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by

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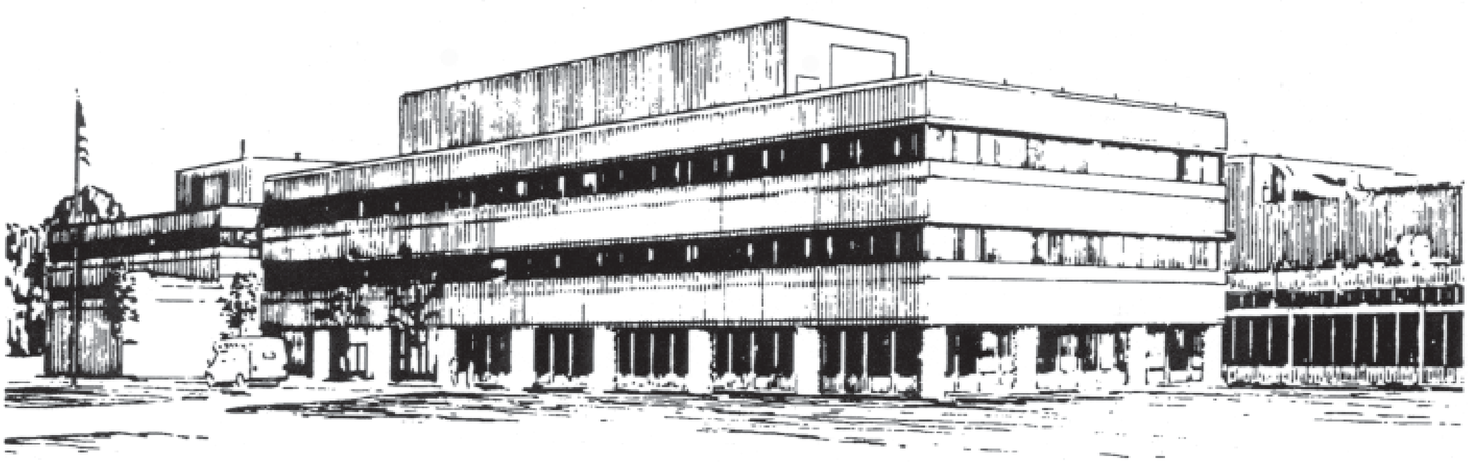

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# Strange Attractors in Drift Wave Turbulence

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# 1 Introduction

There are growing experimental, numerical and theoretical evidences that the anomalous transport observed in tokamaks and stellarators is caused by slow, drift-type modes (such as trapped electron modes and ion-temperature gradient-driven modes). Although typical collision frequencies in hot, magnetized fusion plasmas can be quite low in absolute values, collisional effects are nevertheless important since they act as dissipative sinks. As it is well known, dissipative systems with many (strictly speaking more than two) degrees of freedom are often chaotic and may evolve towards a so-called attractor.

This paper shows that strange attractors in collisional, electrostatic drift wave turbulence with kinetic electrons can exist and that their fractal dimension are actually quite small; this result suggests the presence of deterministic dynamics with *few* key variables but displaying chaotic behavior (because of the fractal dimensionality of the attractor). Another important conclusion is that our observation of a low-dimensional attractor for this specific model of drift wave turbulence has been achieved using an accurate scheme for kinetic electrons (splitting scheme; see next section). In the presence of kinetic electrons, standard schemes (e.g. $\delta f$ scheme [2]) fail to resolve the underlying dynamics of the system, that is the fractal dimension cannot be measured.

# 2 Splitting Scheme

As mentioned in the Introduction, the measurement of the fractal dimension of the attractor in electrostatic drift wave turbulence has been made possible by using an accurate electron scheme. In order to stress the relevance of strange attractors to drift-wave turbulence, we consider a shearless slab model for electrostatic drift waves. We start from the collisionless, electrostatic, gyrokinetic Vlasov equation, in the long-wavelength limit, for particles species $j$ with mass $m_j$ and charge $q_j$

$$\frac{dF_j}{dt} \equiv \frac{\partial F_j}{\partial t} + \left( v_{||} \widehat{\mathbf{b}}_0 + \mathbf{V}_E \right) \cdot \boldsymbol{\nabla} F_j - \frac{q_j}{m_j} \widehat{\mathbf{b}}_0 \cdot \boldsymbol{\nabla} \Phi \frac{\partial F_j}{\partial v_{||}} = C\left(F_j\right) \tag{1}$$

