength. In the unstable region, large-amplitude density perturbations with low azimuthal harmonic numbers, concentrated near the beam surface, are observed. Their nonlinear consequences are discussed.

Understanding the nonlinear dynamics of an intense ion beam propagating in an alternating-gradient focusing lattice is a critical scientific problem in heavy ion fusion, which relies on high-brightness and high-current heavy ion beams to deliver high power to the target.[1–4] Extensive theoretical[5–10] and experimental[11,12] efforts have been devoted to the investigation of the stability and transport properties of intense beams over the past two decades. For an ideal beam focusing system, a primary source of instability is the collective space-charge force due to the un-neutralized beam current and charge. It has been shown in earlier studies[1,6] utilizing the linearization approximation that, under certain conditions, the Kapchinskij-Vladimirskij (KV) beam distribution[5], the only known collisionless beam equilibrium for periodically focused intense ion beams, exhibits space-charge-induced instabilities, resulting in emittance growth and possible beam particle losses.

In this paper, a new particle simulation method based upon the perturbative (δf) scheme is developed to study the detailed stability properties of the KV beam equilibrium, or other choices of initial distribution function, and identify the parameter regime for optimal beam transport. The δf simulation scheme was initially developed for tokamak plasma simulations[13,14] and is extended here for application to intense charged particle beams. In the δf-scheme, the particle distribution function f is separated into a background part f_0 and a perturbed part δf according to $f = f_0 + \delta f$. Here, the dynamics of the background particles associated with the distribution f_0 is assumed to be known, and the linear and nonlinear evolution of δf is computed numerically from the dynamics of a collection of simulation particles. The δf-scheme completely removes statistical noise associated with the representation of f_0 by a finite number of discrete particles, which is the merit of the δf-scheme in comparison with conventional particle-in-cell (PIC) simulations.

In this Letter, the δf simulation scheme is used both to confirm qualitatively the results from previous studies that the KV beam equilibrium is unstable in some parameter regimes, and to reveal, for the first time, the detailed mode structure in the plane transverse to the beam propagation direction. The linear growth rate of the instability is determined, as well as the nonlinear saturation properties. Preliminary results from this δf simulation scheme are reported in this Letter, and a more detailed description will be provided in a future publication.

We consider an intense ion beam propagating in the z-direction through an alternating-gradient quadrupole magnetic field described by $\boldsymbol{B}^q(x,y,s) = B_q'(s)(y\boldsymbol{e}_x + x\boldsymbol{e}_y)$ and $B_q'(s+S) = B_q'(s)$. The dynamics of the beam particles in the transverse phase space (x, y, x', y') can be described by the collisionless Vlasov equation

$$\frac{df}{ds} \equiv \frac{\partial f}{\partial s} + \frac{d\boldsymbol{x}}{ds}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{d\boldsymbol{x}'}{ds}\cdot\frac{\partial f}{\partial \boldsymbol{x}'} = 0, \tag{1}$$

where $s = \beta_b ct$ is the axial coordinate,

$$d\boldsymbol{x}/ds = \boldsymbol{x}' = x'\boldsymbol{e}_x + y'\boldsymbol{e}_y \tag{2}$$

is the velocity in the transverse direction, and the combined forces of the external focusing magnetic field and defocusing space-charge field yield the transverse accelerations

$$\frac{d\boldsymbol{x}'}{ds} = \alpha\kappa_q(s)\boldsymbol{x} - \frac{q}{\gamma_b^3\beta_b^2 mc^2}\frac{\partial}{\partial \boldsymbol{x}}\phi(x,y,s), \tag{3}$$

Here, $\alpha \equiv (-1, 1)$ in the (x, y) directions, respectively, $\kappa_q(s) \equiv qB_q'(s)/\gamma_b\beta_b mc^2 = \kappa_q(s+S)$ is a periodic function describing the quadrupole focusing field, $\phi(x,y,s)$ and $\beta_b\phi(x,y,s)\boldsymbol{e}_z$ are the scalar and vector potentials associated with the space charge and current of the beam, q and m are the ion charge and rest mass, respectively, c is the speed of light *in vacuo*, $\beta_b c$ is the average axial beam velocity, and $\gamma_b = (1-\beta_b^2)^{-1/2}$ is the relativistic mass factor.

