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NONLINEAR GYROKINETIC EQUATIONS

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Nonlinear gyrokinetic equations are derived from a systematic Hamiltonian theory. The derivation employs Lie transforms and a noncanonical perturbation theory first used by Littlejohn for the simpler problem of asymptotically small gyroradius. For definiteness, we emphasize the limit of electrostatic fluctuations in slab geometry; however, there is a straightforward generalization to arbitrary field geometry and electromagnetic perturbations. An energy invariant for the nonlinear system is derived, and various of its limits are considered. The weak turbulence theory of the equations is examined. In particular, the wave kinetic equation of Galeev and Sagdeev is derived from an asystematic truncation of the equations, implying that this equation fails to consider all gyrokinetic effects. The equations are simplified for the case of small but finite gyroradius and put in a form suitable for efficient computer simulation. Although it is possible to derive the Terry-Horton and Hasegawa-Mima equations as limiting cases of our theory, several new nonlinear terms absent from conventional theories appear and are discussed. The resulting theory is very similar in content to the recent work of Lee. However, the systematic nature of our derivation provides considerable insight into the structure and interpretation of the equations.

I. INTRODUCTION

It is generally believed that the anomalous transport observed in magnetized fusion plasmas is related to the existence of turbulent fluctuations of frequency much lower than the ion gyrofrequency. However, the description of such fluctuations involves complex nonlinear equations without simple analytic solutions. Furthermore, since such equations often describe collective motions on extremely disparate time scales, straightforward numerical methods are not viable because of practical limitations on computer time and memory. Therefore, in this paper we shall consider a powerful method for the derivation of reduced, nonlinear equations appropriate specifically for the description of low frequency fluctuations in a magnetized plasma, and which are in a form suitable for efficient numerical analysis.

Our principal concern is with the so-called gyrokinetic equations, defined by requiring that the characteristic frequency of the fluctuations be small compared to the ion gyrofrequency, but that the average spatial scale of the fluctuations perpendicular to the magnetic field, $\overline{k_\perp}^{-1}$, be of the same order as the average ion Larmor radius ρ_i ($\overline{k}_\perp \rho_i \approx 1$). For longer wavelengths, $\overline{k}_\perp \rho_i \ll 1$, the equations reduce to the more familiar drift kinetic equations. Although linear gyrokinetic theory is well-understood,[1-3] nonlinear theories necessary to describe possibly turbulent phenomena are still in a state of infancy. Recently, Lee[4] obtained a nonlinear generalization of the linear gyrokinetic equations for the Vlasov-Poisson system which have the desirable property of being in a form suitable for efficient numerical analysis by means of the so-called particle pushing technique. That is, his nonlinear gyrokinetic Vlasov equation can be written as a total time derivative taken along a characteristic in phase space and conserves phase space volume along this characteristic; we call such equations "phase space preserving." However, by construction his equations are valid only for small but finite Larmor radius for the nonlinear terms. Our principal contribution in this paper is to provide equations valid for $k_\perp \rho_i \approx 1$ for the nonlinear terms as well as the linear terms, while retaining the important phase space preserving property. In earlier significant, pioneering work, Frieman and Chen[5,6] followed a perturbative approach to obtain fully gyrokinetic equations, but in doing so they lost the phase space preserving property. They also retained only the $\mathbf{E} \times \mathbf{B}$ nonlinearity, which may not be the only important term in certain interesting regimes. Wong[7] has derived a set of nonlinear gyrokinetic equations which are phase space preserving. However, his formalism, which involves the use of mixed variable generating functions and perturbative expansions of the equations of motion, is algebraically involved and rather opaque, and his final result is missing several terms (related to $\mathbf{E} \times \mathbf{B}$ drift motion).

In this paper we attempt to set nonlinear gyrokinetic theory on a firmer and more transparent theoretical foundation through the use of covariant (non-canonical) Hamiltonian techniques and Lie transformations, a methodology pioneered by Littlejohn[8,9] for the problem of single particle drifts in a specified (non-self-consistent) potential in the drift kinetic ordering ($\bar{k}_\perp \rho_i \ll 1$). The Hamiltonian method has many advantages. Aside from its elegance and simplicity, the approach automatically ensures that the equations will be phase space preserving, and permits a clearer understanding of the underlying dynamical structure (terms in the Vlasov equation can be immediately and easily linked to gyro-center drifts and accelerations; constants of the motion and adiabatic invariants are conspicuous). The covariant structure of Hamilton's equations, long appreciated by mathematical physicists but only recently exploited by plasma physicists, allows new freedom in the choice of coordinates and momenta when one constructs the perturbation theory upon which the averaging procedure depends. Lie transformations, which replace the transformations based on mixed variable generating functions used in more conventional formulations, greatly simplify the form and manipulation of the perturbation series, especially at high order. We employ these powerful mathematical tools to average away the fast gyromotion time scale and so construct the nonlinear gyrokinetic equations governing low frequency fluctuations in a magnetized plasma. Unlike the earlier applications, the resulting equations are self-consistent—that is, the gyrokinetic evolution equation for the distribution function of the gyrocenters involves effective potentials which are self-consistently determined by a gyrokinetic transformation of Maxwell's equations. In fact, the determination of the self-consistent potentials introduces complexity absent from the non-self-consistent problem. To isolate this complexity, and to be as pedagogical as possible, we have chosen to describe here the case of straight constant magnetic field and electrostatic fluctuations (which still describes a wealth of nonlinear physics). However, there is no conceptual difficulty with including electromagnetic and curvature effects; the general theory will be presented elsewhere.

To reiterate, by employing noncanonical coordinates and a gyroaveraged ion distribution function F_i we maximize the simplicity of the gyrokinetic ion Vlasov equation. Other averaging procedures leave the Vlasov equation in a very complicated form because they either rely on cumbersome canonical coordinates or they fail to renormalize the distribution function (see Appendix A, for example). The distribution function which we use has intuitive physical significance. In fact, in the quasineutral approximation, the difference between the gyroaveraged ion density N_i (the velocity space moment of F_i) and the laboratory ion density n_i will be shown to be equal to the contribution to the density fluctuations due to the polarization drift of the ions. Since one effect of the transformation to gyro-center coordinates is to remove the polarization drift from the equations of motion,[10] it is satisfying to see the effects of the drift reappear in the Poisson equation.

The remainder of this paper is organized as follows. In Sec. II we use Littlejohn's technique of noncanonical variables, Darboux transformations, and Lie transformations to construct a gyrokinetic Hamiltonian for a single particle in a potential temporarily assumed to be given. In Sec. III we enforce self-consistency between the particle motion and the potential and use the averaging transformation constructed in Sec. II to derive the gyrokinetic equations for the Vlasov-Poisson system. In Sec. IV we construct an energy invariant for the system and discuss several limiting forms. We devote Sec. V to an exploration of various limits of the equations. For $\overline{k}_\perp \rho_i$ small we obtain what are basically Lee's equations, although there are differences between his equations and ours, which we discuss. In the limit of negligible ion temperature ($T_i \to 0$) we obtain fluid equations from which the Terry-Horton[11] and Hasegawa-Mima[12] equations can be derived. In Sec. VI we find it instructive to consider briefly the weak turbulence theory of our equations. In particular, we point out that the wave kinetic equation of Galeev and Sagdeev[13] follows from a certain truncated set of equations which is formally inconsistent with the gyrokinetic ordering. This indicates that they failed to consider all gyrokinetic effects. We state our conclusions in Sec. VII. In Appendix A we rederive our gyrokinetic equations using a more complicated recursive formalism, and we derive the relationship between the distribution functions used in the two approaches. The recursive method is perhaps more familiar to workers in drift kinetic theory,[14] but has several disadvantages. Along with its relative complexity compared to the Hamiltonian approach, the resulting Vlasov equation is not phase space preserving until a subtle renormalization of the distribution function is effected. In Appendices B and C we quote several intemediate algebraic results, and in Appendix D we sketch the weak turbulence calculation.

II. A SINGLE PARTICLE GYROKINETIC HAMILTONIAN

In all that follows we adopt the well-known "gyrokinetic ordering":

$$\frac{e\phi}{m\bar{v}^2} = O(\epsilon), \qquad \bar{k}_\perp \rho_{\rm avg} = O(1),$$
$$\frac{\rho_{\rm avg}}{L_{eq}} = O(\epsilon), \qquad \bar{k}_\parallel \rho_{\rm avg} = O(\epsilon),$$
$$\frac{\bar{\omega}}{\Omega} = O(\epsilon),$$

where m is the particle's mass, e is the signed charge (in this section we do not commit ourselves to a particular species), $\rho_{\rm avg} \equiv \bar{v}/\Omega, \bar{v}$ is a characteristic particle speed, $\Omega \equiv eB/c$ is the cyclotron frequency, L_{eq} is an equilibrium scale length, $\bar{\omega}$ and $\bar{\mathbf{k}}$ are the characteristic frequency and wavenumber of the perturbed electric field given by $\mathbf{E} = -\nabla\phi$, and ϵ is a small ordering parameter. We define a Larmor radius involving a characteristic speed rather than a temperature since there is at this point only one particle, travelling through externally imposed fields, and temperature is a statistical concept useful only for an ensemble of particles. The gyrokinetic ordering is motivated, in part, by the nonlinear behavior of drift waves. It is consistent with experimental observations, and agrees with simple theories of nonlinear saturation (taking $e\phi/T_e \approx 1/\bar{k}_\perp L_{eq}$). Furthermore, it allows for wave-particle resonance effects, since both $\bar{\omega}$ and $\bar{k}_\parallel$ enter at the same order. It is instructive to compare this gyrokinetic ordering with the so-called drift kinetic ordering, which several of us have discussed elsewhere.[15,16]

The Hamiltonian K for a nonrelativistic charged particle in an electrostatic field is

$$K(\mathbf{x},\mathbf{p},t) = \frac{1}{2}\left(\mathbf{p} - \frac{e}{\epsilon mc}\mathbf{A}(\mathbf{x})\right)^2 + \epsilon\frac{e}{m}\phi(\mathbf{x},t), \tag{1}$$

in canonical coordinates $(\mathbf{x},\mathbf{p})$, where $\mathbf{x}$ is the position, $\mathbf{p}$ is the conjugate momentum variable related to the velocity $\mathbf{v}$ by $\mathbf{p} \equiv \mathbf{v} + (e/\epsilon mc)\mathbf{A}$, $\mathbf{A}$ is the magnetic vector potential (which we take to be time-independent), and t is the time. This Hamiltonian is time-dependent, a property which tends to complicate the averaging procedure. We can circumvent this problem by introducing so-called "extended phase space" canonical coordinates[17] $(\mathbf{x},\mathbf{p},t,w)$, where now t is a coordinate conjugate to w in an extended eight-dimensional phase space. In these coordinates we write the Hamiltonian as

$$H(\mathbf{x},\mathbf{p},t,w) = \frac{1}{2}\left(\mathbf{p} - \frac{e}{\epsilon mc}\mathbf{A}(\mathbf{x})\right)^2 - w + \epsilon\frac{e}{m}\phi(\mathbf{x},t). \tag{2}$$

Since w equals the particle energy along the particle's trajectory through the extended phase space (as can easily be seen by application of Hamilton's equations), the numerical value of the Hamiltonian is zero, and it is thus a constant of the motion.

