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by

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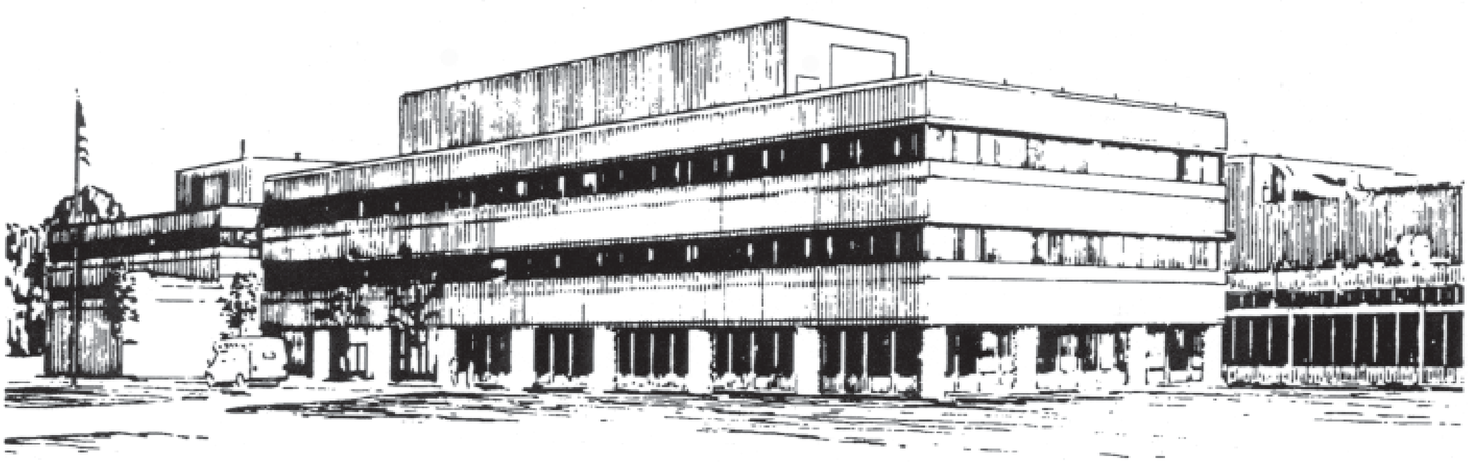

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Alfvén Waves in Gyrokinetic Plasmas

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Abstract

A brief comparison of the properties of Alfvén waves that are based on the gyrokinetic description with those derived from the MHD equations is presented. The critical differences between these two approaches are the treatment of the ion polarization effects. As such, the compressional Alfvén waves in a gyrokinetic plasma can be eliminated through frequency ordering, whereas geometric simplifications are needed to decouple the shear Alfvén waves from the compressional Alfvén waves within the context of MHD. Theoretical and numerical procedures of using gyrokinetic particle simulation for studying microturbulence and kinetic-MHD physics including finite Larmor radius effects are also presented.

I. INTRODUCTION

Recently, Qin et al. [1] has generalized the conventional low-frequency gyrokinetic theory to the high frequency regime. According to this new *gyrocenter-gauge kinetic theory*, the most critical ordering one needs in order to separate the fast gyromotion from the slow-moving gyrocenter motion is to assume that the ion gyroradii are much smaller than the scale lengths of the equilibrium magnetic field, i.e., $\rho_i/L_{B_0} \ll 1$. Under this assumption, they proceed to show that the kinetic description of magnetized plasmas in the gyrocenter coordinates is fully equivalent to the Vlasov-Maxwell system in the particle coordinates. Thus, in this view, the low-frequency gyrokinetic theory is a subset of the new gyrocenter-gauge kinetic theory when one averages out the gyrophase information. Using this simple concept of separating gyromotion from gyrocenter motion, we have first developed in this paper a fully electromagnetic gyrokinetic theory in the limit of $\rho_i \rightarrow 0$ based on a more intuitive approach rather than the usual Lie-perturbation [2] and pull-back transformation [1,3,4] methodology. The purpose is to demonstrate in a more transparent fashion that the unique treatment of polarization effects of the ions in the gyrokinetic theory is the key that enables us to add and suppress shear and compressional Alfvén waves without resorting to additional geometrical simplifications. These additional orderings are apparently needed to separate the shear Alfvén waves from the compressional Alfvén waves in the one-fluid MHD theory [5]. The unique treatment of separating ion polarization drift from the rest of gyrocenter motion in the electrostatic low-frequency gyrokinetic theory was first pointed out by Lee [6] and has been studied by many others [1,3,4,7–10]. It culminates with the recovery of the compressional Alfvén waves and the Bernstein harmonics in the gyrokinetic formalism [3] and the development of gyrokinetic equilibrium [4]. The first part of the article is an attempt to make contact with the MHD theory from the gyrokinetic point of view. We then proceed to discuss the gyrokinetic formulations and numerical issues for finite ρ_i that enable us to compliment the existing numerical tools for gyrokinetic particle simulation [11–13] to be used on massively parallel computers for studying electromagnetic turbulent transport and the related kinetic-MHD physics. The present paper is organized as follows: In Sec. II, Alfvén waves based on the MHD equations are re-visited. Their gyrokinetic counterparts are discussed in Sec. III. The electromagnetic gyrokinetic Vlasov-Maxwell equations in general geometry and the related numerical issues are presented in Sec. IV. The possibility of transport time scale simulation on massively parallel computers and the conclusions are given in Sec. V.

II. MHD ALFVÉN WAVES

In order to understand gyrokinetic Alfvén physics, let us first re-visit Alfvén waves using the one-fluid MHD description. The particular derivations presented here are for the purpose of facilitating the comparisons between the two approaches. The governing equations are: the continuity equation,

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0,$$

the momentum equation,

$$\rho_m \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \frac{1}{c} \mathbf{J} \times \mathbf{B} - \nabla p,$$

Ohm's law,

$$\mathbf{E} + \frac{1}{c} \mathbf{V} \times \mathbf{B} = \eta \mathbf{J},$$

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},$$

and Ampere's law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J},$$

where ρ_m is the mass density, $\mathbf{V}$ is the fluid velocity, $\mathbf{J}$ is the current, and $p = p_i + p_e$ is the pressure. For $\eta = 0$, $\delta\rho_m = 0$, $\delta p = 0$ and $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$, the governing equations in simple geometry take the familiar form of

$$\rho_{mo} \left(\frac{\partial \delta \mathbf{V}}{\partial t} + \delta \mathbf{V} \cdot \nabla \delta \mathbf{V} \right) + \frac{\mathbf{B}}{4\pi} \times (\nabla \times \delta \mathbf{B}) = 0, \quad (1)$$

and

$$\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (\delta \mathbf{V} \times \mathbf{B}), \tag{2}$$

where $\mathbf{B}_0$ is the external magnetic field, and the prefixed δ variables are the perturbed quantities. To facilitate the comparisons with the gyrokinetic approach, let us take time derivative of the linearized Faraday's law, Eq. (2), and substitute the resulting $\partial\delta\mathbf{V}/\partial t$ term by the linearized momentum equation, Eq. (1), to obtain

$$\frac{\partial^2 \delta \mathbf{B}}{\partial t^2} + v_A^2 [\nabla \times (\nabla \times \delta \mathbf{B})_\perp] = 0, \tag{3}$$