where $\widehat{\mathbf{b}}_0 = \mathbf{B}_0 / B_0$ is a unit vector, $\mathbf{V}_E = c\widehat{\mathbf{b}}_0 \times \boldsymbol{\nabla}\Phi / B_0$ is the $\mathbf{E}\times\mathbf{B}$ drift velocity, and $C\left(F_j\right)$ is the collision operator. The confining magnetic field is taken to be of the form $\mathbf{B}_0 = B_0\left(\widehat{\mathbf{z}} + \theta \widehat{\mathbf{y}}\right)$ where $\theta$ is a small parameter, together with the simplification of $\partial/\partial z \mapsto k_z \equiv 0$. Collisional effects on the ion distribution are neglected, $C(F_i) = 0$; the effects of electron-ion collisions can be represented by the number-conserving, energy-conserving Lorentz collision operator [1] including only pitch-angle scattering in the velocity space for the electrons

$$C\left(F_e\right) = \frac{\nu_{ei}}{2} \frac{1}{\sin\zeta} \frac{\partial}{\partial \zeta} \left( \sin\zeta \frac{\partial F_e}{\partial \zeta} \right) , \tag{2}$$

where $\nu_{ei} = 4\pi n_0 e^4 \ln\Lambda / m_e^2 V_{the}^3$ is the collision frequency and $\zeta = \cos^{-1} v_{||} / \left( v_{||}^2 + v_\perp^2 \right)^{1/2}$. Although the standard $\delta f$ scheme [2] works well for the ion dynamics, an accuracy problem arises when the scheme is used to treat the electron dynamics. The origin of this accuracy problem is related to the fact that the bulk of the electrons do not interact with the low-frequency waves but may (and usually do) transfer noise if their dynamics is not treated accurately. Therefore, it is natural to separate the electrons into two groups (adiabatic and nonadiabatic) to reflect their different responses to the low-frequency waves. To do so, we write the distribution $F_j$ as [4]

$$F_j = \exp\left(-\frac{q_j \Phi}{T_j}\right) F_{Mj} + h_j \ , \tag{3}$$

where $F_{Mj}$ is the Maxwellian distribution for particle species $j$ and $h_j$ is the nonadiabatic response. Substituting representation (3) in Eq.(1) and using the relations of $(\partial/\partial t + v_{||}\widehat{\mathbf{b}}_0 \cdot \boldsymbol{\nabla}) F_{Mj} = 0$ and $\mathbf{V}_E \cdot \boldsymbol{\nabla}\Phi \equiv 0$, we obtain an evolution equation for the nonadiabatic response

$$\frac{dh_j}{dt} = C\left(F_j\right) + F_{Mj} \exp\left(-\frac{q_j \Phi}{T_j}\right)\left(\boldsymbol{\kappa}_j \cdot \mathbf{V}_E + \frac{q_j}{T_j}\frac{\partial \Phi}{\partial t}\right) \ , \tag{4}$$

where $\boldsymbol{\kappa}_j = \boldsymbol{\kappa}\left[1 - \frac{\eta_j}{2}\left(1 - \overline{v}_{||}{}^2\right)\right]$, $\overline{v}_{||} = v_{||}/V_{thj}$ and $\boldsymbol{\kappa} = -\boldsymbol{\nabla} n_0/n_0$. As it is evident from Eq.(4) the contribution due to the free streaming particles will not, unlike the corresponding $\delta f$ scheme, appear in the equation for the marker weight. The field equations related to $\Phi$ and $\partial\Phi/\partial t$ have been solved using a multigrid solver; the details are given in Ref. [4]. The initial loading of the markers in velocity space is carried out using a low-noise technique described in Ref. [5]. The linear properties of the splitting scheme [4] and the energy conservation properties have been presented elsewhere [3] and these results are not reproduced here.

# 3 Characterization of Strange Attractors

It is already an accepted notion that many nonlinear dissipative dynamical systems do not approach stationary or periodic states asymptotically. Instead, with appropriate values of their parameters, they tend towards strange attractors on which the motion is chaotic, i.e. not periodic and unpredictable over long times, being extremely sensitive on the initial conditions [7, 8, 6]. Typically a strange attractor arises when the flow in phase space does not contract a volume element in all directions, but stretches it in some. In order to remain confined to a bounded domain, the volume element gets folded at the same time, so that it has after some time a multisheeted structure [7, 6]. In our model, dissipation through collisions is what allows for phase space contraction.