We define the weight function $w \equiv \delta f/f = (f - f_0)/f$, and obtain from Eq. (1)

$$\frac{dw}{ds} = -(1-w)\frac{1}{f_0}\frac{df_0}{ds}. \tag{4}$$

Equation (4) is the governing equation for the δf simulation scheme, and can be further simplified, provided the background distribution function f_0 is an *equilibrium* solution to Eq. (1). For present purposes, f_0 is chosen to be the KV distribution,[1,6]

$$f_0(x,y,x',y',s) = \frac{N}{\pi^2\varepsilon_{x0}\varepsilon_{y0}}\delta(W-1), \tag{5}$$

where, $N = \int f_0 dxdydx'dy'$ is the number of particles per unit length, ε_{x_0} and ε_{y_0} are the unnormalized emittances, and the variable W is defined by

$$W \equiv \frac{x^2}{a^2} + \frac{1}{\varepsilon_{x_0}^2}(ax' - xa')^2 + \frac{y^2}{b^2} + \frac{1}{\varepsilon_{y_0}^2}(by' - yb')^2. \tag{6}$$

Here, $a(s)$ and $b(s)$ are the matched-beam envelope functions for the KV beam equilibrium,[10] and $a' = da(s)/ds$ and $b' = db(s)/ds$. Denoting $\phi(x,y,s) = \phi_0(x,y,s) + \delta\phi(x,y,s)$, where $\phi_0(x,y,s)$ is the equilibrium electrostatic potential corresponding to f_0, it follows that

$$\frac{\partial f_0}{\partial s} + \frac{d\boldsymbol{x}}{ds}\cdot\frac{\partial f_0}{\partial \boldsymbol{x}} + \left.\frac{d\boldsymbol{x}'}{ds}\right|_0 \cdot \frac{\partial f_0}{\partial \boldsymbol{x}'} = 0, \tag{7}$$

where $dx'/ds|_0$ and $dy'/ds|_0$ are the equilibrium accelerations of the beam particles, which can be determined from Eq. (3) by replacing $\phi(x,y,s)$ by $\phi_0(x,y,s)$. Using Eqs. (1) and (7) to evaluate df_0/ds, and substituting into Eq. (4) gives

$$\frac{dw}{ds} = (1-w)\frac{q}{\gamma_b^3\beta_b^2 mc^2}\nabla\delta\phi\cdot\frac{1}{f_0}\frac{\partial f_0}{\partial \boldsymbol{x}'}. \tag{8}$$

Here, ϕ_0 and $\delta\phi$ are determined self-consistently from Poisson's equations

$$\nabla^2\phi_0 = -4\pi q n_0, \qquad \nabla^2\delta\phi = -4\pi q\delta n. \tag{9}$$

In Eq. (9), $\delta n(x,y,s) = \int \delta f dx'dy' = \int wf dx'dy'$ is the perturbed density about the equilibrium profile $n_0(x,y,s) = \int f_0 dx'dy'$. For the KV distribution in Eq. (5), n_0 is uniform (equal to $\hat{n}_0 = N/\pi ab$) and $\phi_0 = -2qN(x^2/a + y^2/b)/(a+b) + const.$ within the elliptical boundary $x^2/a^2 + y^2/b^2 = 1$. Equations (2), (3), (8) and (9) are the set of fully nonlinear equations used in the *nonlinear* δf-scheme. For a linear stability analysis, the *linear* δf-scheme also consists of solving Eqs. (2), (3), (8) and (9), except that ϕ is replaced by ϕ_0 in Eq. (3), and $(1-w)$ is replaced by unity in Eq. (8).

