PREPARED FOR THE U.S. DEPARTMENT OF ENERGY,
UNDER CONTRACT DE-AC02-76CH03073

PPPL-3625
UC-70

PPPL-3625

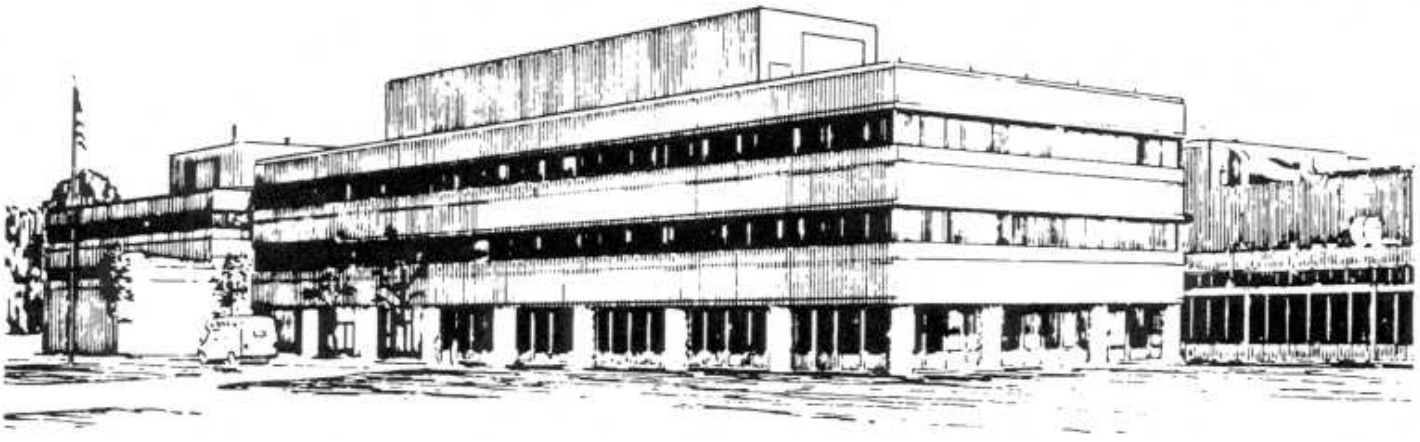
Implications of the Electrostatic Approximation in the Beam Frame on the Nonlinear Vlasov-Maxwell Equations for Intense Beam Propagation

by

Ronald C. Davidson, W. Wei-li Lee,
Hong Qin, and Edward Startsev

November 2001

PPPL
PRINCETON PLASMA PHYSICS LABORATORY



PRINCETON PLASMA PHYSICS LABORATORY
PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY

PPPL Reports Disclaimer

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Availability

This report is posted on the U.S. Department of Energy's Princeton Plasma Physics Laboratory Publications and Reports web site in Fiscal Year 2002. The home page for PPPL Reports and Publications is: http://www.pppl.gov/pub_report/

DOE and DOE Contractors can obtain copies of this report from:

U.S. Department of Energy
Office of Scientific and Technical Information
DOE Technical Information Services (DTIS)
P.O. Box 62
Oak Ridge, TN 37831

Telephone: (865) 576-8401

Fax: (865) 576-5728

Email: reports@adonis.osti.gov

This report is available to the general public from:

National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161

Telephone: 1-800-553-6847 or
(703) 605-6000

Fax: (703) 321-8547

Internet: http://www.ntis.gov/ordering.htm

Implications of the Electrostatic Approximation in the Beam Frame on the Nonlinear Vlasov-Maxwell Equatons for Intense Beam Propagation

Ronald C. Davidson, W. Wei-li Lee, Hong Qin and Edward Startsev

Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543

Abstract

This paper develops a clear procedure for solving the nonlinear Vlasov-Maxwell equations for a one-component intense charged particle beam or finite-length charge bunch propagating through a cylindrical conducting pipe (radius $r = r_w = const.$), and confined by an applied focusing force $\mathbf{F}_{foc}$. In particular, the nonlinear Vlasov-Maxwell equations are Lorentz-transformed to the beam frame ('primed' variables) moving with axial velocity $V_b = \beta_b c = const.$ relative to the laboratory. In the beam frame, the particle motions are nonrelativistic for the applications of practical interest, already a major simplification. Then, in the beam frame, we make the electrostatic approximation ($\mathbf{E}'_s = -\nabla'\phi'$, $\mathbf{E}'_T \simeq 0 \simeq \mathbf{B}'_s$) which fully incorporates beam space-charge effects, but neglects any fast electromagnetic processes with transverse polarization (e.g., light waves). The resulting Vlasov-Maxwell equations are then Lorentz-transformed back to the laboratory frame, and properties of the self-generated fields and resulting nonlinear Vlasov-Maxwell equations in the laboratory frame are discussed.