The motion generated by this Hamiltonian has a fast time scale component describing the Larmor gyrations at the cyclotron frequency. Our goal is to systematically average away these gyrations. To this end, the gyromotion must be isolated. As Littlejohn has pointed out in a series of fundamental papers,[9,18] it is most inconvenient to restrict oneself to canonical coordinates at this point, since the canonical momentum contains both slow and fast time scale effects which greatly complicate the perturbation procedure. Instead, we follow his approach by introducing noncanonical coordinates $(\mathbf{x}, v_\perp, v_\parallel, \theta, t, w)$, where θ is the gyrophase of the gyrating particle:

$$\theta \equiv \tan^{-1}\left(\frac{\mathbf{v}\cdot\hat{\mathbf{x}}}{\mathbf{v}\cdot\hat{\mathbf{y}}}\right).$$

Here $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are arbitrary orthogonal unit vectors in the plane perpendicular to $\mathbf{B}$, and $v_\parallel \equiv \mathbf{v}\cdot\hat{\mathbf{b}}$ and $v_\perp \equiv |\mathbf{v}\times\hat{\mathbf{b}}|$ are the parallel and perpendicular components of the velocity ($\hat{\mathbf{b}} \equiv \mathbf{B}/|\mathbf{B}|$) (see Fig. 1). The fast motion now arises implicitly through the coordinate θ; so if we remove the θ dependence from the equations of motion,we will have achieved our goal of finding equations for the evolution on the slow time scale.

The fact that the coordinates are no longer canonical in no way vitiates the Hamiltonian nature of the equations of motion. Hamiltonian theory can, in fact, be couched in a coordinate-free form; such covariant formulations have been discussed extensively[19,20] and we shall not attempt a detailed examination of the subject. For our purposes it will be sufficient to state several of the main results of this theory; interested readers are referred to the literature.

In generalized (not necessarily canonical) coordinates, $\mathbf{z}$, Hamilton's equations take the form

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, H\}, \tag{3}$$

where $H(\mathbf{z})$ is the Hamiltonian and we have introduced "Poisson bracket" notation. The Poisson bracket of two phase functions f and g is defined by

$$\{f,g\} \equiv \frac{\partial f}{\partial \mathbf{z}} \cdot \mathbf{J} \cdot \frac{\partial g}{\partial \mathbf{z}}, \tag{4a}$$

where $\mathbf{J}$ is an antisymmetric contravariant tensor called the Poisson tensor. This tensor can be defined by its form in canonical coordinates $\mathbf{z}_c \equiv (\mathbf{x}, \mathbf{p})$:

$$\mathbf{J}(\mathbf{z}_c) \equiv \begin{pmatrix} \mathbf{O} & -\mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}, \tag{4b}$$

where $\mathbf{O}$ and $\mathbf{I}$ are the 3×3 null and unit matrices. Since $\mathbf{J}$ transforms contravariantly, it is possible to find its form in any set of coordinates $\bar{\mathbf{z}}$ connected via a diffeomorphism to canonical coordinates:

$$\mathbf{J}(\bar{\mathbf{z}}) = \frac{\partial \bar{\mathbf{z}}}{\partial \mathbf{z}} \cdot \mathbf{J}(\mathbf{z}) \cdot \frac{\partial \bar{\mathbf{z}}}{\partial \mathbf{z}}. \tag{5}$$

In our chosen coordinates $(\mathbf{x}, v_\perp, v_\parallel, \theta, t, w)$, the elements of $\mathbf{J}$ are given in Appendix B. In these coordinates the Hamiltonian becomes

$$H'(\mathbf{x}, v_\perp, v_\parallel, \theta, t, w) = \frac{v_\perp^2}{2} + \frac{v_\parallel^2}{2} - w + \epsilon \frac{e}{m} \phi(\mathbf{x}, t). \tag{6}$$

Although H' is θ-independent, θ dependence, and hence a fast time scale, is introduced into the equations of motion by the Poisson tensor. However, in order to expedite an averaging procedure to remove the fast time scale, we want all of the θ dependence in the Hamiltonian and none in the Poisson brackets. We can remove all θ dependence from the Poisson brackets by using a Darboux transformation[18] to a new set of "semi-canonical" coordinates

$$\mathbf{Z} \equiv (\mathbf{X}, \mu, \theta, U, T, W).$$

(More general approaches using an action form[21] could also be applied; but for the present case of straight magnetic field and electrostatic fluctuations, our method is quite adequate, and perhaps more physical.) This transformation is defined by the following prescription: given a coordinate (say θ), find the coordinate (call it μ) which is canonically conjugate to it, and construct the other variables of the system by requiring that they commute with θ and μ. This decouples the fast scale from the slow scale to lowest order in ϵ, allowing for a more "natural" description of the motion as a fast gyration superimposed on a slow drift both along the field lines (due to the near-constant parallel velocity) and across them (due to the electrostatic perturbation). In mathematical terms, we must solve the following set of coupled differential equations:

$$\{\theta, \mu\} = 1,$$
$$\{\theta, \mathbf{Z}\} = 0 \quad \text{for all } \mathbf{Z} \text{ except } \mu,$$
$$\{\mu, \mathbf{Z}\} = 0 \quad \text{for all } \mathbf{Z} \text{ except } \theta,$$

subject to given initial conditions; we take $\mu = 0, \mathbf{X} = \mathbf{x}, U = v_{\|}, T = t$, and $W = w$ at $v_{\perp} = 0$. We are guaranteed that these equations do, in fact, have a solution as long as the phase space manifold is "symplectic"—i.e., that a closed nondegenerate two-form exists on the manifold.[19] Since the inverse of the Poisson tensor is just such a two-form, we can solve these equations. We obtain

$$\mathbf{X} = \mathbf{x} - \boldsymbol{\rho}, \quad U = v_{\|}, \quad T = t, \quad W = w, \quad \text{and} \quad \mu = \frac{v_{\perp}^2}{2\Omega}, \tag{7}$$

where $\boldsymbol{\rho} \equiv v_{\perp}\hat{\mathbf{a}}/\Omega$ and $\hat{\mathbf{a}}$ is a unit vector defined in Fig. 1 and Appendix B. We see that $\mathbf{X}$ is the *lowest order* (in ϵ) guiding center position, and μ is the *lowest order* adiabatic invariant;[15] i.e., μ is merely the first term in an asymptotic series for the exact invariant, which we call $\overline{\mu}$ and shall use later. Elements of the Poisson tensor in Darboux-transformed coordinates are displayed in Appendix B. The Hamiltonian transforms to

$$\hat{H}(\mathbf{Z}) = \mu\Omega + \frac{1}{2}U^2 - W + \epsilon\frac{e}{m}\phi(\mathbf{X} + \boldsymbol{\rho}, t). \tag{8}$$

Since θ dependence now appears only in the perturbation Hamiltonian (the term proportional to ϕ), $\hat{H}$ is now in a form suitable for averaging. Although mixed variable generating functions[17] could be employed, we find that Lie transformations greatly simplify the algebra involved in the averaging procedure. Several very good elucidations of the theory and application of Lie transforms may be found in the literature;[22,23] one particularly readable elementary account is that of Littlejohn.[24] The transformation equations from coordinates $\mathbf{Z}$ to gyroaveraged coordinates $\overline{\mathbf{Z}}$ are

$$\overline{\mathbf{Z}} = \mathcal{T}\mathbf{Z}; \tag{9a}$$

the gyrokinetic Hamiltonian is therefore given by

$$\overline{H} = \mathcal{T}^{-1}\hat{H}, \tag{9b}$$

since $\mathcal{T}$ is an area preserving (symplectic) near-identity transformation. The transformation $\mathcal{T}$ is defined to be

$$\mathcal{T} \equiv \exp(-\textstyle\int d\epsilon\, L), \tag{9c}$$

where

$$L \equiv \sum_{n=1}^{\infty} \epsilon^{n-1} L_n,$$

$$L_n \equiv \{G_n,\ \}.$$

The G_ns are called the generating functions of the transformation. Upon expanding $\mathcal{T}$, $\overline{H}$, and $\hat{H}$ as power series in ϵ, we find that

$$\overline{H}_0 = \hat{H}_0, \tag{10a}$$

$$\overline{H}_1 = \hat{H}_1 + L_{10}\hat{H}_0, \tag{10b}$$

$$\overline{H}_2 = \hat{H}_2 + L_{10}\hat{H}_1 + \frac{1}{2}(L_{20} + L_{10}^2 + 2L_{11})\hat{H}_0, \tag{10c}$$

where we have broken L_n up into a zeroth order part and a first order part:

$$L_n \equiv L_{n0} + \epsilon L_{n1},$$
$$L_{n0} = \frac{\partial G_n}{\partial \theta}\frac{\partial}{\partial \mu} - \frac{\partial G_n}{\partial \mu}\frac{\partial}{\partial \theta} + \frac{\nabla G_n}{\Omega}\cdot \hat{\mathbf{b}} \times \nabla,$$
$$L_{n1} = \hat{\mathbf{b}}\cdot \nabla G_n \frac{\partial}{\partial U} - \frac{\partial G_n}{\partial U}\hat{\mathbf{b}}\cdot\nabla - \frac{\partial G_n}{\partial T}\frac{\partial}{\partial W}.$$