Grasseberger and Procaccia [9] have introduced an important measure of an attractor known as the correlation exponent, which is based on correlations between random points on the attractor. The basic idea behind the correlation exponent measure is that trajectories belonging to an attractor, although not dynamically correlated, are spatially correlated. Introducing the correlation integral $C(\ell)$ these authors have shown that, for small enough $\ell$, $C(\ell) \sim \ell^\alpha$, where $\alpha$ is the so-called correlation exponent. Grassberger and Procaccia have proved that the information dimension, $\sigma$, the Hausdorff dimension, $D$, and the correlation exponent, $\alpha$, satisfy the inequality

$$\alpha \leq \sigma \leq D \ . \tag{5}$$

In most cases, the inequality (5) is rather tight. To measure the spatial correlation of the attractor, Grassberger and Proccacia consider a time series $\{\mathbf{X}_i \equiv \mathbf{X}\left(t + i\Delta t\right); i = 1, \cdots, M\}$ of points on the attractor, where $\Delta t$ is the (fixed) time step; they define the correlation integral [9] as

$$C\left(\ell\right) \equiv \lim_{M \mapsto \infty} \frac{\widehat{M}(\ell)}{M^2} \ , \tag{6}$$

where

$$\widehat{M}(\ell) \equiv \sum_{ij} H\left(|\mathbf{X}_i - \mathbf{X}_j| - \ell\right) \ , \tag{7}$$

is the number of pairs $(i, j)$ whose distance $d_{ij} = |\mathbf{X}_i - \mathbf{X}_j|$ is less than $\ell$; in Eq.(7) $H(x)$, denotes the Heaviside function. One important conclusion of the work by Grassberger and Proccacia is that, for small $\ell$, the correlation integral $C(\ell)$ grows like a power

$$C\left(\ell\right) \sim l^\alpha \ ,$$

and that this correlation exponent ($\alpha$) can be taken as a measure of the local structure of a strange attractor [9]. The usefulness of this measure for a system with many degrees of freedom is highlighted in the next section.

# 4 Numerical Results

The implementation of the Grassberger-Procaccia algorithm has been tested against known results for the (one-dimensional non-invertible) logistic map [8] and the (two-dimensional invertible) map [11]. In both cases the measured correlation exponent provides a close lower bound to the Hausdorff dimension [10].

Having tested the implementation of the Grassberger-Proccacia algorithm, we consider the case of fully developed electrostatic drift wave turbulence. Since there is no explicit source of dissipation (no phase space contraction) for the ion population, we measure the correlation exponent of the electron dynamics only. We randomly select a set of $M$ electron markers from the electron distribution function. Each sample $\mathbf{X}_q = \left(x_k^{(n)}, v_{||k}^{(n)}\right)$ is recorded for each marker $k$ at time step $n$. In order to prevent spurious spatial correlations, the system must be in the fully nonlinear state; in this paper, the positions in phase space $\mathbf{X}_q$ were recorded for $\omega_{ci}t \geq 3000$ (fully turbulent regime) for $N_s$ time steps. The distance in phase space between $\mathbf{X}_q$ and $\mathbf{X}_{q'}$ is simply given by

$$d_{q,q'} = |\mathbf{X}_q - \mathbf{X}_{q'}| = \left\{ \left[x_k^{(n)} - x_{k'}^{(n')}\right]^2 + \left[v_{||k}^{(n)} - v_{||k'}^{(n')}\right]^2 \right\}^{1/2} , \tag{8}$$

and the correlation integral is computed as in Eq.(6). In a typical simulation, both the number of sampling markers $M$ and the number of time steps $N_s$ is varied to ensure convergence. Figure 1 (left) shows the electron correlation integral, $C_e(\ell)$, as a function of $\ell/\ell_0$, where $\ell_0$ is arbitrary; the collision frequency is $\nu_{ei} = 10^{-4}$. For very small distances, the data for $C_e(\ell)$ deviate from a power law, but that was to be expected: the values of $\mathbf{X}_q$ and $\mathbf{X}_{q'}$ are strongly correlated. For larger $\ell$ the correlation integral follows a power law over 7 orders of magnitude. The $\chi^2$ fit yields a correlation exponent of $\alpha = 0.0126$. This means that the low-dimensional attractor is somewhere between a point ($D = 0$) and a line ($D = 1$). Since the system has many degrees of freedom, such a low-dimensional may seem surprising; however, for a very different physical system, Nicolis and Nicolis [12] have found a strange attractor with a small dimension $D$ in a system with many degrees of freedom (see next section) The key factor here is the rate of phase space contraction.