In the simulations presented here, the periodic focusing channel has a step-function lattice with filling factor $\eta = 1/2$, axial periodicity length $S = 66\ cm$, and vacuum phase advance $\sigma_v = 85°$. A KV beam distribution is loaded at $s = 0$ with transverse emittance $\varepsilon_{x0} = \varepsilon_{y0} = \varepsilon_0 = 5.72\times 10^{-2} cm\cdot rad$, and different values of beam current I_b as measured by the perveance $K = 2qI_b/mc^3\beta_b^3\gamma_b^3$. The beam propagates through a perfectly conducting rectangular pipe with square cross section, and transverse dimension about four times the average matched-beam radius $\bar{a}$, centered on the beam axis at $(x, y) = (0, 0)$. Also, $\delta\phi = 0$ is imposed at the conducting wall. A typical simulation run with a 64×64 grid in the transverse direction and 100,000 simulation particles takes about 1000 seconds on the NERSC Cray Y/MP computer to propagate 50 lattice periods (50×96 steps). Simulations have been carried out for cases where the number of particles is varied from $10,000$ to $200,000$, and it is found that simulations with $50,000$ particles provide more than adequate accuracy.

For small-amplitude perturbations, it is well known from analytical studies[1,6] that the KV beam equilibrium is *linearly stable* at sufficiently low beam current, but can exhibit strong collective instability at high beam current where the self fields due to the beam space charge are more intense. As an example of beam propagation when the equilibrium is linearly stable, we have used the nonlinear δf-scheme with $50,000$ simulation particles, to follow the beam propagation for 500 lattice periods from $s = 0$ to $s = 500S$. For system parameters corresponding to $\sigma_v = 30°$, and space-charge-depressed phase advance $\sigma = \varepsilon_0 \int_{s_0}^{s_0+S} ds/a^2(s) = 11.58°$, which is consistent with normalized beam current $KS/\varepsilon_0 = 1.15$, the initial noise level is $(\delta n)_{s=0}/\hat{n}_0 \lesssim 0.886 \times 10^{-7}$, and the beam propagates quiescently with $(\delta n)_{s=500S}/\hat{n}_0 \lesssim 2.4 \times 10^{-7}$ at 500 lattice periods. As a second example of stable beam propagation, we have considered the choice of system parameters $\sigma_v = 85°$, $\sigma = 71.76°$ and $KS/\varepsilon_0 = 0.5$, corresponding to a stronger quadrupole field. In this case, the beam propagates stably with the noise associated with the density perturbations $\delta n/\hat{n}_0$ increasing from $0.52 \times 10^{-7}$to 0.875×10^{-6} over 500 lattice periods. Standard particle-in-cell simulations of intense beam propagation exhibit significant noise and accuracy problems after one or two hundred lattice periods. Therefore, the δf-scheme used in the present simulations offers substantial advantages.

At sufficiently high beam current, the KV beam equilibrium exhibits strong collective instability.[1,6] Typical numerical results obtained with the nonlinear δf-scheme, using $50,000$ simulation particles, are illustrated in Fig. 1 for beam propagation in the unstable regime, for the choice of system parameters $\sigma_v = 85°$, $\sigma = 20.1°$, and $KS/\varepsilon_0 = 5.77$. Figure 1 shows three-dimensional plots in the transverse (x, y) plane of the normalized perturbed density $\delta n/\hat{n}_0$ at $s = 0$ [Fig. 1(a)] and $s = 30S$ [Fig. 1(b)], and the self-consistent perturbed electrostatic potential $\delta\phi$ at $s = 30S$ [Fig. 1(c)]. Initially, at $s = 0$, it is evident that the noise level satisfies $(\delta n)_{s=0}/\hat{n}_0 \lesssim 1.0 \times 10^{-7}$. By 30 lattice periods ($s = 30S$), however, we note that the maximum density perturbation has grown by 3 orders-of-magnitude to $(\delta n)_{\max}/\hat{n}_0 \simeq 2.77 \times 10^{-4}$, concentrating near the outer edge of the beam where density gradients are largest. From Figs. 1(b) and 1(c), it is also evident that the density perturbation and corresponding self-consistent electrostatic potential have developed a strong coherent mode structure with azimuthal mode number $m = 1$. After about 50 lattice periods ($s = 50S$), the instability saturates nonlinearly in the simulation (see Fig. 3), but the coherent $m = 1$ mode structure persists in the nonlinear regime as the beam propagates further.