Up to this point the transformation has been arbitrary. We now determine the generating functions of the transformation by solving Eqs. (10) subject to the conditions that the new Hamiltonian $\overline{H}$ be θ-independent and that the generating functions contain no θ-independent parts (which would lead to secularities). Thus, by absorbing all θ dependence into the generating functions, we are able to make $\overline{H}$ gyrophase-independent order by order. Furthermore, since this Lie transform is area-preserving, the functional form of the Poisson tensor remains unchanged under the transformation.[18] The Hamiltonian becomes

$$\begin{aligned}\overline{H}(\overline{\mathbf{Z}}) =& \overline{\mu}\Omega + \frac{1}{2}\overline{U}^2 - \overline{W} + \epsilon\frac{e}{m}\overline{\phi} \\ & - \frac{\epsilon^2 e^2}{2m^2\Omega}\left(\frac{\partial}{\partial\overline{\mu}}\langle\tilde{\phi}^2\rangle + \langle\frac{\overline{\nabla}\tilde{\Phi}}{\Omega}\cdot\hat{\mathbf{b}}\times\overline{\nabla}\tilde{\phi}\rangle\right) + O(\epsilon^3),\end{aligned} \quad (11)$$

where

$$\begin{aligned}\overline{\phi}(\overline{\mathbf{X}},\overline{\mu},\overline{T}) &\equiv \langle\phi(\overline{\mathbf{X}}+\overline{\boldsymbol{\rho}},\overline{T})\rangle \\ &\equiv \frac{1}{2\pi}\int_0^{2\pi} d\overline{\theta}\ \phi(\overline{\mathbf{X}}+\overline{\boldsymbol{\rho}},\overline{T}) \\ &= \int \frac{d^3k}{(2\pi)^3}\phi_{\mathbf{k}} J_0(k_\perp\overline{\rho})\exp(i\mathbf{k}\cdot\overline{\mathbf{X}}),\end{aligned}$$

$$\tilde{\phi}(\overline{\mathbf{X}},\overline{\mu},\overline{\theta},\overline{T}) \equiv \phi(\overline{\mathbf{X}}+\overline{\boldsymbol{\rho}},\overline{T}) - \overline{\phi},$$
$$\tilde{\Phi}(\overline{\mathbf{X}},\overline{\mu},\overline{\theta},\overline{T}) \equiv \int^{\overline{\theta}} \tilde{\phi}\ d\overline{\theta},$$

where $\overline{\boldsymbol{\rho}} \equiv (2\overline{\mu}/\Omega)^{1/2}\hat{\mathbf{a}}(\overline{\theta})$. The generating functions G_1 and G_2 are also determined at this order; they are displayed in Appendix C.

The quantity $\overline{\mu}$ is the true adiabatic invariant (correct to all orders in ϵ in our formulation): $\overline{\mu} = \Upsilon\mu$; the other barred quantities have similar interpretations. By assuming that the series for Υ converges, we are ignoring the possibility of stochasticity[10] arising from resonant interactions between the gyromotion and other motions of the system. This stochasticity can often be shown to be unimportant.[10,25] The first order contribution $\overline{\phi}$ to $\overline{H}$ is the potential averaged over a Larmor orbit, familiar to workers in gyrokinetics. The second order contributions to $\overline{H}$ are related to the change in the lowest order invariant, $v_\perp^2/2\Omega$, due to the electric field fluctuations, and to the change in position of the guiding center due to the $\mathbf{E}\times\mathbf{B}$ drift. This can by seen by performing a simple analysis of the unaveraged equations of motion, obtained from Eqs. (3), (8), and (B2):

$$\frac{d\theta}{dT} = \Omega + \epsilon\frac{e}{m}\frac{\partial}{\partial\mu}\phi(\mathbf{X}+\boldsymbol{\rho},T),$$
$$\frac{d\mu}{dT} = -\epsilon\frac{e}{m}\frac{\partial}{\partial\theta}\phi(\mathbf{X}+\boldsymbol{\rho},T),$$
$$\frac{d\mathbf{X}}{dT} = -\epsilon\frac{e}{m}\frac{\nabla\phi}{\Omega}(\mathbf{X}+\boldsymbol{\rho},T)\times\hat{\mathbf{b}}.$$

Solving perturbatively, we find that

$$\theta_0 = \Omega T, \quad \mu_0, \mathbf{X}_0 \quad \text{are constant,}$$

and

$$\mu_1 = -\epsilon\frac{e}{m\Omega}\tilde{\phi},$$
$$\mathbf{X}_1 = -\epsilon\frac{e}{m\Omega}(\nabla\overline{\phi}\times\hat{\mathbf{b}}T + \frac{\nabla\tilde{\Phi}}{\Omega}\times\hat{\mathbf{b}}) \equiv \overline{\mathbf{v}}_E T + \tilde{\mathbf{X}}_1,$$
$$\theta_1 = \epsilon\frac{e}{m\Omega}\frac{\partial}{\partial\mu}(\overline{\phi}T + \frac{\tilde{\Phi}}{\Omega}) \equiv \Omega_1 T + \tilde{\theta}_1.$$

Thus

$$\frac{d\theta_2}{dT} = \epsilon\frac{e}{m}(\tilde{\mathbf{X}}_1\cdot\nabla\frac{\partial\hat{\phi}}{\partial\mu} + \mu_1\frac{\partial^2\hat{\phi}}{\partial\mu^2} + \theta_1\frac{\partial^2\hat{\phi}}{\partial\mu\partial\theta}),$$

where $\tilde{\phi} \equiv \phi[\mathbf{X}_0 + \overline{\mathbf{v}}_E T + \boldsymbol{\rho}(\overline{\Omega}T), T]$ and $\overline{\Omega} \equiv \Omega + \Omega_1$. Upon averaging this equation over the fast time scale $\overline{\Omega}T$ we find

$$\begin{aligned}\langle\frac{d\theta_2}{dT}\rangle &= \epsilon\frac{e}{m}\left(-\frac{e}{m\Omega^2}\langle\nabla\tilde{\Phi}\times\hat{\mathbf{b}}\cdot\nabla\frac{\partial\tilde{\phi}}{\partial\mu}\rangle + \frac{e}{m\Omega}\langle\frac{\partial\tilde{\Phi}}{\partial\mu}\frac{\partial^2\tilde{\phi}}{\partial\mu\partial\theta}\rangle - \frac{e}{m\Omega}\langle\tilde{\phi}\frac{\partial^2\tilde{\phi}}{\partial\mu^2}\rangle\right)\\ &= -\frac{\epsilon e^2}{2m^2\Omega}\frac{\partial}{\partial\mu}\left(\langle\frac{\nabla\tilde{\Phi}}{\Omega}\times\hat{\mathbf{b}}\cdot\nabla\tilde{\phi}\rangle + \frac{\partial}{\partial\mu}\langle\tilde{\phi}^2\rangle\right),\end{aligned}$$

which is the same as we would obtain using Eq. (3) with $\overline{H}$.

III. THE GYROKINETIC VLASOV-POISSON SYSTEM

In this section we use the single particle gyrokinetic Hamiltonian to find a Vlasov equation for the gyroaveraged ion distribution function F_i, and we enforce self-consistency by writing the Poisson equation in terms of F_i. The electrons are assumed to be drift kinetic and the appropriate Vlasov equation is derived by taking the drift kinetic limit of the gyrokinetic Hamiltonian.

In canonical coordinates the Vlasov and Poisson equations are

$$\{f_i(\mathbf{x},\mathbf{p},t), H_i(\mathbf{x},\mathbf{p},t,w)\} = 0, \tag{12}$$

$$\nabla^2\phi(\mathbf{x},t) = -4\pi e\left[\int f_i(\mathbf{x}',\mathbf{p}',t)\delta(\mathbf{x}-\mathbf{x}')d^6z' - n_e\right], \tag{13}$$

where n_e is the electron density, f_i is the ion distribution function, and H_i is the ion Hamiltonian. We have inserted the delta function in Eq. (13) in order to expedite the coordinate transformations in the six-dimensional phase space $(\mathbf{x},\mathbf{p})$. We need not integrate over the full extended phase space because t is never changed during the transformations and the integrand is not a function of w. Defining a distribution function g_i in terms of the Darboux transformed coordinates,

$$g_i(\mathbf{X},\mu,U,\theta,T) \equiv f_i(\mathbf{x},\mathbf{p},t),$$

Eqs. (12) and (13) become

$$\{g_i, \hat{H}_i\} = 0, \tag{14}$$

$$\nabla^2 \phi = -4\pi e \left[\int g_i(\mathbf{Z}) \delta(\mathbf{X} - \mathbf{x} + \boldsymbol{\rho}) d^6 Z - n_e \right], \tag{15}$$

where $d^6 Z \equiv \|\partial z / \partial Z\| d^3 X d\mu dU d\theta$. We now apply the averaging transformation $\mathcal{T}^{-1}$ to Eq. (14). Since $\mathcal{T}^{-1}$ is area preserving, it commutes with the Poisson brackets (the form of the Poisson tensor remains unchanged):

$$\begin{aligned} \mathcal{T}^{-1}\{g_i, \hat{H}_i\} = 0 &= \{\mathcal{T}^{-1} g_i, \mathcal{T}^{-1} \hat{H}_i\} \\ &= \{F_i, \overline{H_i}\}, \end{aligned} \tag{16}$$

where we have defined $F_i \equiv \mathcal{T}^{-1} g_i$ and have used Eq. (9b).

We may now average Eq. (16) over θ to obtain equations for the average part, $\overline{F}_i$, and the fluctuating part, $\tilde{F}_i$, of F_i:

$$\{\overline{F}_i, \overline{H}_i\} = 0, \quad \{\tilde{F}_i, \overline{H}_i\} = 0.$$

Since we are not interested in $\tilde{F}_i$, and since $\tilde{F}_i$ appears nowhere in the equation for $\overline{F}_i$, we can set $\tilde{F}_i = 0$ to obtain the gyrokinetic Vlasov equation:

$$\{F_i, \overline{H}_i\} = 0, \tag{17}$$

where $F_i = \langle \mathcal{T}^{-1} g_i \rangle$ and $\mathcal{T}^{-1} g_i = \langle \mathcal{T}^{-1} g_i \rangle$. Actually, setting $\tilde{F}_i = 0$ is equivalent, by definition of F_i, to choosing a particular set of initial conditions for f_i. Since choice of a particular gyrophase distribution at $t = 0$ has negligible effect on the long-time evolution of the system, we are justified in doing this.