Typeset using REVTEX

I. INTRODUCTION

Periodic focusing accelerators and transport systems [1–7] have a wide range of applications ranging from basic scientific research in high energy and nuclear physics, to applications such as coherent radiation sources, heavy ion fusion, tritium production, nuclear waste transmutation, and spallation neutron sources for materials and biological research [8,9]. At the high beam currents and charge densities of practical interest, of particular importance are the effects of the intense self fields produced by the beam space charge and current on determining the detailed equilibrium, stability and transport properties, and the nonlinear dynamics of the system. Through analytical studies based on the nonlinear Vlasov-Maxwell equations for the distribution function $f_b(\mathbf{x}, \mathbf{p}, t)$ and the self-generated electric and fields $\mathbf{E}_s(\mathbf{x}, t)$ and $\mathbf{B}_s(\mathbf{x}, t)$, and numerical simulations using particle-in-cell models and nonlinear perturbative simulation techniques, considerable progress has been made in developing an improved understanding of the collective processes and nonlinear beam dynamics characteristic of high-intensity beam propagation in periodic focusing and uniform focusing transport systems [10–28]. In almost all applications of the Vlasov-Maxwell equations to intense beam propagation, the analysis is carried out in the laboratory frame, and various simplifying approximations are made, ranging from the electrostatic-magnetostatic approximation [29], to the Darwin-model approximation [30–35] which neglects fast transverse electromagnetic perturbations.

Given the general importance of model assumptions in affecting the detailed outcome of calculations, in this paper we develop a clear procedure for solving the nonlinear Vlasov-Maxwell equations for a one-component intense charged particle beam or finite-length charge bunch propagating through a cylindrical conducting pipe (radius $r = r_w = const.$), and confined by an applied focusing force $\mathbf{F}_{foc}$. In particular, the nonlinear Vlasov-Maxwell equations are Lorentz-transformed to the beam frame ('primed' variables) moving with axial velocity $V_b = \beta_b c = const.$ relative to the laboratory [24]. In the beam frame, the particle motions are nonrelativistic for the applications of practical interest, already a major simpli-

fication. Then, in the beam frame, we make the electrostatic approximation ($\mathbf{E}'_s = -\nabla'\phi'$, $\mathbf{E}'_T \simeq 0 \simeq \mathbf{B}'_s$) which fully incorporates beam space-charge effects, but neglects any fast electromagnetic processes with transverse polarization (e.g., light waves). The resulting Vlasov-Maxwell equations are then Lorentz-transformed back to the laboratory frame, and properties of the self-generated fields and resulting nonlinear Vlasov-Maxwell equations in the laboratory frame are discussed.

II. VLASOV-MAXWELL EQUATIONS AND TRANSFORMATION TO THE BEAM FRAME

In the present analysis, we consider an intense charged particle beam with characteristic transverse dimensions a and b propagating in the z-direction with average axial velocity $V_b = \beta_b c = const.$ and characteristic directed kinetic energy $(\gamma_b - 1)m_b c^2$. Here, c is the speed of light in *vacuo*, $\gamma_b = (1-\beta_b^2)^{-1/2}$ is the relativistic mass factor, and e_b and m_b are the charge and rest mass, respectively, of a beam particle. A perfectly conducting cylindrical wall is located at radius $r = r_w$, where $r = (x^2 + y^2)^{1/2}$ is the radial distance from the beam axis. The particle motion in the beam frame ('primed' coordinates) is assumed to be nonrelativistic with $|\mathbf{v}'| \ll c$. Furthermore, the beam current density and charge density are allowed to be arbitrarily large, subject only to the requirement that the beam particles be confined by the applied focusing fields $\mathbf{E}_{foc}(\mathbf{x}, t)$ and $\mathbf{B}_{foc}(\mathbf{x}, t)$. The specific forms of $\mathbf{E}_{foc}(\mathbf{x}, t)$ and $\mathbf{B}_{foc}(\mathbf{x}, t)$ of course depend on the particular focusing field configuration under consideration (quadrupole, solenoidal, rf, etc.). Finally, in the present analysis, the beam can be continuous in the z-direction, or correspond to a finite-length charge bunch.

Within the context of the above assumptions, a complete description of the collective processes and nonlinear dynamics of the charged particle beam is provided by the nonlinear Vlasov-Maxwell equations [1], which describe the evolution of the distribution function $f_b(\mathbf{x}, \mathbf{p}, t)$ in the six-dimensional laboratory-frame phase space $(\mathbf{x}, \mathbf{p})$, and the corresponding self fields, $\mathbf{E}_s(\mathbf{x}, t)$ and $\mathbf{B}_s(\mathbf{x}, t)$, generated self-consistently by the beam space charge and

current. In laboratory-frame variables, the nonlinear Vlasov-Maxwell equations describing the self-consistent evolution of $f_b(\mathbf{x},\mathbf{p},t)$, $\mathbf{E}_s(\mathbf{x},t)$ and $\mathbf{B}_s(\mathbf{x},t)$ can be expressed as

$$\frac{\partial f_b}{\partial t}+\mathbf{v}\cdot\frac{\partial f_b}{\partial \mathbf{x}}+\left[\mathbf{F}_{foc}+e_b\Big(\mathbf{E}_s+\frac{1}{c}\mathbf{v}\times\mathbf{B}_s\Big)\right]\cdot\frac{\partial f_b}{\partial \mathbf{p}}=0\ , \tag{1}$$