Additional simulation results are presented in Fig. 2 for beam propagation in the unstable regime. Here, the system parameters are identical to those in Fig. 1. Figure 2 shows a plot of the logarithm of the maximum perturbed density, $\log(\delta n)_{\max}$, versus axial distance s as the beam propagates over 100 lattice periods. Numerical results from both the nonlinear δf-scheme and the linear δf-scheme are presented. Figure 2 also shows the corresponding unnormalized beam emittance in the x-direction, $\varepsilon_x = 4\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)^{1/2}$, obtained in the nonlinear δf-scheme by averaging over the distribution of particles. Note from Fig. 2 that the growth during the linear phase is *exponential*, and $(\delta n)_{\max}$ increases by 6 orders-of-magnitude from the initial noise level. Moreover, the linear and nonlinear δf-schemes are in excellent agreement during the linear growth phase. By $s \simeq 50S$, however, it is evident from the curve in Fig. 2 (nonlinear δf-scheme) that the instability saturates nonlinearly with $(\delta n)_{\max}$ reaching a relatively steady large-amplitude level with $(\delta n)_{\max} \simeq 0.5\ \hat{n}_0$. Following $s = 50S$, the beam continues to evolve nonlinearly. While the coherent $m = 1$ structure persists during the nonlinear phase, at least through 100 lattice periods, the beam quality deteriorates. Indeed, by 80 lattice periods ($s = 80S$), it is found that the transverse dimension of the beam has increased by about 10% relative to the transverse dimension at

$s = 0$. Correspondingly, from Fig. 2, there is a growth in transverse beam emittance, which has increased about 42% during the same propagation interval(from $s = 0$ to $s = 80S$).

Finally, the linear (or nonlinear) δf-schemes can be used to calculate the linear growth rate as a function of beam current over a wide range of system parameters. Typical numerical results are presented in Fig. 3 for $\sigma_v = 85°$. Here, the growth rate (gain) Γ is defined during the linear phase by $\delta n_{\max} = \delta n_0 \exp(\Gamma s/S)$, and the measured value of Γ is plotted in Fig. 3 versus normalized beam current KS/ε_0. For completeness, also shown are self-consistent plots of the space-charge-depressed phase advance σ versus KS/ε_0 calculated for a KV beam equilibrium and $\sigma_v = 85°$. As expected, at sufficiently low values of KS/ε_0 and high values of σ, the KV beam equilibrium is linearly stable. However, as the beam current is increased and the phase advance σ is depressed to sufficiently low values, the system is linearly unstable, and the growth rate Γ can be substantial.

In summary, a new simulation method utilizing the δf-scheme has been used to study the transverse stability properties of an intense ion beam propagating through an alternating-gradient focusing lattice. The δf-scheme is found to be highly effective in describing detailed properties of beam stability and propagation over long distances. It has been shown that the KV beam equilibrium is indeed unstable in certain parameter regimes, particularly at sufficiently high beam current. Furthermore, the mode structure and linear and nonlinear evolution of the instability have been explored over a wide range of system parameters. The optimal parameter regime for stable beam propagation, and the determination of beam propagation properties for other choices of injected distribution function f_0 (e.g., uniform charge density in the $x - y$ cross-section, but Gaussian distribution in transverse $x' - y'$ momentum) will be the subject of a future publication. We also intend to use the δf simulation scheme to study issues such as emittance growth, halo formation, stochasticity, charge homogenization, entropy production and collisionless dissipation.