By virtue of the relation between g_i and F_i given above, we can write the Poisson equation as

$$\nabla^2 \phi(\mathbf{x}, t) = -4\pi e \left[\int \left[\mathcal{T} F_i(\overline{\mathbf{Z}}) \right] \delta(\overline{\mathbf{X}} - \mathbf{x} + \overline{\boldsymbol{\rho}}) d^6 \overline{Z} - n_e \right]. \tag{18}$$

This is the gyrokinetic Poisson equation. The quasineutrality condition is obtained by setting the right hand side of Eq. (18) equal to zero.

So far in this section we have made no approximations, and have introduced no ordering parameters. We need only find the form of $\mathcal{T}$ and $\overline{H}$ using Lie transformations to obtain gyrokinetic equations good to any order in ϵ that we wish. Of course, this again assumes that the time scales are sufficiently disparate so as to render negligible the stochastic regions around resonances. To $O(\epsilon^3)$ the Vlasov and Poisson equations are

$$\frac{\partial F_i}{\partial \overline{T}} + (\overline{U}\hat{\mathbf{b}} - \epsilon\frac{e}{m_i\Omega_i}\overline{\nabla}\psi \times \hat{\mathbf{b}}) \cdot \overline{\nabla}F_i - \epsilon\frac{e}{m_i}\hat{\mathbf{b}} \cdot \overline{\nabla}\psi\frac{\partial F_i}{\partial \overline{U}} = O(\epsilon^3), \qquad (19a)$$

where

$$\psi = \overline{\phi} - \frac{\epsilon e}{2m_i\Omega_i}\left(\frac{\partial}{\partial\overline{\mu}}\langle\tilde{\phi}^2\rangle + \langle\frac{\overline{\nabla}\tilde{\Phi}}{\Omega_i} \cdot \hat{\mathbf{b}} \times \overline{\nabla}\tilde{\phi}\rangle\right), \qquad (19b)$$

and

$$\nabla^2\phi(\mathbf{x},t) = -4\pi e\Bigg[\int \Bigg(F_i + \epsilon\frac{e}{m_i\Omega_i}(\tilde{\phi}\frac{\partial F_i}{\partial\overline{\mu}} + \frac{\overline{\nabla}\tilde{\Phi}}{\Omega_i} \cdot \hat{\mathbf{b}} \times \overline{\nabla}F_i)$$
$$+ \frac{\epsilon^2 e^2}{2{m_i}^2\Omega_i^2}(-[2\tilde{\phi}\frac{\partial\overline{\phi}}{\partial\overline{\mu}} - \frac{\partial}{\partial\overline{\mu}}\langle\tilde{\phi}^2\rangle - \langle\frac{\overline{\nabla}\tilde{\Phi}}{\Omega_i} \cdot \hat{\mathbf{b}} \times \overline{\nabla}\tilde{\phi}\rangle + 2\frac{\overline{\nabla}\tilde{\Phi}}{\Omega_i} \cdot \hat{\mathbf{b}} \times \overline{\nabla\phi}]\frac{\partial F_i}{\partial\overline{\mu}}$$
$$+ \tilde{\phi}^2\frac{\partial^2 F_i}{\partial\overline{\mu}^2}) + O(\epsilon^3)\Bigg)\delta(\overline{\mathbf{X}} - \mathbf{x} + \overline{\boldsymbol{\rho}})d^6\overline{Z} - n_e\Bigg], \qquad (20)$$

where $\overline{\nabla}F \equiv \partial F/\partial\overline{\mathbf{X}}$, $\nabla F \equiv \partial F/\partial\mathbf{x}$, ρ and μ are now specifically defined in terms of the ions, $\overline{\rho} \equiv (2\overline{\mu}/\Omega_i)^{1/2}$, m_i is the ion mass, and Ω_i is the ion gyrofrequency.

Although there are $O(\epsilon^3)$ corrections to Eq. (19a), we are guaranteed that it will be phase space preserving by the Hamiltonian approach. A word of caution in interpreting the order of the terms of Eq. (19a): taking $|\overline{\nabla}F| = O(\epsilon)$ has allowed us to divide out a factor of ϵ from the equation. This must be taken into account when determining what terms may be kept at any particular order. Thus, to $O(\epsilon^3)$ we must keep terms like $\hat{\mathbf{b}} \cdot \overline{\nabla\phi} \partial(\delta F_i)/\partial\overline{U}$.

The electron drift kinetic equation can be easily derived by taking the drift kinetic limit of the gyrokinetic Vlasov equation. Since the electron polarization drift is much smaller than that of the ions, we can neglect its effects in the gyrokinetic Poisson equation (see Sec. V) and use the simplest possible equations to govern the drift kinetic electron motion:

$$\frac{\partial f_e}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \frac{e}{m_e\Omega_e}\nabla\phi \times \hat{\mathbf{b}}) \cdot \nabla f_e + \frac{e}{m_e}\hat{\mathbf{b}} \cdot \nabla\phi \frac{\partial f_e}{\partial v_{\|}} = 0, \tag{21}$$

where Ω_e is the (signed) electron gyrofrequency, m_e is the electron mass, and where $n_e = \int f_e d^3v$ in the gyrokinetic Poisson equation. This equation, together with Eqs. (19) and (20), constitute a closed set of equations describing low frequency electrostatic plasma fluctuations. In the next section we will examine the energy conservation properties of this system.

Although we have accomplished our goal of removing the fast time scale from the equations of motion, it is not necessarily true that the equations, when used self-consistently, generate solutions with no high frequency component. We may be sure that fluctuations such as the ion or electron Bernstein modes, which depend on the cyclotron resonance, will not appear, but other high frequency modes (such as plasma oscillations) are not ruled out *a priori*. Since a thorough treatment of the normal modes of the system is outside the scope of this paper, which is primarily concerned with the form of the nonlinear interactions, we will only briefly touch upon this interesting subject. By linearizing Eqs. (19) and (20), it is a straightforward excercise to construct the linear gyrokinetic dispersion relation for a shearless slab with a density gradient. Assuming that the electron and ion background distributions are Maxwellian, we find two modes: the usual drift-ion-acoustic mode, and a finite-$k_\parallel$ variant of the convective cell which turns into an electron plasma oscillation in the limit of small $k_\perp$. Only the latter mode exhibits possible high frequency behavior, and so it is to this mode that we devote our attention. In the high frequency limit ($\omega \gg k_\parallel v_i, k_\parallel v_e$, where v_i and v_e are the thermal velocities), the dispersion relation for the real part of this mode can be written as

$$\omega^2 = \frac{k_\parallel^2 v_e^2}{k^2 \lambda_{D_e}^2 + k_\perp^2 \rho_s^2},$$

in the limit of small $k_\perp \rho_i$, where $\lambda_{D_e}^{-2} \equiv (4\pi e^2 n_0)/T_e$, n_0 is the background density, $\rho_s^2 \equiv T_e/(m_i \Omega_i)$, and T_e is the electron temperature. Thus, as long as $k_\perp \rho_s > k\lambda_{De}$ and $k_\parallel/k_\perp < (m_e/m_i)^{1/2}$ there are no high frequency ($\omega \gtrsim \Omega_i$) roots. We may therefore use the full Poisson equation rather than the quasineutral approximation with no fear of the equations generating high frequency noise. (Although quasineutrality is often adequate, it is sometimes not uniformly satisfactory in inhomogeneous systems.) However, if the numerical scheme being used is such that there is no control over the size of $k_\perp$, it is important to be on guard against this possibility.

IV. ENERGY CONSERVATION

The Hamiltonian nature of the system and the elegance of the Lie transform approach allow us to find simple general expressions for the conservation laws of the gyrokinetic system. The method we employ is applicable to all the conservation laws; as an example of the general technique we consider energy conservation. There are several ways to attack this problem. One is to seek a conserved moment of the gyrokinetic equation of motion, where the moment is taken over all phase space coordinates but time and energy. However, there exists no general procedure which dictates the appropriate moment. Therefore, we adopt an alternative procedure in which we begin with the conservation law in laboratory coordinates, then transform this law into the gyrokinetic variables. The integrands appearing in the moment equations of this section are sometimes written in terms of barred (averaged) variables, and sometimes in terms of the unbarred variables; it really makes no difference since they appear as dummy variables in the integrations and it should be clear from the context which set is being used in any particular equation. The well-known energy constant of the Vlasov-Poisson system is:

$$\mathcal{E} = \int \frac{m_i v^2}{2} f_i d^6 z + \int \frac{m_e v^2}{2} f_e d^6 z + \int \frac{|\mathbf{E}|^2}{8\pi} d^3 x. \tag{22}$$

Since we employ the simplest possible electron drift kinetic equation, $v_\perp$ is a constant of the electron motion and only the parallel electron kinetic energy plays a role in the energy conservation of the system. Using the averaging transformation $\mathcal{T}$ it is not difficult to write the ion kinetic energy in terms of F_i:

$$K_i \equiv \int \frac{m_i v^2}{2} f_i d^6 z = \int m_i(\mu\Omega_i + \frac{U^2}{2}) \mathcal{T} F_i d^6 Z.$$

This can be couched in a more useful form by means of the following "integration by parts" theorem:

$$\int g \mathcal{T} f d^6 z = \int f \mathcal{T}^{-1} g d^6 z \quad \text{for all } f,\, g \text{ independent of } w.$$

Applying this to the expression for K_i yields the gyrokinetic energy conservation law:

$$\mathcal{E} = \int F_i(\mathbf{Z})\mathrm{T}^{-1} m_i(\mu\Omega_i + \frac{U^2}{2})d^6Z + \int f_e \frac{m_e v^2}{2} d^6 z + \int \frac{|\mathbf{E}|^2}{8\pi} d^3 x = \text{constant.} \tag{23}$$

Of course, one never deals with the exact gyrokinetic system, but rather with some asymptotic approximation to it, worked out to some order in ϵ. The averaging transformation can then be expanded out to obtain an expression for the energy as accurate as needed for any particular application. In fact, in at least two cases Eq. (23) provides exact invariants for approximate gyrokinetic systems. By dropping the terms which are quadratic in ϕ in the Poisson equation Eq. (20), we obtain

$$\nabla^2\phi(\mathbf{x},t) = -4\pi e\left[\int [F_i + \epsilon\frac{e}{m_i\Omega_i}(\tilde{\phi}\frac{\partial F_i}{\partial\overline{\mu}} + \frac{\overline{\nabla}\tilde{\Phi}}{\Omega_i}\cdot\hat{\mathbf{b}}\times\overline{\nabla}F_i)]\delta(\overline{\mathbf{X}} - \mathbf{x} + \overline{\boldsymbol{\rho}})d^6\overline{Z} - n_e\right]. \tag{24}$$