valid for $v_A (\equiv B_0/\sqrt{4\pi\rho_{mo}}) \gg c_s$. Consequently, we obtain

$$\frac{\partial^2 \delta B_\parallel}{\partial t^2} - v_A^2 \nabla^2 \delta B_\parallel = 0, \tag{4}$$

and

$$\frac{\partial^2 \delta \mathbf{B}_\perp}{\partial \mathbf{t^2}} - v_A^2 [\nabla_\parallel^2 \delta \mathbf{B}_\perp + \nabla_\perp (\nabla_\perp \cdot \delta \mathbf{B}_\perp)] = 0, \tag{5}$$

where $\parallel$ and $\perp$ denote directions parallel and perpendicualer to the external magentic field. Thus, for $\delta B_\parallel \neq 0$, the normal modes from Eq. (4) are the compressional Alfvén waves with

$$\omega^2 = k^2 v_A^2,$$

where $k^2 = k_\perp^2 + k_\parallel^2$. In the case of $\nabla_\perp \cdot \delta \mathbf{B}_\perp = 0$, the waves propagating perpendicular to both $\hat{\mathbf{b}}_0$ and $\mathbf{k}_\perp$ according to Eq. (5) are the shear-Alfvén waves with

$$\omega^2 = k_\parallel^2 v_A^2.$$

Otherwise, for $\nabla_\perp \cdot \delta \mathbf{B}_\perp \neq 0$, we can take $\nabla_\perp$ of Eq. (5) to obtain

$$\frac{\partial^2}{\partial t^2} (\nabla_\perp \cdot \delta \mathbf{B}_\perp) - v_A^2 \nabla^2 (\nabla_\perp \cdot \delta \mathbf{B}_\perp) = 0.$$

This equation is related to Eq. (4) through the condition of $\nabla \cdot \delta \mathbf{B} = 0$ and, hence, they both have compressional Alfvén waves of $\omega^2 = k^2 v_A^2$ as the normal modes. Consequently, for $\delta \mathbf{B}_\parallel = 0$ and $\nabla_\perp \cdot \delta \mathbf{B}_\perp = 0$, this system of equations has only shear-Alfvén waves.

These conditions can be satisfied by introducing

$$\delta \mathbf{B}_\perp = \nabla \times \mathbf{A}_\parallel = \nabla A_\parallel \times \hat{\mathbf{b}}_o,$$

where $\hat{\mathbf{b}}_0$ is the unit vector along $\mathbf{B}_0$. Substituting it into Eqs (1) and (2), we obtain

$$\frac{d\delta \mathbf{V}_\perp}{dt} + \frac{v_A^2}{B_0}\left[\hat{\mathbf{b}}_0 \times \nabla(\nabla \cdot \mathbf{A}_\parallel) + \frac{1}{B_0}\nabla^2 A_\parallel \nabla_\perp A_\parallel\right] = 0, \tag{6}$$

and

$$\frac{\partial \mathbf{A}_\parallel}{\partial t} = \delta \mathbf{V}_\perp \times (\mathbf{B}_0 + \delta \mathbf{B}_\perp) - c\nabla\phi, \tag{7}$$

where $\delta \mathbf{V}_\parallel = 0$ from $\mathbf{J} \times \mathbf{B} \cdot \hat{\mathbf{b}}_0 = 0$ and $d/dt = \partial/\partial t + \delta \mathbf{V}_\perp \cdot \nabla$. From $(\partial \mathbf{A}_\parallel/\partial t) \times \hat{\mathbf{b}}_o = 0$, we find from Eq. (7) that

$$\delta \mathbf{V}_\perp = -\frac{c}{B_o}\nabla\phi \times \hat{\mathbf{b}}_o$$

and $\nabla \cdot \delta \mathbf{V}_\perp = 0$. Equation (7) also gives

$$\frac{1}{c}\frac{\partial A_\parallel}{\partial t} + \mathbf{b} \cdot \nabla\phi = 0, \tag{8}$$

which is essentially the parallel part of the collisionless Ohm's law or Faraday's law, where $\mathbf{b} = \hat{\mathbf{b}}_0 + \delta \mathbf{B}_\perp/B_0$. Taking the curl of Eq. (6) and keeping only the parallel components along $\hat{\mathbf{b}}_o$, we obtain the so-called vorticity equation as

$$\frac{d\nabla_\perp^2\phi}{dt} + \frac{v_A^2}{c}(\hat{\mathbf{b}} \cdot \nabla)\nabla_\perp^2 A_\parallel = 0, \tag{9}$$

where $k_\parallel^2 \ll k_\perp^2$ is used together with the approximation of

$$\nabla_\perp^2(\hat{\mathbf{b}}_o \cdot \nabla)A_\parallel \approx (\hat{\mathbf{b}}_o \cdot \nabla)\nabla_\perp^2 A_\parallel, \tag{10}$$

which is only valid for $\hat{\mathbf{b}}_0 \neq \hat{\mathbf{b}}_0(\mathbf{x}_\perp)$. Equations (8) and (9) are the well-known reduced MHD equations [5]. For $exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$, the corresponding normal modes are the shear Alfvén waves of $\omega^2 = k_\parallel^2 v_A^2$. Thus, we indeed eliminate the compressional Alfvén waves and recover the previous analysis based on $\delta \mathbf{B}_\parallel = 0$ and $\nabla_\perp \cdot \delta \mathbf{B}_\perp = 0$. However, we have to make geometric simplifications in order to achieve this. We should remark here that the system conserves energy, i.e.,

$$\frac{\partial}{\partial t}\int\left[|\nabla_\perp\phi|^2+\frac{v_A^2}{c^2}|\nabla_\perp A_\parallel|^2\right]d\mathbf{x}=0,$$

which can be obtained by multiplying the vorticity equation, Eq. (9), by ϕ before integrating in $\mathbf{x}$. Strauss [5] has used Eqs. (8) and (9) for studying kink modes in slab geometry. However, the use of Eq. (10) in the derivation makes the regime of validity for these reduced MHD equation rather limited. The difficulty comes from the treatment of the $\nabla_\perp^2\phi$ term in the vorticity equation, Eq. (9), which is the contribution from the ion polarization drift in the gyrokinetic theory [6,7]. Thus, from one fluid MHD point of view, the elimination of compressional Alfvén waves cannot be accomplished by simply assuming that $\omega^2 \ll k^2v_A^2$ and it needs additional geometrical ordering in order to extract the $\nabla_\perp^2\phi$ term from the formulation. On the other hand, for gyrokinetic plasmas, by relating $\nabla_\perp^2\phi$ as the ion polarization drift, a totally different treatment of the term is employed without additional ordering, which we will explain.