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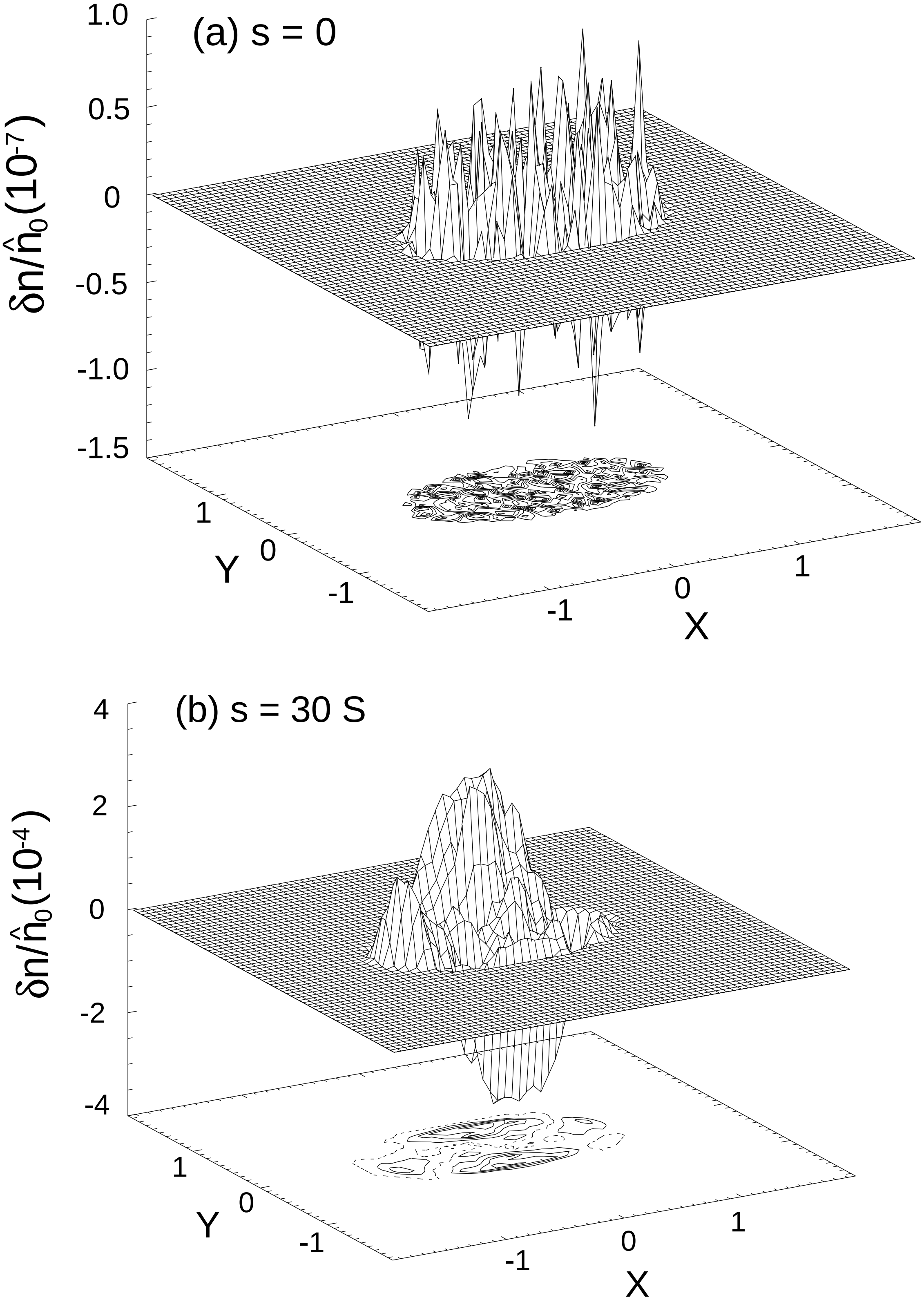
Fig. 1. Plots of perturbed density profiles $\delta n(x,y,s)/\hat{n}_0$ in the transverse (x,y) plane obtained in nonlinear δf-simulation with $50,000$ particles. Profiles are shown at (a) $s=0$, and (b) $s=30S$, and system parameters correspond to $\sigma_v=85°$, $\sigma=20.1°$, and $KS/\varepsilon_0=5.77$. Plot of perturbed electrostatic potential $\delta\phi(x,y,s)$ at $s=30S$ is given in (c).

