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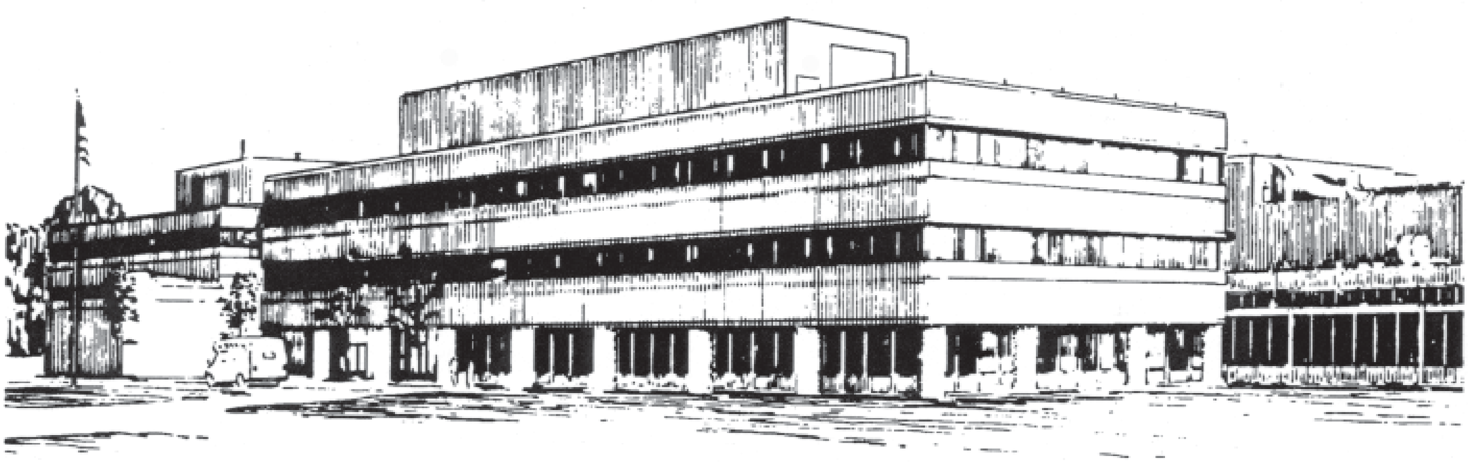

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Gabor wave packet method to solve plasma wave equations

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Abstract. A numerical method for solving plasma wave equations arising in the context of mode conversion between the fast magnetosonic and the slow (e.g ion Bernstein) wave is presented. The numerical algorithm relies on the expansion of the solution in Gaussian wave packets known as Gabor functions, which have good resolution properties in both real and Fourier space. The wave packets are ideally suited to capture both the large and small wavelength features that characterize mode conversion problems. The accuracy of the scheme is compared with a standard finite element approach.

INTRODUCTION

The problem of computing RF wave dynamics in fusion plasmas is numerically challenging due to locally fine-scale resonance and short-wavelength mode conversion effects [1]. The conversion of fast magnetosonic into ion Bernstein waves, for instance, requires the resolution of waves with dramatically different wavelengths.

Many codes written (e.g. the Mets code [2]) rely on a Fourier decomposition of the waves, requiring many modes to capture short-wavelength phenomena. Thus, there is a need to explore other more efficient numerical approaches, which provide better local resolution, and can take better advantage of windowing or multiple scale-length aspects of the problem. Due to the constraint that the dielectric tensor is most easily expressed analytically for sinusoidal waves, the extension of Fourier to using wave packets with a Gaussian envelop (Gabor functions) is most natural.

In this article, we explore a novel numerical approach based on expanding the solution in Gabor functions, which combines the advantages of the Fourier and finite element methods. The method, which we refer to as the Gabor element method (GEM), allows for a high degree of flexibility in the specification of boundary conditions. The process of discretization leads to, effectively, a sparse matrix due to the limited support of the Gabor functions. Therefore, GEM shares many similarities with the finite element method (FEM). However, GEM differs from FEM in that a single set of basis functions can be used to solve differential equations, in principle, of arbitrary order.

To validate GEM, we focus on two test problems. First, GEM is applied to solve a second order, Airy-type equation with a linear turning point (cut-off). The solution is then compared to that obtained using linear FEM. Next, a fourth-order, Wasow-type model equation describing the mode coupling between fast and slow waves, with two well separated wavelengths, is solved.