III. GYROKINETIC ALFVÉN WAVES

The basic idea of gyrokinetic formulation of the Vlasov-Maxwell system is to first transform the distribution function F from the particle coordinates $(\mathbf{x},\mathbf{v},t)$ to the gyrocenter coordinates $(\mathbf{R},v_\parallel,\mu,\varphi,t)$, where $\mu\equiv v_\perp^2/2$, φ is the gyro-angle, and subscripts $\parallel$ and $\perp$ denote the directions parallel and perpendicular to the $\mathbf{B}_0$ field, respectively. This is valid in the limit of $\rho_i \ll L_{B_0}$ as stated earlier [1], and, in doing so, we separate the gyrocenter motion from the gyromotion. Here, ρ_i is the ion gyroradius and L_{B_0} is the scale length of the external magnetic field. In the low-frequency limit of $\omega \ll \Omega_i$, i.e., the frequency of interest is much smaller than the ion cyclotron frequency, along with the additional ordering of $e\phi/T_e \ll 1$, $\delta B \ll B$, $k_\parallel\rho_i \ll 1$, and $k_\perp\rho_i \approx 1$, a gyrophase-averaging process can then be used to eliminate φ as a phase variable, so that each particle can be viewed as a charged ring centered at the gyrocenter. When the distribution function F in the gyrocenter coordinates is transformed back to the particle coordinates to obtain number and current densities used by Maxwell's equations, one then recovers the ion polarization effects through the pull-back transformation. For detail, please refer to Refs. [1,3,4,6–11].

Here, we will present a derivation of the electromagnetic gyrokinetic-Maxwell equations based on the drift kinetic equation in the limit of $\rho_i \to 0$. This is a special case consistent with the condition of $\rho_i \ll L_{B0}$ given by Ref. [1] for the separation of the gyrocenter motion from the gyromotion itself. In fact, this is the only ordering we need for the rest of the derivation. Moreover, the gyromotion is inconsequential for now since gyroradius is zero. (The physics associated with $\rho_i \neq 0$ will be discussed later in Sec. IV.) We start first from the Vlasov equation,

$$\frac{dF_\alpha}{dt} \equiv \frac{\partial F_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial F_\alpha}{\partial \mathbf{x}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial F_\alpha}{\partial \mathbf{v}} = 0,$$

and let $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_{E\times B}$ with

$$\mathbf{v}_{E\times B} = \frac{c}{B}\mathbf{E}^L \times \hat{\mathbf{b}}_0,$$

we then obtain the usual drift kinetic equation in simple geometry in the electrostatic limit as,

$$\frac{\partial F_{\alpha gc}}{\partial t} + (\mathbf{v}_\parallel + \mathbf{v}_{E\times B}) \cdot \frac{\partial F_{\alpha gc}}{\partial \mathbf{x}} + \frac{q_\alpha}{m_\alpha} E_\parallel \frac{\partial F_{\alpha gc}}{\partial v_\parallel} = 0. \tag{11}$$

In the electrostatic limit, we have

$$\mathbf{E}^L = -\nabla\phi, \quad E_\parallel^L = -\hat{\mathbf{b}}_0 \cdot \nabla\phi,$$

where ϕ is the electrostatic potential, $\hat{\mathbf{b}}_0$ is the unit vector along the external magnetic field and the superscript L denotes the longitudinal quantity since $\nabla \times \nabla\phi = 0$.

However, the parallel drift, the perpendicular $\mathbf{E}^L \times \mathbf{B}$ drift and the parallel acceleration alone are not sufficient to capture all the relevant physics. Specifically, we have to include the ion polarization effects as well. In the electrostatic limit, the polarization drift is

$$\mathbf{v}_p^L = -\frac{mc^2}{eB^2}\frac{\partial \nabla_\perp \phi}{\partial t}. \tag{12}$$

It is interesting to note that this drift does not show up in the drift kinetic equation, because $v_p^L/v_{E\times B} \propto \omega/\Omega_i \ll 1$, where ω is the frequency of interest and Ω_i is the ion cyclotron frequency. To recover the ion polarization response, let us first use the drift kinetic equation including only the polarization drift to obtain

$$\frac{\partial n_p}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \int \mathbf{v}_p^L F_{igc} d\mathbf{v} = 0,$$

where $d\mathbf{v} \equiv dv_\parallel d\mu$. It then yields

$$\rho_p = \frac{1}{4\pi}\frac{\omega_{pi}^2}{\Omega_i^2}\nabla_\perp^2\phi,$$

where $\rho_p (= en_p)$ is the ion polarization density. From Poisson's equation of $\nabla^2\phi = 4\pi\rho$ we have

$$\nabla^2\phi + \frac{\omega_{pi}^2}{\Omega_i^2}\nabla_\perp^2\phi = -4\pi\rho_{gc}, \tag{13}$$

where $\rho = \rho_{gc} + \rho_p$ and

$$\rho_{gc} \equiv n_{igc} - n_{egc} = e\int (F_{igc} - F_{egc}) d\mathbf{v}$$

comes from the drift kinetic equation, Eq. (11). Since $\omega_{pi}^2/\Omega_i^2 \gg 1$ and $k_\parallel \ll k_\perp$ for our regime of interest, the first term in Eq. (13) can be ignored and we then recover the well-known quasineutral gyrokinetic Poisson's equation in the limit of $\rho_i \approx 0$ [6]. Combining Eq. (13) with the drift kinetic equation, Eq. (11), we have the governing gyrokinetic equations in the electrostatic approximation.

For the finite-β effects, let us now turn our attention to Ampere's law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

where $\mathbf{E} = \mathbf{E}^L + \mathbf{E}^T$ and the superscript T denotes the transverse quantity, i.e., $\nabla \cdot \mathbf{E}^T = 0$. For $\omega^2 \ll k^2c^2$, the transverse induction current can be ignored, i.e., $\partial \mathbf{E}^T/\partial t \approx 0$, where ω is the frequency of interest. This is the so-called Darwin model [14]. Furthermore, by letting $\partial \mathbf{E}^L/\partial t \approx 0$ and taking $\nabla\cdot$ of Ampere's law, we recover the well-known quasineutrality condition of

$$\nabla \cdot \mathbf{J} = 0.$$

Thus, if we are only interested in the low-frequency waves, we can neglect displacement currents altogether in Ampere's law. We will discuss this point later in this section. By using the Coulomb gauge of $\mathbf{B} = \nabla \times \mathbf{A}$ and $\nabla \cdot \mathbf{A} = 0$, we then have

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J},$$

where

$$\mathbf{J} = \mathbf{J}_{gc} + e\int \mathbf{v}_p^T F_{igc} d\mathbf{v},$$

$\mathbf{J}_{gc}$ the the gyrocenter current from the drift kinetic equation and

$$\mathbf{v}_p^T = -\frac{mc^2}{eB^2}\frac{1}{c}\frac{\partial \mathbf{A}_\perp}{\partial t} \tag{14}$$

is the transverse polarization drift for the ions, as first pointed out by Qin et al. [3]. Consequently, gyrokinetic Ampere's law becomes

$$\nabla^2 \mathbf{A} - \frac{1}{v_A^2}\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}_{gc}, \tag{15}$$

which, for $\mathbf{J}_{gc} = 0$ and $\mathbf{A}_\perp \neq 0$ and with the ansatz of $exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$, give rises to the compressional Alfvén normal modes as

$$\omega^2 = k^2 v_A^2,$$

where $v_A \equiv c\Omega_i/\omega_{pi}$. However, for $\mathbf{J}_{gc} \neq 0$ and $\omega^2 \ll k^2 v_A^2$, the explicitly time-dependent part of Eq. (15) can be ignored, where ω is the frequency of interest. Thus, when the compressional Alfvén waves are not essential, e.g., in low-frequency gyrokinetic (or drift-kinetic) plasmas, they can be suppressed easily without invoking additional approximation as in the case for MHD discussed in Sec. II. We will refer to this approximate form of the equation as the low-frequency gyrokinetic Ampere's law. This important feature will be discussed later. The drift kinetic equation, Eq. (11), is now modified by