Fig. 2. Plots of normalized maximum density perturbation $(\delta n)_{\mathrm{max}}/\hat{n}_0$, and transverse emittance $\varepsilon_x/\varepsilon_{x0}$ versus s/S for simulation with $50,000$ particles using the linear and nonlinear δf-schemes. System parameters correspond to $\sigma_v=85°$, $\sigma=20.1°$, and $KS/\varepsilon_0=5.77$.

Fig. 3. Plot of linear growth rate Γ versus normalized beam current KS/ε_0 obtained from linear δf-simulation with $50,000$ particles, for vacuum phase advance $\sigma_v=85°$. Also shown is self-consistent plot of space-charge-depressed phase advance σ versus KS/ε_0.


(a) s = 0
1.0
0.5
0
-0.5
-1.0
-1.5
δn/n̂0(10-7)
1
0
-1
Y
-1
0
1
X
(b) s = 30 S
4
2
0
-2
-4
δn/n̂0(10-4)
1
0
-1
Y
-1
0
1
X

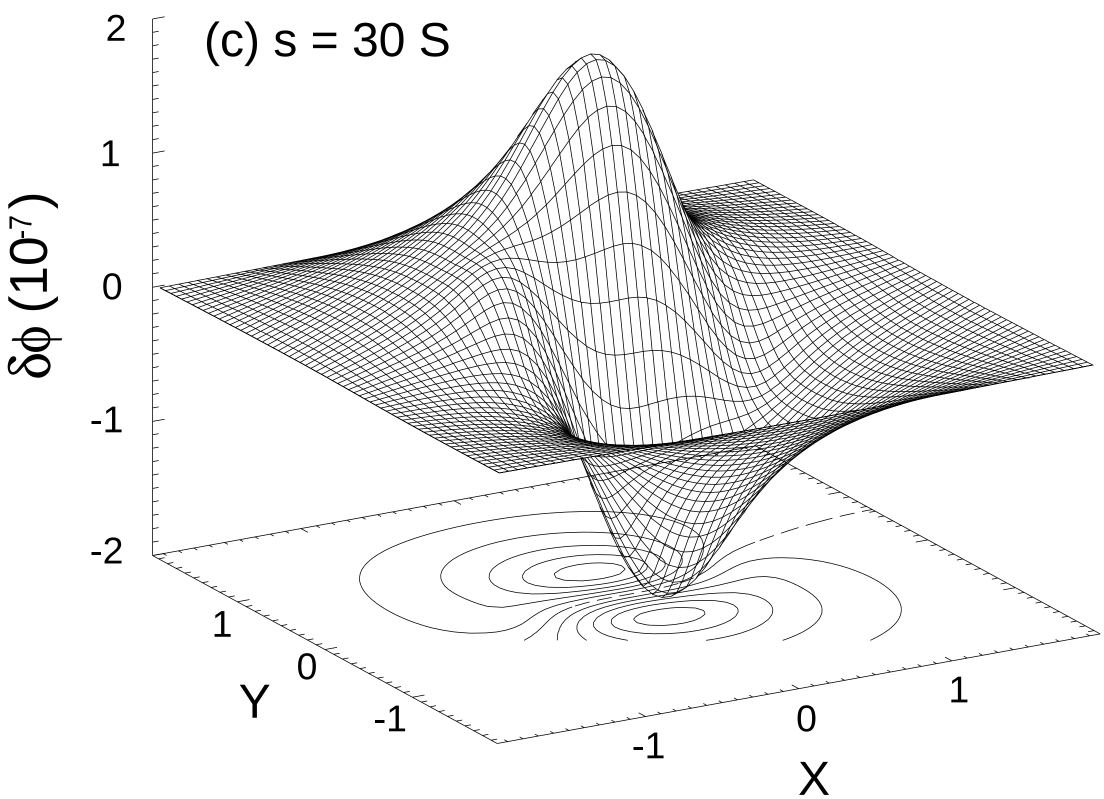
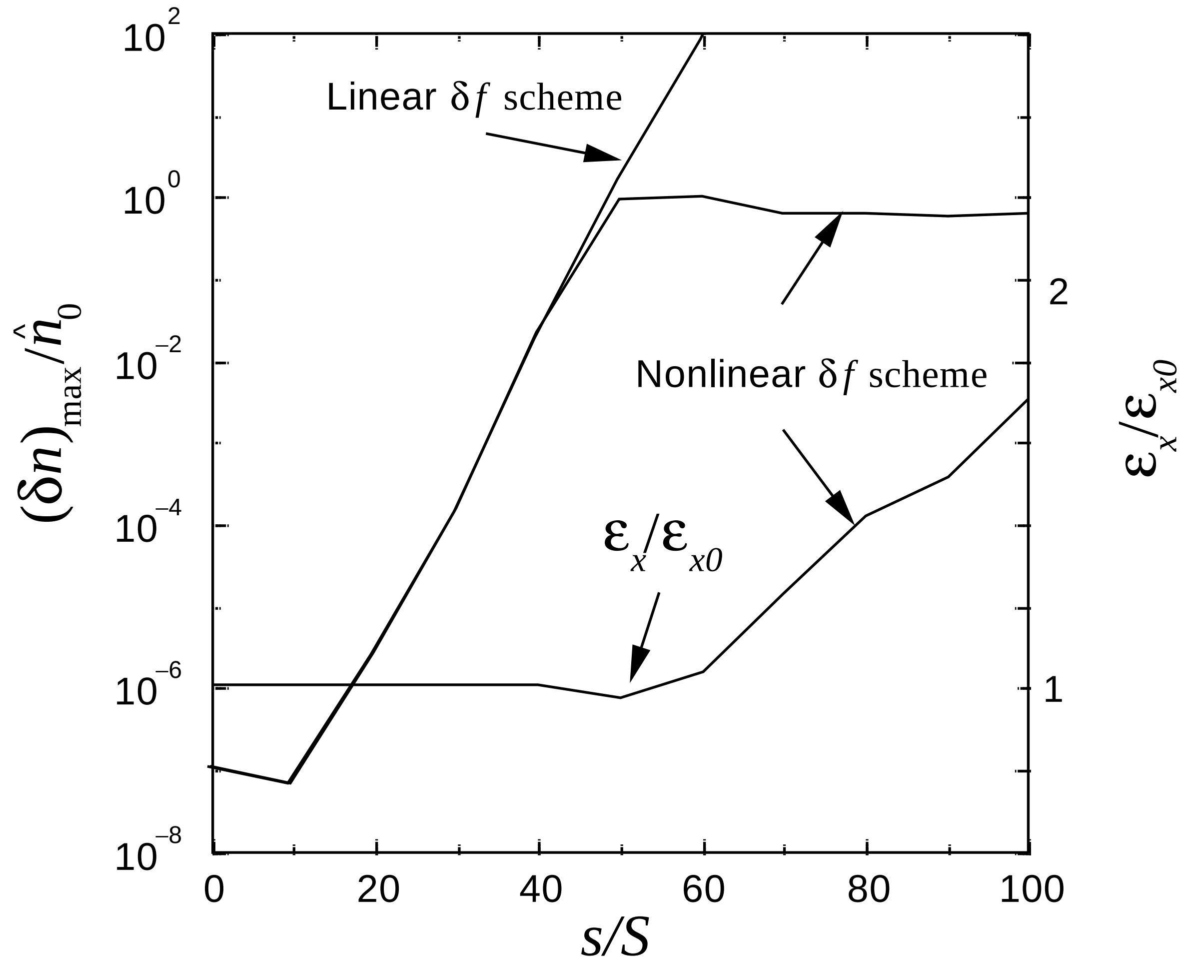


FIG. 1.



Linear δf scheme
Nonlinear δf scheme
$\varepsilon_x/\varepsilon_{x0}$
$(\delta n)_{max}/\hat{n}_0$
s/S
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
0
20
40
60
80
100
2
1

FIG. 2.