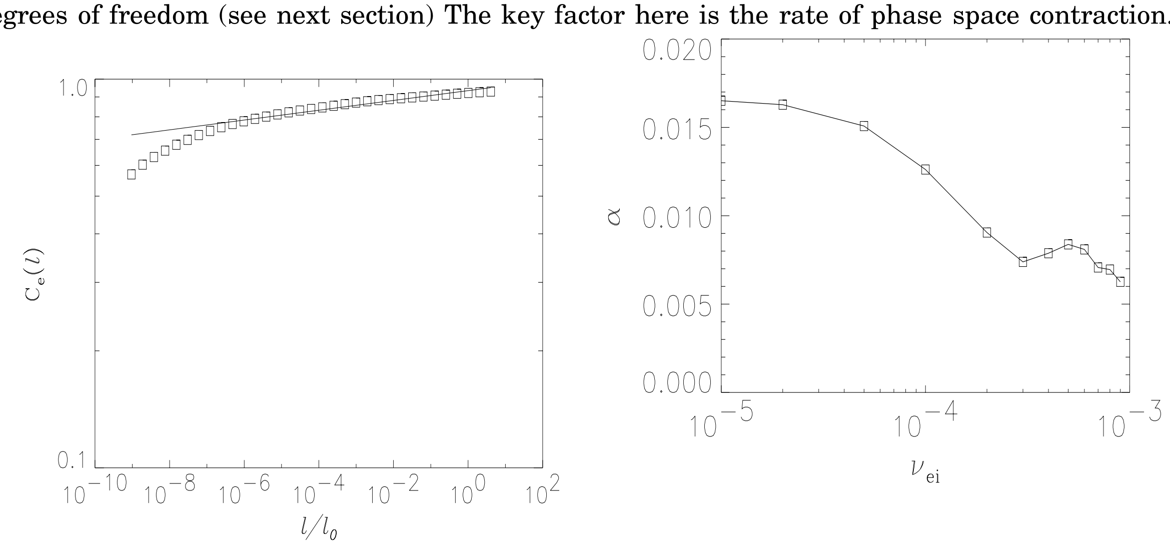


Figure 1: Correlation integral for $\nu_{ei} = 10^{-4}$ as a function of $\ell/\ell_0$ (left); dependence of $C(\ell)$ on $\nu_{ei}$.

To pursue this argument, we have measured the dependence of the correlation exponent $\alpha$ on the collision frequency (Figure 1; right). The general trend is a decrease in the correlation exponent, and therefore a decrease in the Hausdorff dimension, with increasing collision frequency. This is not surprising as the phase space contraction rate is related to, but not necessarily directly proportional to, the collision frequency.

# 5 Conclusions

We have identified the existence of a low-dimensional strange attractor in particle-in-cell, electrostatic drift-wave turbulence. The dimension of the attractor has been estimated based on the measurement of the correlation exponent [9] (a lower bound to the usual Hausdorff dimension). It has been shown that the dimension of the attractor is sensitive to the electron-ion collision frequency since this quantity is related to the contraction rate in phase.

Numerical results have shown the presence of a low-dimensional attractor in a system with many degrees of freedom. In a different context, Nicolis and Nicolis [12] have studied the attractor associated with the climatic evolution over the past million years based on isotope records of deep-sea cores. The surprising result of Nicolis and Nicolis's work is that, although the climate has very many degrees of freedom, a well-defined low-dimensional attractor was identified based on the experimental time series. Their results and our results suggest that some physical systems with many degrees of freedom can possess low-dimensional attractors, implying the presence of deterministic dynamics with few key variables but displaying unpredictable behavior (because of the fractal dimensionality of the attractor).

As a final remark, we note that, since the Grassberger-Procaccia algorithm is based on the information contained in one (or many) time series, their method can be useful to analyze and characterize strange attractors from experimental measurements in fusion plasmas.

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