$$\mathbf{v}_{E\times B} = \frac{c}{B_o}\mathbf{E}^L \times \hat{\mathbf{b}}_o, \quad \mathbf{v}_\parallel = v_\parallel \mathbf{b},$$

$$E_\parallel = E_\parallel^L + E_\parallel^T = -\mathbf{b}\cdot\nabla\phi - \frac{1}{c}\frac{\partial A_\parallel}{\partial t}, \tag{16}$$

$$\mathbf{b} \equiv \frac{\mathbf{B}}{B} \approx \hat{\mathbf{b}}_o + \frac{\delta\mathbf{B}}{B_0} = \hat{\mathbf{b}}_o + \frac{\nabla\times\mathbf{A}}{B_o},$$

and ϕ and $\mathbf{A}$ are given by Eqs. (13) and (15), respectively. Here, the conservation of $\mu_B(\equiv v_\perp^2/2B_0)$ is assumed, which we will discuss later. Consequently, only the parallel part of Faraday's law, $\mathbf{E} = (1/c)(\partial\mathbf{A}/\partial t) - \nabla\phi$, is used. These equations are similar to those presented in the earlier work based on more rigorous derivations [8,9] and are the electromagnetic version of the gyrokinetic system in slab geometry.

We have so far shown how, in the drift kinetic limit, the transverse part of the polarization drift gives rise to the compressional Alfvén waves. Let us now proceed to show how the longitudinal part of the polarization drift plays such an essential role for the shear Alfvén waves. The zeroth-order velocity moments of the drift kinetic equation, Eq. (11), gives

$$\frac{d\rho_{gc}}{dt} + \mathbf{b}\cdot\nabla J_{\parallel gc} = 0, \tag{17}$$

where

$$\rho_{gc} = e\int (F_{igc} - F_{egc})d\mathbf{v},$$

$$J_{\parallel gc} = e\int v_\parallel (F_{igc} - F_{egc})d\mathbf{v},$$

and

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} - \frac{c}{B}\nabla\phi\times\hat{\mathbf{b}}_0\cdot\nabla.$$

With the substitution of the gyrokinetic Poisson's equation, Eq. (13), and the gyrokinetic Ampere's law, Eq. (15), in the quasineutral low-frequency limit, we have

$$\frac{d\nabla_\perp^2\phi}{dt} + \frac{v_A^2}{c}(\mathbf{b}\cdot\nabla)\nabla^2 A_\parallel = 0. \tag{18}$$

This is the gyrokinetic version of the vorticity equation, Eq. (9), and the first term on the RHS comes from the polarization density in Eq. (13). Most importantly, as one can see, unlike for the MHD case, no additional geometric approximation described by Eq. (10) is needed in obtaining Eq. (18), neither do we need to assume $k_\parallel^2 \ll k_\perp^2$. Letting $E_\parallel = 0$ in the drift kinetic equation, we recover the collisionless parallel Ohm's law, Eq. (8), in its nonlinear form as

$$\frac{1}{c}\frac{\partial A_\parallel}{\partial t} + \mathbf{b}\cdot\nabla\phi = 0. \tag{19}$$

These two equations, Eqs. (18) and (19) are often referred as the reduced two-field MHD equations, which, for the ansatz of $exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$, give the shear-Alfvén normal modes as

$$\omega^2 = k_\parallel^2 v_A^2.$$

Thus, we easily make contact with the reduced MHD equations, Eqs. (8) and (9). However, Eqs. (18) and (19) are more general by including terms associated with $\hat{\mathbf{b}}$, as defined in Eq. (16), and with ∇^2 instead of $\nabla_\perp^2$. Moreover, we should remark here, Eqs. (18) and (19) are valid for sheared slab, but not Eqs. (8) and (9). As we know, $E_\parallel = 0$ is a special case for the drift kinetic equation. In general, to study the kinetic shear Alfvén waves, we have to solve the drift kinetic equation, Eq. (11), along with Eq. (13), Eq. (15) and Eq. (16). (See, e.g., Ref. [13]) In the cold electron limit, the shear Alfvén eigenmodes are $\omega^2 = k_\parallel^2 v_A^2/(1 + c^2k^2/\omega_{pe}^2)$, while, in the warm electron limit, they become $\omega^2 = k_\parallel^2 v_A^2(1 + k_\perp^2\rho_s^2)$, where $\rho_s \equiv \rho_i\sqrt{T_e/T_i}$.

The parallel Ampere's law can be calculated as

$$\nabla^2 A_\parallel = -\frac{4\pi}{c}\sum_\alpha q_\alpha \int v_\parallel F_{\alpha gc} d\mathbf{v} \tag{20}$$

Equations (20) together with the gyrokinetic Poisson's equation, Eq. (13) as well as the drift kinetic equation, Eqs. (11) and (16), form a complete set of equations describing low-frequency physics for magnetically confined plasmas for studying both gradient-driven microinstabilities for $k_\perp\rho_i \ll 1$. The reason that the longitudinal induction current, $\partial\mathbf{E}^L/\partial t$, can be ignored in Ampere's law for low-frequency waves is as follows. From Eq. (19), which gives $k_\parallel \phi = \omega A_\parallel/c$ since $E_\parallel = 0$, and from $\nabla \cdot \mathbf{A} = 0$, the term in question is small if $\omega^2 \ll k^2c^2$ and/or $\omega^2 \ll k_\parallel^2 c^2$. This set of equations is energy conserving. It can be shown that

$$\frac{\partial}{\partial t}\left[\langle\sum_\alpha \frac{m_\alpha}{2}\int v_\parallel^2 F_{\alpha gc} dv_\parallel d\mu\rangle_{\mathbf{x}} + \frac{\omega_{pi}^2}{\Omega_i^2}\frac{1}{8\pi}\langle|\nabla_\perp \phi|^2\rangle_{\mathbf{x}} + \frac{1}{8\pi}\langle|\nabla A_\parallel|^2\rangle_{\mathbf{x}}\right] = 0,$$

where $\langle\cdots\rangle_{\mathbf{x}}$ denotes spatial average.

The inclusion of the gyrocenter current in the perpendicular direction in $\mathbf{J}_{gc} = \mathbf{J}_{\parallel gc} + \mathbf{J}_{\perp gc}$ is related to general geometry and will be carried out in the next section.

IV. ELECTROMAGNETIC GYROKINETIC VLASOV-MAXWELL EQUATIONS IN GENERAL GEOMETRY

For our purpose so far, we have assumed that particle gyroradius $\boldsymbol{\rho} \rightarrow 0$ in calculating n_{gc} and $\mathbf{J}_{\parallel gc}$ and have shown the origin of polarization effects in the gyrokinetic Maxwell's equations. However, from the earlier analyses [7,11], it has been pointed out that the drift kinetic equation, Eq. (11), actually describes the evolution of the distribution function $F(\mathbf{R})$ in gyrocenter coordinates, whereas Maxwell's equations, Eqs. (13) and (15), require $n_{gc}(\mathbf{x})$ and $\mathbf{J}_{gc}(\mathbf{x})$ in particle coordinates. The two coordinates are related through $\mathbf{x} = \mathbf{R} + \boldsymbol{\rho}$, and $\boldsymbol{\rho} \neq 0$ gives rise to the finite Larmor radius (FLR) effects. It is the transformation of the distribution function F from $\mathbf{R}$ to $\mathbf{x}$ that first captured the polarization effects [6,7] in gyrokinetic Poisson's equations. Qin et. al [1,3,4] call it the pullback transformation. We have demonstrated in Sec. III that ion polarization effects in the limit of small $k_\perp \rho_i$ can be recovered without resorting to cooridinates transformation. However, to capture the FLR effects in the other aspects of gyrokinetics, we have to let $\boldsymbol{\rho} \neq 0$ in calculating field quantities and pushing particles in general geometry.