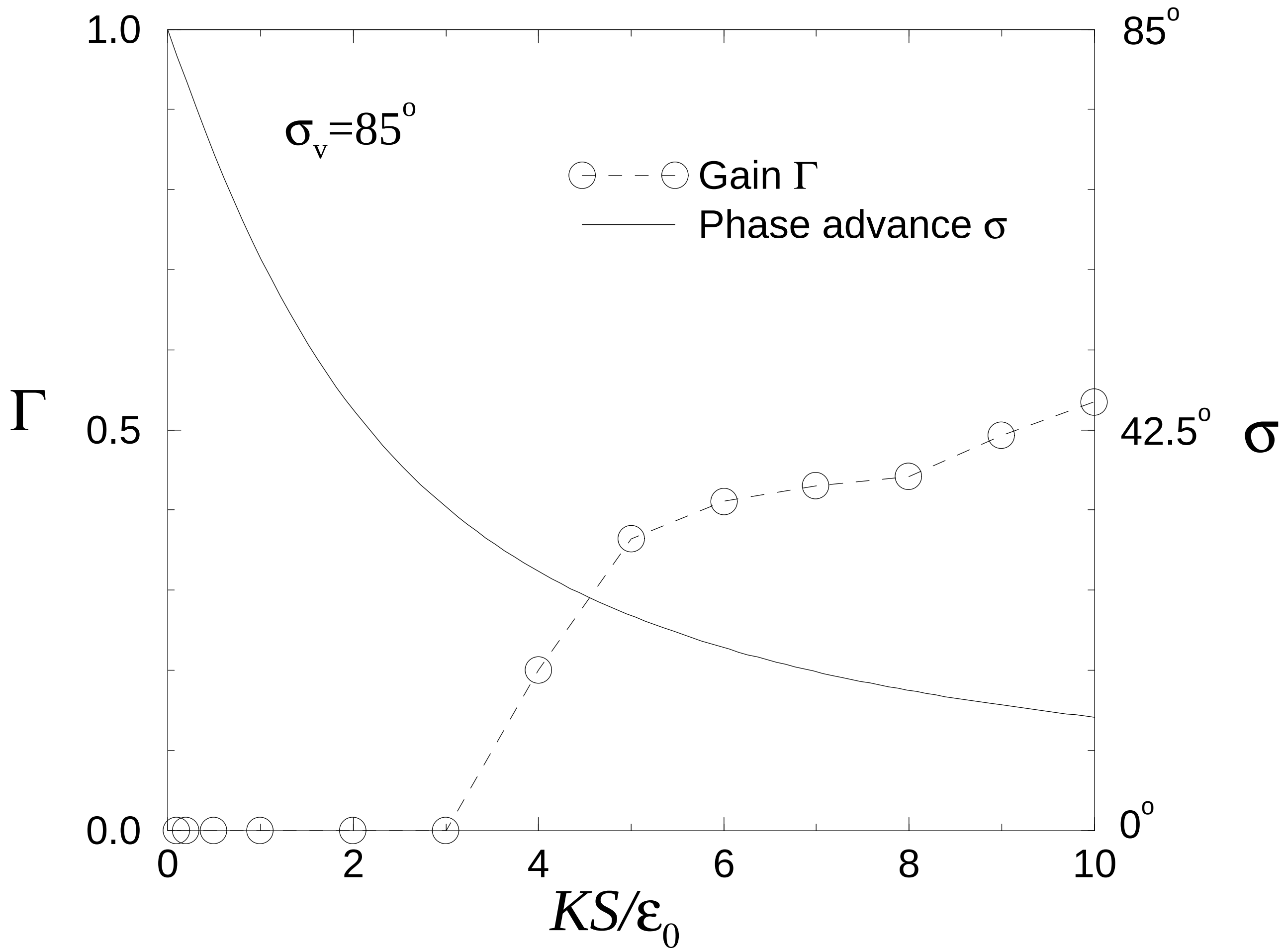


FIG. 3.