This equation, along with the gyrokinetic ion Vlasov equation (19) and the electron drift kinetic equation (21), form a system with the following exact energy invariant:

$$\frac{\partial}{\partial t}\Bigg(\int m_i(\mu\Omega_i + \frac{U^2}{2})F_i d^6Z + \int \frac{m_e v^2}{2} f_e d^6 z + \int \frac{|\mathbf{E}|^2}{8\pi}d^3x + \frac{e^2}{2m_i\Omega_i}\int (\frac{\partial}{\partial\mu}\langle\tilde{\phi}^2\rangle + \langle\frac{\nabla\tilde{\Phi}}{\Omega_i}\cdot\hat{\mathbf{b}}\times\nabla\tilde{\phi}\rangle)F_i d^6Z\Bigg) = 0. \tag{25}$$

This formula can easily be verified by taking the kinetic energy moments of Eqs. (19) and (21), subtracting them and substituting for the ion density using Eq. (24). The system can be further simplified while preserving energy conservation by dropping the nonlinear terms in ψ in Eq. (19b) and by linearizing the Poisson equation. The gyrokinetic Vlasov and Poisson equations are then

$$\frac{\partial F_i}{\partial\overline{T}} + (\overline{U}\hat{\mathbf{b}} - \frac{e}{m_i\Omega_i}\overline{\nabla\phi}\times\hat{\mathbf{b}})\cdot\overline{\nabla}F_i - \frac{e}{m_i}\hat{\mathbf{b}}\cdot\overline{\nabla\phi}\frac{\partial F_i}{\partial\overline{U}} = 0, \tag{26}$$

and

$$\left(k^2\lambda_{Di}^2+1-\Gamma_0(b)+[\Gamma_1(b)-\Gamma_0(b)]i\mathbf{k}_\perp\cdot\nabla_\perp\ln n_0\rho_i^2\right)\frac{e\phi_{\mathbf{k}}}{T_i}n_0$$
$$=\int J_0(k_\perp\rho)F_{i\mathbf{k}}2\pi\Omega_i d\mu dU-n_{e\mathbf{k}}, \tag{27}$$

where, for the purposes of computation, we have assumed that the background distribution is Maxwellian in μ with temperature T_i and linearly varying density n_0, $\Gamma_n(b)\equiv I_n(b)\exp(-b)$, I_n is a modified Bessel function of order n , $\lambda_{Di}^{-2}\equiv(4\pi e^2n_0)/T_i$, $\rho_i^2\equiv T_i/m_i\Omega_i^2$, $b\equiv k_\perp^2\rho_i^2$, and we have Fourier transformed the Poisson equation. The term in Eq. (27) proportional to $i\mathbf{k}_\perp$ is neglected by most authors. These equations describe drift waves in the linear stage of growth, and they have the following exact energy invariant:

$$\frac{\partial}{\partial t}\left(\int m_i(\mu\Omega_i+\frac{U^2}{2})F_i d^6Z+\int\frac{m_e v^2}{2}f_e d^6z+\int\frac{|\mathbf{E}|^2}{8\pi}d^3x\right.$$
$$\left.+\frac{e^2n_0}{2T_i}\int\frac{d^3k}{(2\pi)^3}(1-\Gamma_0)|\phi_{\mathbf{k}}|^2\right)=0. \tag{28}$$

The gyroaveraged ion and electron kinetic energies and the electrostatic potential energy are apparent in all of the energy invariants presented. The last term in the invariants represents the perpendicular "sloshing" energy of the ions. This can be most easily seen by comparing the invariant for the linearized system with the expression for the energy of an electrostatic wave:[26]

$$\mathcal{E}=\frac{\partial}{\partial\Omega_{\mathbf{k}}}(\Omega_{\mathbf{k}}D_h)\frac{E_{\mathbf{k}}^2}{8\pi},$$

where $\Omega_{\mathbf{k}}$ is the frequency of the mode, and D_h is the Hermitian part of the linear dielectric [see, for instance, Eq. (D2b)]. Electrostatic fluctuations cause the ions to oscillate as the potential wells pass by; the energy associated with this sloshing motion is equal to the extra term in Eq. (28). The electrons do not contribute since their polarization drift is negligible compared to that of the ions.

Although the approximate gyrokinetic systems presented in this section all conserve energy exactly, they neglect terms which are quadratic in ϕ in the Poisson equation. These terms are of the same order as terms which we kept in Eq. (24), and thus may be important to the nonlinear evolution of the system. It is possible to retain these quadratic terms and improve the energy conservation by adding in the next order $[O(\phi^3)]$ contributions to the Hamiltonian from which the gyrokinetic Vlasov equation is generated. These $O(\phi^3)$ terms balance the quadratic terms in Eq. (20); however, energy is no longer exactly conserved. As we will see, these $O(\phi^3)$ terms play no role in the perturbation and weak turbulence theory of the next sections, so their importance is questionable. However, the $O(\phi^2)$ terms in the Vlasov and Poisson equations (19) and (20) do enter into the weak turbulence theory. Although it remains to be seen which set, Eqs. (19) and (20), or Eqs. (19) and (24), best approximates the actual dynamics, on the preliminary evidence of numerical simulations involving the small $k_\perp \rho$ limit of the latter set, we feel that the latter set is adequate for numerical work, and captures the dominant physics.

V. LIMITING FORMS

We shall now examine two limiting forms of Eqs. (19) and (20). By taking $k_\perp \rho$ small, we obtain equations similar to those of Lee:[4]

$$\frac{\partial F_i}{\partial \overline{T}} + \left(\overline{U}\hat{\mathbf{b}} - \epsilon \frac{e}{m_i \Omega_i} \overline{\nabla} \psi' \times \hat{\mathbf{b}}\right) \cdot \overline{\nabla} F_i - \epsilon \frac{e}{m_i} \hat{\mathbf{b}} \cdot \overline{\nabla} \psi' \frac{\partial F_i}{\partial \overline{U}} = O(\epsilon^3), \qquad (29)$$

where

$$\psi' = \overline{\phi} - \epsilon \frac{e\rho_i^2}{2T_i}(\overline{\nabla}_\perp \phi)^2 + [O(k_\perp \rho)^4],$$

and

$$\nabla^2 \phi + \frac{\rho_i^2}{n_0 \lambda_{D_i}^2}(N_i \nabla_\perp^2 \phi + \nabla_\perp \phi \cdot \nabla_\perp N_i) + 4\pi e(\hat{N}_i - n_e)$$
$$= O(\epsilon^3) + O[(k_\perp \rho_i)^4], \qquad (30)$$

where

$$N_i \equiv \int d^6 \overline{Z} \delta(\overline{\mathbf{X}} - \mathbf{x}) F_i, \quad \hat{N}_i \equiv \int 2\pi \Omega_i d\overline{\mu} d\overline{U} \frac{d^3 k}{(2\pi)^3} \exp(i\mathbf{k} \cdot \mathbf{x}) J_0(k_\perp \overline{\rho}) F_{i\mathbf{k}}.$$

There are two differences between Lee's equations and Eqs. (29) and (30). The nonlinear contribution to ψ' derived by Lee involves ρ instead of ρ_i. This is traceable to an inconsistent renormalization of the distribution function (see Appendix A), a subtle issue. However, the difference is unimportant as long as the equilibrium distribution function is Maxwellian in μ. Furthermore, Lee's Poisson equation overlooks the $\nabla_\perp \phi \cdot \nabla_\perp N_i$ term. The addition of this term is required for energy conservation, and also allows us to write the quasineutrality condition as

$$\hat{N}_i - n_e = -\frac{e}{m_i \Omega_i^2} \nabla_\perp \cdot (N_i \nabla_\perp \phi). \qquad (31)$$

Thus, in the long-wavelength limit, the difference between the gyroaveraged density $\hat{N}_i$ and the laboratory ion density n_i is the right-hand side of Eq. (31), which is the lowest order contribution to the density fluctuations provided by the polarization drift. It is also interesting to note the emergence in Eq. (31) of a term proportional to $\delta N_i \nabla_\perp^2 \phi$, which Lee also retains in his Poisson equation, and which seems to have a large effect on the saturated state of drift waves in certain cases.[4] We call this term the "nonlinear polarization density" term. This is not the same as the "nonlinear polarization drift" term discussed by many authors, proportional to $\mathbf{E} \times \mathbf{B} \cdot \nabla \mathbf{E}_\perp$, which gives rise to the mode coupling term of the Hasegawa-Mima equation. The nonlinear polarization density term is a higher order effect that is not contained in the Hasegawa-Mima equation.

Another enlightening limiting form of Eqs. (19) and (20) is obtained by taking the perpendicular ion temperature to zero, i.e., $\mu \to 0$. In this case one can take velocity-space moments of the resulting Vlasov equation to obtain a hierarchy of fluid equations, the first two of which are:

$$\frac{\partial N_i}{\partial t} - \nabla \psi_f \times \hat{\mathbf{b}} \cdot \nabla N_i + \hat{\mathbf{b}} \cdot \nabla J_i = O(\epsilon^3), \qquad (32)$$

$$\frac{\partial J_i}{\partial t} - \nabla \psi_f \times \hat{\mathbf{b}} \cdot \nabla J_i + \hat{\mathbf{b}} \cdot \nabla(\langle U \rangle J_i + P_i) + \hat{\mathbf{b}} \cdot \nabla \psi_f N_i = O(\epsilon^3), \qquad (33)$$

where

$$\psi_f \equiv \phi - \frac{1}{2}(\nabla_\perp \phi)^2, \qquad J_i \equiv \int U F_i 2\pi \Omega_i d\mu dU,$$

$$\langle U \rangle \equiv \frac{J_i}{N_i}, \qquad P_i \equiv \int (U - \langle U \rangle)^2 F_i 2\pi \Omega_i d\mu dU,$$

and for the remainder of this section it is convenient to normalize distances to $\rho_s \equiv (T_e/m_i\Omega_i^2)^{1/2}$, times to Ω_i^{-1}, and $e\phi$ to T_e. The Poisson equation becomes

$$n_0(\frac{\lambda_{De}}{\rho_s})^2 \nabla^2 \phi + N_i - n_e + \nabla_\perp \cdot (N_i \nabla_\perp \phi) + N_i[(\nabla_\perp \nabla_\perp \phi)^2 - (\nabla_\perp^2 \phi)^2] = O(\epsilon^3), \qquad (34)$$

where the norm of the tensor $\nabla_\perp \nabla_\perp \phi$ appears. These equations are good for $k_\perp \rho_s = O(1)$ and $k_\parallel \rho_s = O(\epsilon)$. It is instructive to take the limit $k_\parallel \rho_s \to 0$ and $\lambda_{De}/\rho_s \to 0$. In this case an equation for the time development of the potential can be derived which contains all terms necessary to perform a consistent weak turbulence analysis on the fluid system. Taking the time derivative of the quasineutrality condition obtained from Eq. (34) and substituting for $\partial N_i/\partial t$ yields

$$\nabla\psi_f \times \hat{\mathbf{b}}\cdot\nabla N_i - \frac{\partial n_e}{\partial t} + \nabla_\perp\cdot[\nabla\psi_f\times\hat{\mathbf{b}}\cdot\nabla N_i\nabla_\perp\phi + N_i\nabla_\perp\frac{\partial\phi}{\partial t}]$$
$$+ N_i\frac{\partial}{\partial t}[(\nabla_\perp\nabla_\perp\phi)^2 - (\nabla_\perp^2\phi)^2] = O(\epsilon^4).$$