Thus, the characteristics of the nonlinear gyrokinetic Vlasov equation,

$$\frac{\partial F_{\alpha gc}}{\partial t} + \frac{d\mathbf{R}}{dt}\cdot\frac{\partial F_{\alpha gc}}{\partial \mathbf{R}} + \frac{dv_\parallel}{dt}\frac{\partial F_{\alpha gc}}{\partial v_\parallel} = 0, \tag{21}$$

in general geometry including toroidal effects for finite $k_\perp \rho_\alpha$ are [8–10]

$$\frac{d\mathbf{R}}{dt} = v_\parallel \mathbf{b}^* + \frac{v_\perp^2}{2\Omega_{\alpha 0}} \hat{\mathbf{b}}_0 \times \nabla ln B_0 - \frac{c}{B_0} \nabla \bar{\phi} \times \hat{\mathbf{b}}_0,$$

$$\frac{dv_\parallel}{dt} = -\frac{v_\perp^2}{2} \mathbf{b}^* \cdot \nabla ln B_0 - \frac{q_\alpha}{m_\alpha} \left(\mathbf{b}^* \cdot \nabla \bar{\phi} + \frac{1}{c} \frac{\partial \bar{A}_\parallel}{\partial t} \right),$$

where

$$\mathbf{b}^* \equiv \mathbf{b} + \frac{v_\parallel}{\Omega_{\alpha 0}} \hat{\mathbf{b}}_0 \times (\hat{\mathbf{b}}_0 \cdot \nabla) \hat{\mathbf{b}}_0,$$

$$\mathbf{b} = \hat{\mathbf{b}}_0 + \frac{\nabla \times \bar{\mathbf{A}}}{B_0},$$

$$F_{\alpha gc} = \sum_{j=1}^{N_\alpha} \delta(\mathbf{R} - \mathbf{R}_{\alpha j}) \delta(\mu - \mu_{\alpha j}) \delta(v_\parallel - v_{\parallel \alpha j}),$$

N_α is the total number of particles, ρ_α is the gyroradius of the species α, $\Omega_{\alpha 0} \equiv q_\alpha B_0 / m_\alpha c$ and the gyrophase averaged potentials are

$$\begin{pmatrix} \bar{\phi} \\ \bar{\mathbf{A}} \end{pmatrix} (\mathbf{R}) = \langle \int \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix} (\mathbf{x}) \delta(\mathbf{x} - \mathbf{R} - \boldsymbol{\rho}) d\mathbf{x} \rangle_\varphi,$$

with $\langle \cdots \rangle_\varphi \equiv \oint d\varphi / 2\pi$. Here, for our present purpose, we assume that

$$\mu_B \equiv v_\perp^2 / 2B_0 \approx cons. \tag{22}$$

and the background Maxwellian distribution is

$$F_{\alpha gc}^M = \frac{n}{\sqrt{2\pi} v_{t\alpha}^3} exp(-\frac{v_\perp^2 + v_\parallel^2}{2 v_{t\alpha}^2}).$$

The validity of Eq. (22) is discussed in Appendix A.

The field equations are still given by Eqs. (13) and (15), where the charge and current densities now become [7–11]

$$\rho_{gc}(\mathbf{x}) = \sum_\alpha q_\alpha \langle \int F_{\alpha gc}(\mathbf{R}) \delta(\mathbf{R} - \mathbf{x} + \boldsymbol{\rho}) d\mathbf{R} dv_\parallel d\mu \rangle_\varphi, \tag{23}$$

and

$$\mathbf{J}_{gc}(\mathbf{x}) = \mathbf{J}_{\parallel gc}(\mathbf{x}) + \mathbf{J}^M_{\perp gc}(\mathbf{x}) + \mathbf{J}^d_{\perp gc}(\mathbf{x})$$

$$= \sum_\alpha q_\alpha \langle \int (\mathbf{v}_\parallel + \mathbf{v}_\perp + \mathbf{v}_d) F_{\alpha gc}(\mathbf{R}) \delta(\mathbf{R} - \mathbf{x} + \rho) d\mathbf{R} dv_\parallel d\mu \rangle_\varphi, \tag{24}$$

where

$$\mathbf{v}_d \equiv \frac{v_\parallel^2}{\Omega_{\alpha 0}} \hat{\mathbf{b}}_0 \times (\hat{\mathbf{b}}_0 \cdot \nabla) \hat{\mathbf{b}}_0 + \frac{v_\perp^2}{2\Omega_{\alpha 0}} \hat{\mathbf{b}}_0 \times \nabla ln B_0,$$

and $\mathbf{J}_{\parallel gc}$, $\mathbf{J}^M_{\perp gc}$, and $\mathbf{J}^d_{\perp gc}$ are calculated from $\mathbf{v}_\parallel$, $\mathbf{v}_\perp$, and $\mathbf{v}_d$, respectively.

If we are only interested in physics with $k_\perp^2 \rho_i^2 \ll 1$, we can assume that $\boldsymbol{\rho} \to 0$ in the evaluation of $\bar{\phi}$, $\bar{A}_\parallel$ in Eq (21) and $\rho(\mathbf{x})$ in Eq. (23), i.e., there is no difference between $\mathbf{x}$ and $\mathbf{R}$. This approximation is also true for the calculations for the parallel current, $\mathbf{J}_{\parallel gc}$, and the magnetic drift current, $\mathbf{J}^d_{\perp gc}$. In this sense, we recover the drift-kinetic approximation of the Vlasov-Maxwell system in general geometry. However, we need to assume that $k_\perp^2 \rho_i^2$ is small but finite in order to account for the finite Larmor radius (FLR) effects which relate diamagnetic current, $\mathbf{J}^M_{\perp gc}$, to the plasma pressure. In the Fourier $\mathbf{k}$-space, it takes the form of

$$\mathbf{J}^M_{\perp gc}(\mathbf{x}) = \sum_\alpha q_\alpha \sum_{\mathbf{k}} \int F_{\alpha gc}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \langle \mathbf{v}_\perp e^{i\mathbf{k}\cdot\boldsymbol{\rho}} \rangle_\varphi dv_\parallel d\mu, \tag{25}$$

where $\mathbf{k}_\perp = k_\perp (cos\theta \hat{\mathbf{e}}_1 + sin\theta \hat{\mathbf{e}}_2)$, $\mathbf{v}_\perp = v_\perp (cos\varphi \hat{\mathbf{e}}_1 + sin\varphi \hat{\mathbf{e}}_2)$, and $\boldsymbol{\rho} = (v_\perp / \Omega_\alpha)(-sin\varphi \hat{\mathbf{e}}_1 + cos\varphi \hat{\mathbf{e}}_2)$. From