THE GABOR ELEMENT METHOD

Our aim is to solve ordinary differential equations of arbitrary order $2N$ (an even integer),

$$\sum_{i,j=0}^{N} (-)^i \frac{d^i}{dx^i}\left[f_{ij}(x)\frac{d^j y(x)}{dx^j}\right] = s(x) \;\; ; \;\; x \in [0,1], \tag{1}$$

subject to N boundary conditions at $x = 0$ and 1

$$\frac{d^i}{dx^i}\left[\sum_{j=0}^{N} f_{N,j}(x)\frac{d^j y(x)}{dx^j}\right]_0^1 = C_i + \sum_{j=0}^{N-1} B_{ij}\frac{d^j y(x)}{dx^j} \;\; ; \;\; i = 0\cdots N-1. \tag{2}$$

In (1), f_{ij} and s are user supplied functions of the independent variable x. Note that conditions (2) are flexible enough to accommodate Dirichlet, Neumann, or Robin boundary conditions by allowing, if required, the C_i's and B_{ij}'s to be infinite.

The Gabor element method is now presented. Following a Galerkin approach, (1) is multiplied by a test function $h(x)$ and integrated over the domain $[0,1]$ to yield

$$a(h,y) = b(h) \tag{3}$$

where

$$\begin{aligned} a(h,y) \equiv \quad & \int_0^1 dx \sum_{ij} \frac{d^i h}{dx^i} f_{ij} \frac{d^j y}{dx^j} + \sum_{i=1}^{N-1}\sum_{\ell=0}^{i} (-)^{i+\ell}\left[\frac{d^\ell h}{dx^\ell}\frac{d^{i-\ell-1}}{dx^{i-\ell-1}}\left(\sum_j f_{ij}\frac{d^j y(x)}{dx^j}\right)\right]_0^1 \\ & + \sum_{\ell=0}^{N} (-)^{\ell+N}\left[\frac{d^\ell h}{dx^\ell}\sum_j B_{N-\ell-1,j}\frac{d^j y}{dx^j}\right]_0^1 \end{aligned} \tag{4}$$

and

$$b(h) \equiv \int_0^1 dx\, h\, s - \sum_{\ell=0}^{N} (-)^{\ell+N}\left[\frac{d^\ell h}{dx^\ell} C_{N-\ell-1}\right]_0^1. \tag{5}$$

The first term in (4) represents the energy functional while the two subsequent terms arise after integrating by parts i times the term of (1) in [ ]. Next, the solution

$$y(x) = \sum_\gamma g_\gamma(x) y_\gamma \;\; ; \;\; g_\gamma(x) \equiv \mathrm{e}^{\mathrm{i}2\pi u_j x}\mathrm{e}^{-(x-x_i)^2/(2w_i^2)} \tag{6}$$

is expanded in Gabor wave packets $g_\gamma(x)$ where $\gamma = (i,j)$. Equation (6) is a double expansion in wave-numbers $2\pi u_j$, $j = -(N_F-1)/2\cdots(N_F-1)/2$ and envelop positions $x_i, i = 0,\cdots N_x$. Upon inserting (6) into (3) and choosing $h(x) = g_{\gamma'}$ we then get $N_x N_F$ linear coupled equations

$$\sum_\gamma A_{\gamma',\gamma} y_\gamma = b_{\gamma'}$$

for the unknowns y_γ, where $A_{\gamma',\gamma} \equiv a(g_{\gamma'}, g_\gamma)$ and $b_{\gamma'} \equiv b(g_{\gamma'})$.

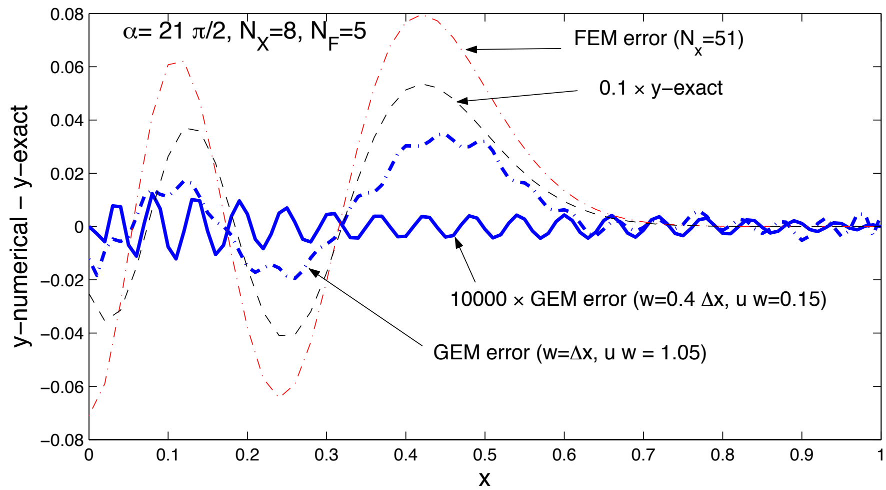

FIGURE 1. Error of the GEM solution for the Airy type equation using two combinations of u and w parameters. Notice the reduction factor of 10 000 used to plot the GEM solution obtained using $w = 0.4\Delta x$ and $uw = 0.15$. The FEM error obtained using linear hat elements (same number of degrees of freedom) is shown for comparison.