Applying the quasineutrality condition once again then implies that

$$\frac{\partial n_e}{\partial t} - \nabla\psi_f\times\hat{\mathbf{b}}\cdot\nabla\Big(n_e - \nabla_\perp\cdot[(n_e - n_0\nabla_\perp^2\phi)\nabla_\perp\phi] - n_0[(\nabla_\perp\nabla_\perp\phi)^2 - (\nabla_\perp^2\phi)^2]\Big)$$
$$- \nabla_\perp\cdot\Big(\nabla\psi_f\times\hat{\mathbf{b}}\cdot\nabla(n_e - n_0\nabla_\perp^2\phi)\nabla_\perp\phi + (n_e - n_0\nabla_\perp^2\phi)\nabla_\perp\frac{\partial\phi}{\partial t}\Big)$$
$$- n_0\frac{\partial}{\partial t}[(\nabla_\perp\nabla_\perp\phi)^2 - (\nabla_\perp^2\phi)^2] = O(\epsilon^4), \qquad (35)$$

where n_0 is the background density. Taking the electron response to be of the general form $\delta n_{e\mathbf{k}} = n_0(1+i\alpha_\mathbf{k})(e\phi_\mathbf{k}/T_e)$, where $\alpha_\mathbf{k}$ represents the nonadiabatic electron response, and keeping only the lowest order terms, we obtain the Terry-Horton equation:

$$(1 + i\alpha_\mathbf{k} + k^2)\frac{\partial\phi_\mathbf{k}}{\partial t} = -\, i\omega_*\phi_\mathbf{k}$$
$$+ \tfrac{1}{2}\int\frac{d^3k'd^3k''}{(2\pi)^3}\mathbf{k}'\times\mathbf{k}''\cdot\hat{\mathbf{b}}\phi_{\mathbf{k}'}\phi_{\mathbf{k}''}$$
$$\times\left[k''^2_\perp - k'^2_\perp + i(\alpha_{\mathbf{k}''} - \alpha_{\mathbf{k}'})\right]\delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}''), \qquad (36)$$

where $\omega_* \equiv -\mathbf{k}\times\hat{\mathbf{b}}\cdot\nabla\log n_0$. Taking $\alpha_\mathbf{k}$ to zero leads to the Hasegawa-Mima equation.

VI. WEAK TURBULENCE THEORY

In this section we briefly consider aspects of the weak turbulence theory of the gyrokinetic system. We do not mean to imply that such a theory is adequate; indeed, for drift fluctuations a strong turbulence calculation is required, at least for certain wavenumbers. However, many structural features of a complete strong turbulence calculation are already present in the simpler weak turbulence limit, and we will consider that limit in order to better understand the power of the gyrokinetic approach and the various gyrokinetic nonlinearities. In particular, we wish to understand how our gyrokinetic formalism is related to the well-known discussion of Galeev and Sagdeev of a weak turbulence theory appropriate for drift waves. Those authors begin with the complete magnetized Vlasov equation, and therefore encounter significant difficulties in systematically performing the required integrations along perturbed trajectories. The gyrokinetic description is much simpler and "cleaner." In fact, we show in Appendix D that the wave kinetic equation corresponding to the following truncated set of equations,

$$\frac{\partial F_i}{\partial \overline{T}} - \frac{e}{m_i\Omega_i}\overline{\nabla\phi} \times \hat{\mathbf{b}} \cdot \overline{\nabla} F_i = 0, \tag{37}$$

and

$$0 = \int \delta F_{i\mathbf{k}} J_0(k_\perp\rho) 2\pi\Omega_i d\mu dU + \{\tau[1-\Gamma_0(b)]+1\}\frac{e\phi_{\mathbf{k}}}{T_e}, \tag{38}$$

where $\tau = T_e/T_i$, is the same as the equations of Galeev and Sagdeev. There are several typographical errors in Ref. 13.

Equations (37) and (38) may be obtained from Eqs. (19) and (20) by linearizing ψ, applying the quasineutrality condition to the linearized gyrokinetic Poisson equation, taking the electrons to be adiabatic and taking $k_\parallel \to 0$, and again assuming that the background distribution function is Maxwellian in μ. The details of the calculation, presented in Appendix D, are similar to those of Kadomtsev.[27] We solve perturbatively for the potential, assuming that the turbulence is almost delta-correlated in frequency around the zeroes of the linear dispersion relation, and that the Fourier amplitudes of the turbulent spectrum are stochastic functions with near-Gaussian statistical properties, which allows us to drop the fourth order cumulants appearing in the equation. We repeat that this calculation is by no means original to this paper and in itself is rather uninteresting; however, it is important to note that it is based on a truncated set of equations. Only the nonlinear $\mathbf{E} \times \mathbf{B}$ drift appears in the equations upon which this result depends; many other gyrokinetic effects have been neglected, including the nonlinear polarization drift term, $\nabla \cdot (\delta N_i \nabla \phi)$, appearing in various forms in Eqs. (20), (24), (30), and (34). Although the $\mathbf{E} \times \mathbf{B}$ nonlinearity is the largest nonlinear term, it is not necessarily a good approximation to leave out the higher order drifts. Although they are small, they are correlated to the lower order fluctuations in such a way that their effect on the nonlinear mode coupling and growth rate is of the same order as that of the $\mathbf{E} \times \mathbf{B}$ drift. Examination of the gyrokinetic equations leads to the conclusion that terms of order ϕ^2 in both the Vlasov and Poisson equations are required for a consistent weak turbulence analysis. Thus, Eqs. (19) and (20) contain all terms necessary; however, this calculation is rather involved and space precludes a discussion here. We reiterate that the discussion of the present section is primarily pedagogical: since weak turbulence theory does contain many of the salient aspects of other renormalized theories, it provides several instructive insights into the structure of the nonlinear equations.

VII. CONCLUSIONS

Through the use of noncanonical Hamiltonian techniques and Lie transforms, we have been able to derive fully gyrokinetic, phase space preserving nonlinear equations governing self-consistent low frequency electrostatic plasma fluctuations in a straight, constant magnetic field. Energy conservation for the nonlinear system was discussed, as was the physics of the nonlinear drifts.

By linearizing both the gyrokinetic potential Eq. (19b), and the gyrokinetic Poisson equation (20), equations may be obtained which are equivalent to the equations of Frieman and Chen. Our gyrokinetic equations also reduce to the familiar Terry-Horton and Hasegawa-Mima equations in the limit of negligible ion temperature, and to equations similar to those of Lee in the small $k_\perp \rho$ limit. They are similar to the equations of Wong, but contain several terms missing from his gyrokinetic potential and his renormalized gyrokinetic distribution function. It is interesting to note, however, that his equations still conserve energy [neglecting the $O(\phi^2)$ terms in his Poisson equation] even though they are not complete. The weak turbulence theory of the equations was briefly investigated and we found that the theory of Galeev and Sagdeev does not include all relevant gyrokinetic effects. In any case, it is clear that the gyrokinetic approach affords a more expeditious route to the derivation of nonlinear statistical descriptions (for either weak or strong fluctuations) than do approaches based on the full Vlasov equation.

From a practical as well as a theoretical point of view, we believe that the most important feature of this work is that it provides self-consistent gyrokinetic equations in phase space preserving form. This allows for particularly efficient solution of the fully nonlinear equations using existing numerical techniques. Since it is also important for the equations to exhibit energy conservation, especially in numerical work, we feel that Eqs. (19) and (24) are appropriate equations to use for a full numerical study of low frequency plasma fluctuations. Although these equations may have defects (cf. Sec. IV), they do contain all the physics of previous formulations, as well as new effects such as the nonlinear polarization density term. Furthermore, preliminary numerical results indicate good agreement with simulations involving the full unaveraged Vlasov equation. The effects of arbitrary magnetic fields and electromagnetic fluctuations can be incorporated into the formalism in a relatively straightforward manner. Such work is in progress, and has already yielded useful insights. Although much work remains to be done, we believe that these tools lay the proper foundation for a detailed analysis, both analytical and numerical, of nonlinear low frequency fluctuations in magnetized plasma.

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APPENDIX A: THE RECURSIVE APPROACH

In this appendix we present an alternative derivation of the gyrokinetic equations, using the so-called recursive method. Although the approach is quite cumbersome, it provides a useful comparison to the more powerful techniques used in the main body of the text, and is probabably more familiar to workers in drift kinetic and gyrokinetic theory. Here, we break the Darboux-transformed distribution function $g_i(\mathbf{Z})$ into an averaged part, $\overline{g}_i$, and a fluctuating part, $\tilde{g}_i$. We then average the equation of motion, Eq. (14), to obtain two coupled equations for $\overline{g}_i$ and $\tilde{g}_i$:

$$\langle\{\overline{g}_i + \tilde{g}_i, \hat{H}_i\}\rangle = 0, \tag{A1a}$$

$$\{\overline{g}_i + \tilde{g}_i, \hat{H}_i\} - \langle\{\overline{g}_i + \tilde{g}_i, \hat{H}_i\}\rangle = 0. \tag{A1b}$$

Since $\hat{H}_i$ is θ-dependent, these equations no longer decouple as they did when we Lie transformed to the gyroaveraged Hamiltonian $\overline{H}_i$; now both equations involve $\overline{g}_i$ and $\tilde{g}_i$.