$$exp(-i\mathbf{k}\cdot\boldsymbol{\rho}) = exp\left[i \frac{k_\perp v_\perp}{\Omega_{\alpha 0}} sin(\varphi - \theta) \right]$$

$$= \sum_{n=-\infty}^{+\infty} J_n(\frac{k_\perp v_\perp}{\Omega_{\alpha 0}}) e^{in(\varphi - \theta)},$$

we have

$$\langle \mathbf{v}_\perp e^{-i\mathbf{k}\cdot\boldsymbol{\rho}} \rangle_\varphi = i v_\perp J_1(\frac{k_\perp v_\perp}{\Omega_{\alpha 0}})(-sin\theta \hat{\mathbf{e}}_1 + cos\theta \hat{\mathbf{e}}_2), \tag{26}$$

where J_n is the Bessel function of the n-th order. Substituting it into the equation for $\mathbf{J}^M_{\perp gc}$, Eq. (25), we obtain, for $k_\perp v_\perp / \Omega_\alpha \ll 1$,

$$\mathbf{J}^M_{\perp gc}(\mathbf{x}) = -\sum_\alpha \nabla_\perp \times \frac{c\hat{\mathbf{b}}_0}{B_0} p_{\alpha\perp}, \tag{27}$$

where $p_{\alpha\perp} = m_\alpha \int (v_\perp^2/2) F_{\alpha gc}(\mathbf{x}) dv_\| d\mu$ and $J_1(k_\perp v_\perp/\Omega_\alpha) \approx k_\perp v_\perp/\Omega_\alpha$ is used. The perpendicular current associated with magnetic drifts becomes

$$\mathbf{J}^d_{\perp gc} = \frac{c}{B_0}\sum_\alpha \left[p_{\alpha\|}(\nabla \times \hat{\mathbf{b}}_0)_\perp + p_{\alpha\perp}\hat{\mathbf{b}}_0 \times (\nabla ln B_0)\right], \tag{28}$$

where $p_{\alpha\|} = m_\alpha \int v_\|^2 F_{\alpha gc}(\mathbf{x}) dv_\| d\mu$, and $\hat{\mathbf{b}}_0 \times (\hat{\mathbf{b}}_0 \cdot \nabla)\hat{\mathbf{b}}_0 = (\nabla \times \hat{\mathbf{b}}_0)_\perp$ is used. Therefore, we have [4]

$$\mathbf{J}_{\perp gc} = \mathbf{J}^M_{\perp gc} + \mathbf{J}^d_{\perp gc} = \frac{c}{B_0}\sum_\alpha \left[\hat{\mathbf{b}}_0 \times \nabla p_{\alpha\perp} + (p_{\alpha\|} - p_{\alpha\perp})(\nabla \times \hat{\mathbf{b}}_0)_\perp\right]. \tag{29}$$

For $p = p_{\alpha\|} = p_{\alpha\perp}$, we recover the usual expression for pressure balance as

$$\mathbf{J}_{\perp gc} = \frac{c}{B_0}\sum_\alpha \hat{\mathbf{b}}_0 \times \nabla p_\alpha. \tag{30}$$

Gyrokinetic Ampere's law Eq. (15), in the low frequency limit, can now be written as

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c}(\mathbf{J}_{\|gc} + \mathbf{J}^M_{\perp gc} + \mathbf{J}^d_{\perp gc}), \tag{31}$$

where the three gyrocenter currents are defined in Eq. (24). Since we assume that gyroradius is much smaller than the scale length of the magnetic field inhomogeneity, the current associated with the magnetic drift given by Eq. (28) is adequate for our purpose. However, the expressions for the diamagnetic currents in Eqs. (27), (29) and (30) are only valid for $k_\perp^2\rho_\alpha^2 \ll 1$. The calculations for $\mathbf{J}_{\|gc}$ and $\mathbf{J}^M_{\perp gc}$ in Eq. (24) for $k_\perp\rho_i \approx 1$ along with the FLR considerations for ρ_{gc} in Eq. (23) as well as $\bar{\phi}$ and $\bar{A}_\|$ in Eq. (21) wiil be discussed later.

Taking the zeroth-order velocity moments of Eq. (21) yields

$$\frac{d}{dt}\rho_{gc} + \hat{\mathbf{b}} \cdot \nabla J_{\|gc} + \nabla_\perp \cdot \mathbf{J}^d_{\perp gc} = 0,$$

valid to the lowest order in terms of gyroradius vs. magnetic inhomogeneity, ρ/L_B. where ρ_{gc}, $J_{\|gc}$, and $\mathbf{J}^d_{\perp gc}$ are given by Eqs. (23),(24), and (28), respectively and

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} - \frac{c}{B}\nabla\bar{\phi} \times \hat{\mathbf{b}}_0 \cdot \nabla.$$

Substituting with Eq. (13) and Eq. (31), we have

$$\frac{d}{dt}\nabla_\perp^2\phi + \frac{v_A^2}{c}(\hat{\mathbf{b}}\cdot\nabla)\nabla^2 A_\parallel - 4\pi\frac{v_A^2}{c^2}\nabla_\perp\cdot\mathbf{J}_{\perp gc}^d = 0 \tag{32}$$

This is the toroidal version of Eq. (18). Assuming that $p_\alpha = p_{\alpha\parallel} = p_{\alpha\perp}$ and

$$\frac{dp_\alpha}{dt} = 0$$

together with Eq. (19), we obtain a more complete version of the Strauss's reduced high β equations [15] without any geometrical simplifications. This example serves to emphasize the point that gyrokinetic Vlasov-Maxwell equations contain all the MHD physics, within the limit of gyrokinetic ordering.

Now let us turn our attention to FLR effects. For $k_\perp\rho_i \approx 1$, the calculation of the perpendicular current, Eq. (25), involves the Bessel function, J_1. As shown earlier [11] for the calculation of the charge density which contains the Bessel Function J_0 , it is best to evaluate these functions in the configuration space rather than the Fourier $\mathbf{k}$ space. Let us follow the same recipe and use the discrete representation of $F(\mathbf{R},\mu,v_\parallel,t)$ in Eq. (21). Therefore, the charge density in Eq. (13), given by

$$\rho_{gc}(\mathbf{x}) = \sum_\alpha q_\alpha\langle\int F_{\alpha gc}(\mathbf{R})\delta(\mathbf{R}-\mathbf{x}+\boldsymbol{\rho})d\mathbf{R}dv_\parallel d\mu\rangle_\varphi \tag{33}$$

in Fourier $\mathbf{k}$ space, becomes

$$\rho_{gc}(\mathbf{x}) = \sum_\alpha q_\alpha\sum_\mathbf{k}\int F_{\alpha gc}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\langle e^{i\mathbf{k}\cdot\boldsymbol{\rho}}\rangle_\varphi dv_\parallel d\mu,$$