RESULTS

For simplicity, we will assume in the following that the phase-space lattice is uniform: $x_i = i\Delta x$ and $u_j = ju$. The accuracy of GEM will depend on the values of the grid spacing Δx, the fundamental frequency u and the Gaussian (half) width w. It can be proved that the Gabors form a frame only under the condition that $u\Delta x < 1$ [3]. Moreover, in order for the Gabors to overlap, we must have $2w \sim \Delta x$.

To determine more precisely the optimal u and w parameters, we solve equation $y'' + \alpha^2(1-2x)y = 0$ whose solution, the Airy function $\mathrm{Ai}[(\alpha/2)^{2/3}(2x-1)]$ is a propagating wave for $x < 1/2$ but evanescent for $x > 1/2$. Figure 1 shows the pointwise error of the Gabor solution for $\alpha = 21\pi/2$, using 8 envelops and 5 Fourier modes $(-2\cdots+2)$. The exact solution (reduced by a factor of 0.1) is shown as a dashed line. The GEM error (dash-dotted line) compares favorably with the FEM error obtained using the same number of degrees of freedom. Notice that the FEM error is proportional to the second derivative y'', as expected for linear hat elements. However, the choice of $w = \Delta x$ and $uw = 1.05$ is suboptimal; changing these parameters to $w/\Delta x = 0.4$ and $uw = 0.15$ suppresses the error by a factor > 10000. This emphasizes the ability of GEM to capture the solution more accurately than FEM with a small number of degrees of freedom.

To model the coupling of fast to slow waves, we solve the Wasow equation

$$\left(\frac{d^2}{dx^2} + k^2[1-0.5(x-0.5)]\right)\left(\frac{d^2}{dx^2} + k^2[1-160(x-0.5)]\right)y + \alpha y = 0$$

subject to boundary conditions $y(0) = 0$, $y(1) = 1$, and $y'(0) = y'(1) = 0$, with $k^2 = 2\times 10^3$ and $\alpha = 8\times 10^6$. To the right of $x = 0.506$, the slow wave is evanescent while to the left it is propagating with $\lambda \to 0.01$. The wavelength of the fast wave ranges

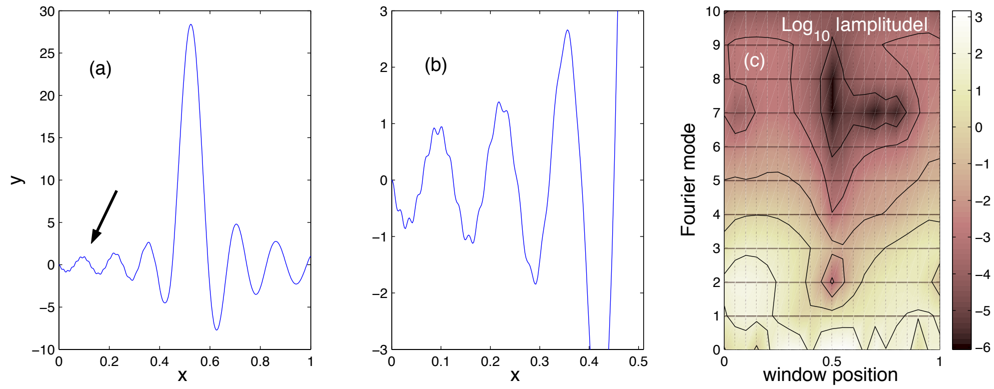

FIGURE 2. Solution of the fourth order equation obtained using $w/\Delta x = 1.0$, $u\Delta x = 0.9$, $N_x = 21$ and $N_F = 21$: (a) solution, (b) blow-up of $x < 0.5$ region, and (c) spectrogram (log of amplitude).

from $0.13 - 0.16$ across. Figure 2 shows the solution (a) with the short wavelength contribution from the slow wave clearly noticeable in (b). Picture (c) shows the mode structure in phase space, which depends on the choice of u and w parameters.

CONCLUSIONS

Good accuracy was achieved with the Gabor element method (GEM) when $u\Delta x < 1$ and $w/\Delta x \sim 0.4 - 1$. A small Gaussian width w yields a sparser matrix system but requires more Fourier modes. A larger $w/\Delta x \sim 1$ can be more efficient but yields a residual error that cannot be suppressed by increasing the resolution in phase space.

When optimally chosen, u and w yield an error that is insensitive to high order derivatives of the solution and so confers to GEM the capability to extract small and large features equally well. For problems with an oscillatory solution, GEM typically outperforms the finite element method error by several orders of magnitudes.

ACKNOWLEDGMENTS

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