Our strategy will be to solve for $\tilde{g}_i$ perturbatively as a functional of $\overline{g}_i$, and use this relation to obtain the evolution equation for $\overline{g}_i$. Solving Eq. (A1b) for $\tilde{g}_i(\overline{g}_i)$ yields $\tilde{g}_i = \tilde{g}_{i1} + \tilde{g}_{i2}$, where

$$\begin{aligned}\tilde{g}_{i1} =& \frac{e}{m_i\Omega_i}\tilde{\phi}\frac{\partial \overline{g}_i}{\partial \mu},\\ \tilde{g}_{i2} =& \frac{e}{m_i\Omega_i^2}\left\{-\mathcal{L}\left(\frac{\partial \overline{g}_i}{\partial \mu}\tilde{\Phi}\right) + \nabla\tilde{\Phi}\times\hat{\mathbf{b}}\cdot\nabla\overline{g}_i + \Omega_i\hat{\mathbf{b}}\cdot\nabla\tilde{\Phi}\frac{\partial \overline{g}_i}{\partial v_\parallel}\right.\\ &\left. - \frac{e}{m_i}\left[\frac{\partial \overline{g}_i}{\partial \mu}\frac{\partial \overline{\phi}}{\partial \mu}\tilde{\phi} - \frac{1}{2}\frac{\partial^2 \overline{g}_i}{\partial \mu^2}(\tilde{\phi}^2 - \langle\tilde{\phi}^2\rangle)\right]\right\},\end{aligned}$$

and $\mathcal{L} \equiv \partial/\partial t + [v_\parallel\hat{\mathbf{b}} - (e/m_i)\nabla\overline{\phi}\times\hat{\mathbf{b}}/\Omega_i]\cdot\nabla - (e/m_i)\hat{\mathbf{b}}\cdot\nabla\overline{\phi}\partial/\partial v_\parallel$. Substituting this expression into Eq. (A1a) yields the following rather messy equation, which is clearly not phase space preserving as it stands:

$$\mathcal{L}\bar{g}_i - \frac{e^2}{2m_i{}^2\Omega_i^2}(\nabla\langle\tilde{\phi}^2\rangle \times \hat{\mathbf{b}} \cdot \nabla\frac{\partial \bar{g}_i}{\partial \mu} - \Omega_i \hat{\mathbf{b}} \cdot \nabla\langle\tilde{\phi}^2\rangle \frac{\partial^2 \bar{g}_i}{\partial \mu \partial v_{\|}})$$
$$- \langle \frac{e}{m_i \Omega_i} \nabla\tilde{\phi} \times \hat{\mathbf{b}} \cdot \nabla \tilde{g}_{i2}\rangle + \frac{e}{m_i}\frac{\partial}{\partial \mu}\langle\tilde{\phi}\frac{\partial \tilde{g}_{i2}}{\partial \theta}\rangle = O(\epsilon^3).$$

It is certainly not obvious that this equation can be put in phase space preserving form, but nevertheless if we substitute the relation

$$\bar{g}_i = F_i + \frac{e^2}{2m_i{}^2\Omega_i^2}\left(\frac{\partial}{\partial \mu}(\frac{\partial F_i}{\partial \mu}\langle\tilde{\phi}^2\rangle) + \langle\frac{\nabla\tilde{\Phi}}{\Omega_i} \cdot \hat{\mathbf{b}} \times \nabla\tilde{\phi}\rangle \frac{\partial F_i}{\partial \mu}\right), \tag{A2}$$

we find, after very lengthy algebra, that we are led again to Eq. (19). We can complete the calculation by writing the Poisson equation in terms of $\bar{g}_i$,

$$\nabla^2 \phi = -4\pi e\left[\int(\bar{g}_i + \tilde{g}_i\{\bar{g}_i\})d^6 Z - n_e\right],$$

where the braces indicate functional dependence. Then substitution of F_i for $\bar{g}_i$ using Eq. (A2) reproduces Eq. (20). Of course, Eq. (A2) is exactly what we would get by solving the coupled equations

$$F_i = \langle \mathcal{T}^{-1} g_i\rangle \quad \text{and} \quad 0 = \mathcal{T}^{-1} g_i - \langle \mathcal{T}^{-1} g_i\rangle,$$

which we derived in Sec. III using Hamiltonian techniques. Equation (A2) would not have been obvious if we had not already been aware of the answer. The power of the Hamiltonian technique lies in the way that it automatically renormalizes $\bar{g}_i$, keeping the Vlasov equation in a simple phase space preserving form.

APPENDIX B: POISSON BRACKETS

In the coordinates $(\mathbf{x}, v_\perp, \theta, v_\parallel, t, w)$ all Poisson brackets except the following vanish:

$$\{\mathbf{x}, v_\perp\} = \hat{\mathbf{c}}, \qquad \{\mathbf{x}, v_\parallel\} = \hat{\mathbf{b}}, \qquad \{\mathbf{x}, \theta\} = -\frac{\hat{\mathbf{a}}}{v_\perp}, \tag{B1a,b,c}$$

$$\{\theta, v_\perp\} = \frac{\Omega}{v_\perp}, \qquad \{w, t\} = 1, \tag{B1d,e}$$

where $\hat{\mathbf{a}} \equiv \hat{\mathbf{x}} \cos\theta - \hat{\mathbf{y}} \sin\theta, \hat{\mathbf{c}} \equiv -\hat{\mathbf{x}} \sin\theta - \hat{\mathbf{y}} \cos\theta$.

In Darboux transformed coordinates $\mathbf{Z} = (\mathbf{X}, \mu, \theta, U, T, W)$, all Poisson brackets except the following vanish:

$$\{\mathbf{X}, \mathbf{X}\} = \frac{\hat{\mathbf{b}} \times \mathbf{I}}{\Omega}, \qquad \{\mathbf{X}, U\} = \hat{\mathbf{b}}, \tag{B2a,b}$$

$$\{\mu, \theta\} = 1, \qquad \{W, T\} = 1. \tag{B2c,d}$$

APPENDIX C: GENERATING FUNCTIONS

The following functions generate the averaging transformation:

$$\frac{\partial G_1}{\partial \theta} = -\frac{e\tilde{\phi}}{m\Omega}, \tag{C1a}$$

$$\frac{\partial G_2}{\partial \theta} = \frac{e^2}{m^2\Omega^2}\Bigg(2\tilde{\phi}\frac{\partial \overline{\phi}}{\partial \mu} + \frac{\partial}{\partial \mu}(\tilde{\phi}^2 - \langle\tilde{\phi}^2\rangle) - \frac{\partial}{\partial \theta}(\tilde{\phi}\frac{\partial \tilde{\Phi}}{\partial \mu})$$
$$+ \frac{\nabla\tilde{\Phi}}{\Omega} \cdot \hat{\mathbf{b}} \times \nabla\tilde{\phi} - \langle\frac{\nabla\tilde{\Phi}}{\Omega} \cdot \hat{\mathbf{b}} \times \nabla\tilde{\phi}\rangle + 2\frac{\nabla\tilde{\Phi}}{\Omega} \cdot \hat{\mathbf{b}} \times \nabla\overline{\phi}$$
$$+ 2\frac{m}{e}(\frac{\partial \tilde{\Phi}}{\partial t} + U\hat{\mathbf{b}} \cdot \nabla\tilde{\Phi})\Bigg). \tag{C1b}$$

APPENDIX D: DETAILS OF THE WEAK TURBULENCE CALCULATION

The object of this section is to obtain an equation for the evolution of the electric field intensity on the nonlinear saturation time scale, assuming that it is almost delta-correlated and that it is small. The latter assumption allows us to solve Eqs. (37) and (38) recursively for ϕ; we find through $O(\phi^3)$ that

$$\int D(\tilde{k},\tilde{k}')\phi_{\tilde{k}'}\frac{d^4k'}{(2\pi)^4}+\int E(\tilde{k},\tilde{k}',\tilde{k}'')\phi_{\tilde{k}'}\phi_{\tilde{k}''}(2\pi)^4\delta(\tilde{k}-\tilde{k}'-\tilde{k}'')\frac{d^4k'd^4k''}{(2\pi)^8}$$
$$+\int F(\tilde{k},\tilde{k}',\tilde{k}'')\phi_{\tilde{k}''}\phi_{\tilde{k}'-\tilde{k}''}\phi_{\tilde{k}-\tilde{k}'}\frac{d^4k'd^4k''}{(2\pi)^8}=0, \qquad \text{(D1)}$$

where

$$D(\tilde{k},\tilde{k}')\equiv\frac{i\mathbf{k}'\times\hat{\mathbf{b}}\cdot i\mathbf{k}}{i\omega}n_{0\,\tilde{k}-\tilde{k}'}\int F_M J_0(k_\perp\rho)J_0(k'_\perp\rho)2\pi\Omega_i\,d\mu dU$$
$$+\{\tau[1-\Gamma_0(b)]+1\}\delta(\tilde{k}-\tilde{k}')(2\pi)^4,$$

$$E(\tilde{k},\tilde{k}',\tilde{k}'')\equiv-\frac{i\mathbf{k}'\times\hat{\mathbf{b}}\cdot i\mathbf{k}}{i\omega}\frac{\omega'_*}{\omega'}\int F_M J_0(k_\perp\rho)J_0(k'_\perp\rho)J_0(k''_\perp\rho)2\pi\Omega_i\,d\mu dU,$$

$$F(\tilde{k},\tilde{k}',\tilde{k}'')\equiv\frac{\mathbf{k}'\times\hat{\mathbf{b}}\cdot\mathbf{k}}{i\omega}\frac{\mathbf{k}'\times\hat{\mathbf{b}}\cdot\mathbf{k}''}{i\omega'}\frac{\omega''_*}{\omega''}$$
$$\times\int F_M J_0(k''_\perp\rho)J_0(|\mathbf{k}_\perp-\mathbf{k}'_\perp|\rho)J_0(|\mathbf{k}''_\perp-\mathbf{k}'_\perp|\rho)J_0(k_\perp\rho)2\pi\Omega_i\,d\mu dU.$$