Substituting the discrete expression into ρ_{gc}, we have

$$\rho_{gc}(\mathbf{x}) = \sum_\alpha q_\alpha\sum_{j=1}^N\langle\delta(\mathbf{x}-\mathbf{x}_{\alpha j})\rangle_\varphi$$

$$= \sum_\alpha q_\alpha\sum_{j=1}^N\langle\int\delta(\mathbf{R}-\mathbf{R}_{\alpha j})\delta(\mathbf{x}-\mathbf{R}-\boldsymbol{\rho}_{\alpha j})d\mathbf{R}\rangle_\varphi$$

$$= \sum_\alpha q_\alpha\sum_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{j=1}^N e^{-i\mathbf{k}\cdot\mathbf{R}_{\alpha j}}\langle e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{\alpha j}}\rangle_\varphi/V,$$

where the relationship of

$$\delta(\mathbf{x}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}/V$$

is used, and V is the volume. The parallel current $\mathbf{J}_{\|gc}$, from Eq. (24), now becomes

$$\mathbf{J}_{\|gc} = \sum_{\alpha} q_{\alpha} \sum_{j=1}^{N} v_{\|\alpha j} \langle \delta(\mathbf{x} - \mathbf{x}_{\alpha j}) \rangle_{\varphi} \tag{34}$$

$$= \sum_{\alpha} q_{\alpha} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{j=1}^{N} v_{\|\alpha j} e^{-i\mathbf{k}\cdot\mathbf{R}_{\alpha j}} \langle e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{\alpha j}} \rangle_{\varphi}/V.$$

Likewise, from Eq. (21), we have

$$\begin{pmatrix} \bar{\phi} \\ \bar{\mathbf{A}} \end{pmatrix} (\mathbf{R}_{\alpha j}) = \langle \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix} (\mathbf{x}_{\alpha j}) \rangle_{\varphi} \tag{35}$$

$$= \sum_{\mathbf{k}} \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix} (\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{R}_{\alpha j}} \langle e^{i\mathbf{k}\cdot\boldsymbol{\rho}_{\alpha j}} \rangle_{\varphi}.$$

The quantity representing gyrophase averaging for ρ_{gc}, $J_{\|gc}$, $\bar{\phi}$, and $\bar{\mathbf{A}}$,

$$\langle e^{\pm i\mathbf{k}\cdot\boldsymbol{\rho}_{\alpha j}} \rangle_{\varphi} = J_0(k_{\perp}\rho_{\alpha j}),$$

can be calculated by a charged ring in the $\mathbf{x}$-space [11] as

$$\langle e^{\pm i\mathbf{k}\cdot\boldsymbol{\rho}_{\alpha j}} \rangle_{\varphi} = \sum_{n=-\infty}^{\infty} J_n(k_{\perp}\rho_{\alpha j}) \frac{1}{L} \sum_{l=1}^{L} exp(\frac{i2\pi nl}{L})$$

$$= \sum_{n=-\infty}^{\infty} J_n(k_{\perp}\rho_{\alpha j}) \frac{1}{2L} sin2\pi n \frac{cos\frac{n\pi}{L}}{sin\frac{n\pi}{L}}$$

$$= J_0 + O(J_{\pm mL}), m = 1, 2, 3, \cdots, \tag{36}$$

where L is the number of points in a ring for the numerical calculation. For $L \to \infty$, we recover J_0. However, only four points are needed ($L = 4$), if we use a grid size of ρ_i in the simulation and, consequently, are only interested in $k_\perp \rho_i < 2$, for which $J_0 \gg J_4$. In such a system, electrons can be represented by the gyrocenters since $\rho_e \ll \rho_i$.

We can use the similar treatment for the perpendicular current density, given by perpendicular part of Eqs. (24) or (25). Thus, we have

$$\mathbf{J}^M_{\perp gc}(\mathbf{x}) = \sum_\alpha q_\alpha \sum_{j=1}^N \langle \mathbf{v}_{\perp\alpha j} \delta(\mathbf{x} - \mathbf{x}_{\alpha j}) \rangle_\varphi \tag{37}$$

$$= \sum_\alpha q_\alpha \sum_{j=1}^N \langle \int \mathbf{v}_{\perp\alpha j} \delta(\mathbf{R} - \mathbf{R}_{\alpha j}) \delta(\mathbf{x} - \mathbf{R} - \boldsymbol{\rho}_{\alpha j}) d\mathbf{R} \rangle_\varphi$$

$$= \sum_\alpha q_\alpha \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{j=1}^N e^{-i\mathbf{k}\cdot\mathbf{R}_{\alpha j}} \langle \mathbf{v}_{\perp\alpha j} e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{\alpha j}} \rangle_\varphi / V,$$

where$\langle \mathbf{v}_{\perp\alpha j} e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{\alpha j}} \rangle_\varphi$ representing gyrophase averaging is given by Eq. (26). Similar to Eq. (36), we can represent this gyrophase averaging process by a rotating charged ring to obtain

$$\langle \mathbf{v}_\perp e^{-i\mathbf{k}\cdot\boldsymbol{\rho}_{\alpha j}} \rangle_\varphi = \sum_{n=-\infty}^{\infty} J_n(k_\perp \rho_{\alpha j}) e^{-in\theta} \frac{v_\perp}{L} \sum_{l=1}^{L} exp(\frac{i2\pi nl}{L})(cos\frac{2\pi l}{L}\hat{\mathbf{e}}_1 + sin\frac{2\pi l}{L}\hat{\mathbf{e}}_2)$$

$$= \sum_{n=-\infty}^{\infty} J_n(k_\perp \rho_{\alpha j}) e^{-in\theta} \frac{v_\perp}{2L} \sum_{+,-} cos[(1+\frac{1}{L})(n \pm 1)\pi] \frac{sin(n\pm 1)\pi}{sin[(n\pm 1)\pi/L]} (\hat{\mathbf{e}}_1 \mp i\hat{\mathbf{e}}_2)$$

$$= iv_\perp J_1(-sin\theta \hat{\mathbf{e}}_1 + cos\theta \hat{\mathbf{e}}_2) + O(J_{\pm 1 \pm mL}), m = 1, 2, 3, \cdots. \tag{38}$$

For $L \to \infty$, we recover Eq. (26). Since $J_1 \gg J_3$ for $k_\perp \rho_i < 2$, we can again use 4 points ($L = 4$) to represent a rotating charged ring as shown in Fig. 1. Thus, Eq. (37) with $L = 4$ is a more accurate way to calculate the perpendicular current for finite $k_\perp \rho_{\alpha j}$ than that of Eq. (27).

With these numerical gyrophase averaging schemes for calculating $\langle \cdots \rangle_\varphi$ in Eqs. (33), (34), (35) and (37) in place, we can then use them in Eqs. (13), (15) or (31), (21) and (35) to push particles by following the procedures outlined in Ref. [11]. We should remark here that Eq. (13) is only valid for small $k_\perp \rho_i$ and we need to use the original form of ion polarization in Refs. [6,7,11] for finite $k_\perp \rho_i$. Namely, we should replace the $(\omega_{pi}^2/\Omega_i^2)\nabla^2\phi(\mathbf{x})$ term in Eq. (13) by solving instead

$$(\tau/\lambda_D^2)(\phi - \tilde{\phi}) = -4\pi\rho_{gc}, \tag{39}$$

where

$$\tilde{\phi}(\mathbf{x}) \equiv \int \bar{\phi}(\mathbf{R})F_i(\mathbf{R},\mu,v_\parallel)\delta(\mathbf{R},\mu,v_\parallel)d\mathbf{R}d\mu dv_\parallel,$$

$\tau \equiv T_e/T_i$, $\lambda_D \equiv \sqrt{T_e/4\pi n_0 e^2}$ and $\bar{\phi}$ is defined in Eq. (35). The use of these simulation techniques described here on electromagentic microturbulence and kinetic MHD physics will be published elsewhere.