We have normalized $e\phi$ to T_e, ω to Ω_i, k^{-1} to ρ_s, and $\tilde{k}$ is here defined to be the four-vector $(-\omega,\mathbf{k})$ with $\mathbf{k}$ the three-vector wavenumber, F_M is a Maxwellian distribution in μ, $\omega'_*\equiv\omega_*(\mathbf{k}')$, and $\omega_*\equiv-[\mathbf{k}\times\hat{\mathbf{b}}\cdot\nabla\log(n_0)]$ is the drift frequency. Multiplying (D1) by $\phi^*_{\tilde{k}}$ and expanding $\phi_{\tilde{k}}$ perturbatively ($\phi=\phi_0+\phi_1+...$, where ϕ_0 is the linear potential, exactly delta-correlated around the zeros of the linear dispersion relation, ϕ_1 represents the first order nonlinear line broadening, etc.), we find that

$$D_{\tilde{k}}\langle\phi\phi\rangle_{\tilde{k}} + \frac{1}{2i}\left(\frac{\partial\langle\phi\phi\rangle_{\tilde{k}}}{\partial\chi}\frac{\partial D_{\tilde{k}}}{\partial\tilde{k}} - \frac{\partial D_{\tilde{k}}}{\partial\chi}\frac{\partial\langle\phi\phi\rangle_{\tilde{k}}}{\partial\tilde{k}}\right)$$

$$= \int\Bigg(\frac{\overline{E}(\tilde{k}-\tilde{k}',\tilde{k},-\tilde{k}')\overline{E}(\tilde{k},\tilde{k}',\tilde{k}-\tilde{k}')}{D_{\tilde{k}-\tilde{k}'}}$$

$$- [F(\tilde{k},\tilde{k}-\tilde{k}',-\tilde{k}') + F(\tilde{k},\tilde{k}-\tilde{k}',\tilde{k})]\Bigg)\langle\phi\phi\rangle_{\tilde{k}}\langle\phi\phi\rangle_{\tilde{k}'}\frac{d^4k'}{(2\pi)^4}$$

$$+ \frac{1}{2}\int\frac{|\overline{E}(\tilde{k},\tilde{k}',\tilde{k}-\tilde{k}')|^2}{D^*_{\tilde{k}'}}\langle\phi\phi\rangle_{\tilde{k}-\tilde{k}'}\langle\phi\phi\rangle_{\tilde{k}'}\frac{d^4k'}{(2\pi)^4}, \qquad \text{(D2a)}$$

where

$$D_{\tilde{k}}(\chi) \equiv -(\omega_*/\omega)\Gamma_0 + \tau(1-\Gamma_0) + 1 \qquad \text{(D2b)}$$

is the linear dielectric function for adiabatic electrons and $k_\parallel \to 0$ (in the quasineutral limit), $\overline{E}(a,b,c) \equiv E(a,b,c) + E(a,c,b)$, and $\chi \equiv (\overline{\tau}, \mathbf{R})$ are the coordinates representing the slow time and space scales of the macroscopic plasma parameters. We have used the "Random Phase Approximation," throwing away the fourth order cumulants to write the four point correlation functions (obtained by expanding ϕ) in terms of the wave intensities, $\langle\phi\phi\rangle_{\tilde{k}}$. Assuming that these intensities are fairly sharply peaked around the zeros of $D_{\tilde{k}}$ has allowed us to use a W.K.B. approximation on the term $D(\tilde{k},\tilde{k}')\langle\phi\phi\rangle_{\tilde{k}'}$ of Eq. (D1) by Fourier transforming over the fast fluctuations in space and time and allowing $D_{\tilde{k}}$ to depend parametrically on the slow scales. This sharp peaking allows us to perform the frequency integrations as well; substituting for $D_{\tilde{k}}$ and writing the equation in terms of the wave action $N_{\mathbf{k}} \equiv \langle\phi\phi\rangle_{\mathbf{k}}\partial D_{\tilde{k}}/\partial\omega|_{\omega=\Omega_k}$, we obtain the well-known Galeev and Sagdeev wave kinetic equation for drift wave turbulence:

$$\frac{\partial N_{\mathbf{k}}}{\partial \bar{\tau}} + \frac{\partial}{\partial \mathbf{R}} \cdot (\mathbf{v}_g N_{\mathbf{k}}) - \frac{\partial}{\partial \mathbf{k}} \cdot (\frac{\partial \Omega_{\mathbf{k}}}{\partial \mathbf{R}} N_{\mathbf{k}}) = 2\gamma_l N_{\mathbf{k}}$$

$$+ N_{\mathbf{k}} \int \frac{d^3k'}{(2\pi)^3} (\mathbf{k} \times \mathbf{k}' \cdot \hat{\mathbf{b}})^2 \frac{(\omega'_* - \omega_*)}{\Omega_{\mathbf{k}}} \Big(\int F_M J_0(k_\perp \rho)^2 J_0(k'_\perp \rho)^2 2\pi \Omega_i d\mu dU$$

$$- \frac{[\int F_M J_0(k_\perp \rho) J_0(k'_\perp \rho) J_0(|\mathbf{k}_\perp - \mathbf{k}'_\perp| \rho) 2\pi \Omega_i d\mu dU]^2}{\Gamma_0(|\mathbf{k}_\perp - \mathbf{k}'_\perp| \rho_i)} \Big) 2\pi \delta(\Omega_{\mathbf{k}} - \Omega_{\mathbf{k}'}) \langle \phi\phi \rangle_{\mathbf{k}'}$$

$$- \frac{1}{2} \int \frac{d^3k' d^3k''}{(2\pi)^6} (\mathbf{k} \times \mathbf{k}' \cdot \hat{\mathbf{b}})^2 \frac{sgn(\mathrm{Im}(D_{(-\Omega_k, \mathbf{k})}))}{\Omega_{\mathbf{k}}} (\frac{\omega'_*}{\Omega_{\mathbf{k}'}} - \frac{\omega''_*}{\Omega_{\mathbf{k}''}})^2$$

$$\times \Big[\int F_M J_0(k_\perp \rho) J_0(k'_\perp \rho) J_0(k''_\perp \rho) 2\pi \Omega_i d\mu dU \Big]^2$$

$$\times \langle \phi\phi \rangle_{\mathbf{k}'} \langle \phi\phi \rangle_{\mathbf{k}''} \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'') \delta(\Omega_{\mathbf{k}} - \Omega_{\mathbf{k}'} - \Omega_{\mathbf{k}''}) (2\pi)^4, \qquad \text{(D3)}$$

where $\mathbf{v}_g \equiv \partial(\mathrm{Re}\, D_{\tilde{k}})/\partial\omega|_{\omega=\Omega_k}$ is the group velocity, $\Omega_{\mathbf{k}}$ is the real linear frequency, satisfying $\mathrm{Re}\, D_{\tilde{k}} = 0$, and γ_l is the well-known linear growth rate.

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Figure Captions

FIG. 1. Geometric representation of gyromotion. The vectors $\boldsymbol{\rho}$, $\hat{\mathbf{a}}$, and $\hat{\mathbf{c}}$ rotate with the particle.

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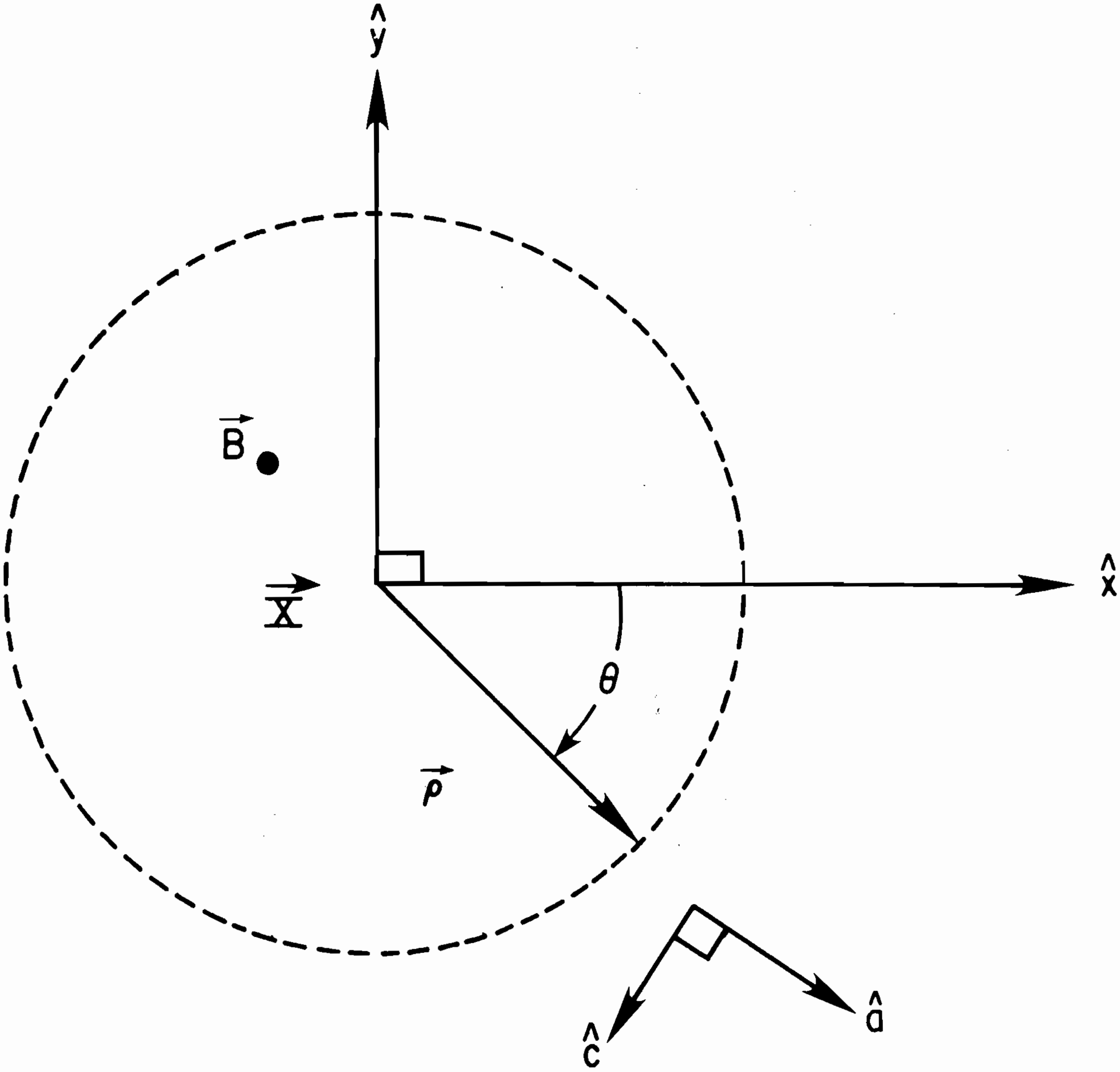

Fig. 1

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