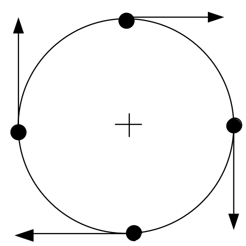

Figure 1. Four-Point Approximation for a Rotating Ion Ring.

V. SUMMARY AND CONCLUSIONS

In this paper, we have shown the basic difference between the gyrokinetic and MHD descriptions of Alfven waves and have also outlined the procedures for calculating $\mathbf{J}_\perp$ as well as other field quantities in general geometry for arbitrary grid size. These new pieces of information are useful for the purpose of simulating electromagnetic microturbulence and kinetic MHD physics in magnetically confined plasmas. In particular, we can use these equations to simulate microturbulence in transport time scale as well. Let us explain.

Since the inception of the gyrokinetic particle simulation [6,11] and the first tokamak microturbulence simulation [16], major progress in computational capability and physics understanding have been made, for example, by using the gyrokinetic Global Toroidal Code (GTC) on the massively parallel computers to study zonal flow physics [17] with collisonal effects [18] and, most recently, to study the size scaling on the reactor size plasmas ($a = 1000\rho_s$) [19] on the IBM SP Power3 at National Energy Research Supercomputing Center (NERSC). For this particular size-scaling study, the largest run took 72 wallclock hours with one billion ion particles (8 particles/cell) with the adiabatic electron approximation for 7000 time steps running on 1024 processors (25M particle*step/sec) with 10% efficiency for each processor. Most recently, GTC has also achieved 5 times the speed per CPU on the Cray's SX6 vector parallel computer [20]. These exercises underscore the importance of using gyrokinetic PIC codes on the parallel architecture to carry out realistic simulations of turbulence transport in tokamaks and stellarators. All these simulations mentioned here in Refs. [16], [17], [18] and [19] cover about $1msec$ of the duration of the tokamak discharge and the results all indicate that the turbulence has developed well into the steady state with well defined background evolutions of density, current and temperature at the end of the simulations. This is a very crucial point that a well established turbulence steady state can be established in such a short time.

These are very encouraging results. However, many steps have to be taken before we can simulate electromagnetic microturbulence using a global gyrokinetic particle code like GTC. First of all, one needs to introduce into the code the all-important electron dynamics and the associated finite-β effects, for example, by using the split-weight particles simulation schemes [12,13]. We also need a very efficient elliptic solver for the field equations, such as Eqs. (13) or (39) and (15) or (31), and those related to the split-weight schemes of Refs. [12] and [13]. A multigrid solver under development seems to be very adequate for the purpose [21]. For realistic simulation of modern-day tokamaks, capablilities of handling shaped plasmas are also needed in a global code like GTC [22].

Beyond that, we can also envision the possibility of developing new algorithms for simulating electromagnetic microturbulence on the transport time scale using a gloabl gyrokinetic particle code, for example, by utilizing the capabilities present in this paper. One possible scenario involves 1) using of prescribed density, current, and temperature profiles to calculate the equilibrium magnetic configurations, 2) loading of particles in the phase space consistent with the equilibrium, 3) carrying out microturbulence simulation as an initial-value problem using a global gyrokinetic particle code, and 4) modifying the density, current and temperature profiles to their new values for a prescribed time duration according to the coefficients given by the turbulent state steady in the simulation. Steps 1) to 4) can then be repeated until the end of the time period of interest. In essence, this procedure can be considered as a conglomerate of three codes: a global particle code, a transport code and an equlibrium code. We have no doubt that many other procedures are also feasible.

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APPENDIX A

In an inhomogeneous plasma with time dependent electromagnetic perturbations, the magnetic moment μ that is conserved is different from the usual $\mu_B = v_\perp^2/2B_0$, which, strictly speaking, is only conserved in a homogeneous time-independent magnetic field. The relationship between μ and μ_B has been derived in Refs. [1,3,4] as

$$\mu = \mu_B \left(1 - \frac{mc}{e}\frac{v_\parallel}{B_0}\hat{\mathbf{b}}_0 \cdot \nabla \times \hat{\mathbf{b}}_0\right) + \frac{e}{m^2 c}\left(\frac{e}{c}\mathbf{A}\cdot\frac{\partial \boldsymbol{\rho}_\mathbf{0}}{\partial\varphi} + \frac{\partial \mathbf{S}}{\partial\varphi}\right) \tag{40}$$

where the second term on the RHS is the correction due to the inhomogeneity of the equilibrium field and the third term is the correction due to time dependent electromagnetic perturbations. $\mathbf{A}$ is the perturbed vector potential, $\boldsymbol{\rho}_0 \equiv -\mathbf{v}\times\hat{\mathbf{b}}_\mathbf{0}/\boldsymbol{\Omega}_\mathbf{0}$ is particle's gyroradius, φ is the gyrophase coordinate, and S is the gyro-center gauge satisfying

$$\frac{\partial S}{\partial t} + \dot{\mathbf{R}}\frac{\partial S}{\partial \mathbf{R}} + \dot{V}_\parallel\frac{\partial S}{\partial V_\parallel} + \dot{\varphi}\frac{\partial S}{\partial\varphi} = e\widetilde{\phi}(\mathbf{R}+\rho_\mathbf{0},\mathbf{t}) - \frac{\mathbf{e}}{\mathbf{c}}\widetilde{\mathbf{V}\cdot\mathbf{A}}(\mathbf{R}+\rho_\mathbf{0},\mathbf{t}). \tag{41}$$

Here $\widetilde{\phi}(\mathbf{R}+\rho_\mathbf{0},\mathbf{t})$ and $\widetilde{\mathbf{V}\cdot\mathbf{A}}(\mathbf{R}+\rho_\mathbf{0},\mathbf{t})$ are the gyrophase dependent parts of $\phi(\mathbf{R}+\rho_\mathbf{0},\mathbf{t})$ and $\mathbf{V}\cdot\mathbf{A}(\mathbf{R}+\rho_\mathbf{0},\mathbf{t})$ respectively.

$$\widetilde{\phi}(\mathbf{R}+\rho_\mathbf{0},\mathbf{t}) = \phi(\mathbf{R}+\rho_\mathbf{0},\mathbf{t}) - \langle\phi(\mathbf{R}+\rho_\mathbf{0},\mathbf{t})\rangle$$

$$\widetilde{\mathbf{V}\cdot\mathbf{A}}(\mathbf{R}+\rho_\mathbf{0},\mathbf{t}) = \mathbf{V}\cdot\mathbf{A}(\mathbf{R}+\rho_\mathbf{0},\mathbf{t}) - \langle\mathbf{V}\cdot\mathbf{A}(\mathbf{R}+\rho_\mathbf{0},\mathbf{t})\rangle. \tag{42}$$

We note that in the present of electromagnetic perturbations, the dynamics of μ_B is gyrophase dependent even thought its definition is gyro-phase independent. That is why the definition of μ has a gyro-phase dependent component so that its time derivative can be gyro-phase independent, which as a matter of fact, vanishes. However, since $\hat{\mathbf{b}}_0 \cdot \nabla \times \hat{\mathbf{b}}_0 = (\nabla \times \mathbf{B}_0)_\parallel / B_0$ is usually small and $\mathbf{A}$ is the perturbation, we can assume that $\mu \approx \mu_B$ is nearly constant and is independent of phase.

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