and

$$\nabla\cdot\mathbf{E}_s=4\pi e_b\int d^3p f_b\ , \tag{2}$$

$$\nabla\times\mathbf{B}_s=\frac{1}{c}4\pi e_b\int d^3p\mathbf{v}f_b+\frac{1}{c}\frac{\partial\mathbf{E}_s}{\partial t}\ , \tag{3}$$

$$\nabla\times\mathbf{E}_s=-\frac{1}{c}\frac{\partial\mathbf{B}_s}{\partial t}\ , \tag{4}$$

$$\nabla\cdot\mathbf{B}_s=0\ . \tag{5}$$

Here, $\mathbf{F}_{foc}=e_b(\mathbf{E}_{foc}+c^{-1}\mathbf{v}\times\mathbf{B}_{foc})$ is the applied focusing force in the laboratory frame, and the velocity $\mathbf{v}$ and momentum $\mathbf{p}$ are related by $\mathbf{p}=\gamma m_b\mathbf{v}$, where $\gamma=(1+\mathbf{p}^2/m_b^2c^2)^{1/2}$. The Vlasov equation (1) describes the incompressible evolution of the distribution function $f_b(\mathbf{x},\mathbf{p},t)$ in the six-dimensional phase space $(\mathbf{x},\mathbf{p})$ as the beam particles interact with the applied focusing fields, $\mathbf{E}_{foc}(\mathbf{x},t)$ and $\mathbf{B}_{foc}(\mathbf{x},t)$, and the average self fields, $\mathbf{E}_s(\mathbf{x},t)$ and $\mathbf{B}_s(\mathbf{x},t)$, generated by the beam particles. Note that the Vlasov equation (1) is highly nonlinear because $\mathbf{E}_s(\mathbf{x},t)$ and $\mathbf{B}_s(\mathbf{x},t)$ are determined self-consistently in terms of the beam charge density, $\rho_b(\mathbf{x},t)=e_b\int d^3p f_b(\mathbf{x},\mathbf{p},t)$, and current density, $\mathbf{J}_b(\mathbf{x},t)=e_b\int d^3p\mathbf{v}f_b(\mathbf{x},\mathbf{p},t)$, from Maxwell's equations (2)–(5). Here, $n_b(\mathbf{x},t)=\int d^3p f_b(\mathbf{x},\mathbf{p},t)$ is the number density of beam particles.

The Vlasov-Maxwell equations (1)–(5) can of course be analyzed directly in laboratory-frame variables. However, for a beam consisting of a single charge component (the case considered here), there is considerable advantage to transforming to the beam frame moving with axial velocity $V_b=\beta_b c$ relative to the laboratory. In the beam frame the particle motion is nonrelativistic ($|\mathbf{v}'|\ll c$) by assumption, which results in a welcome simplification of the

corresponding Vlasov-Maxwell equations in the beam frame. The Lorentz transformation [36,37] relating the primed variables $(\mathbf{x}', \mathbf{p}', t')$ in the beam frame to the unprimed variables $(\mathbf{x}, \mathbf{p}, t)$ in the laboratory frame is given by

$$x' = x \ , \quad y' = y \ , \quad z' = \gamma_b(z - V_b t) \ ,$$

$$t' = \gamma_b(t - V_b z/c^2) \ ,$$

$$p'_x = p_x \ , \quad p'_y = p_y \ , \quad p'_z = \gamma_b(p_z - \gamma m_b V_b) \ ,$$

$$\gamma' = \gamma_b(\gamma - V_b p_z/m_b c^2) \ , \tag{6}$$

where $\gamma_b = (1 - V_b^2/c^2)^{-1/2}$. Here, the particle momentum and velocity are related by $\mathbf{p} = \gamma m_b \mathbf{v}$ and $\mathbf{p}' = \gamma' m_b \mathbf{v}'$, where $\gamma = (1 + \mathbf{p}^2/m_b^2 c^2)^{1/2}$ and $\gamma' = (1 + \mathbf{p}'^2/m_b^2 c^2)^{1/2}$ are the kinematic mass factors. In the beam frame, the nonlinear Vlasov equation for the distribution function $f'_b(\mathbf{x}', \mathbf{p}', t')$ can be expressed as [24]

$$\frac{\partial f'_b}{\partial t'} + \mathbf{v}' \cdot \frac{\partial f'_b}{\partial \mathbf{x}'} + \left[\mathbf{F}'_{foc} + e_b\Big(\mathbf{E}'_s + \frac{1}{c}\mathbf{v}' \times \mathbf{B}'_s\Big)\right] \cdot \frac{\partial f'_b}{\partial \mathbf{p}'} = 0 \ . \tag{7}$$

In Eq. (7), $\mathbf{E}'_s(\mathbf{x}', t)$ and $\mathbf{B}'_s(\mathbf{x}', t')$ are the self-generated fields in the beam frame, and we approximate $\gamma' = 1 + \mathbf{p}'^2/2m_b^2 c^2$ and $\mathbf{p}' = m_b \mathbf{v}'$ because the particle motion in the beam frame is assumed to be nonrelativistic. Furthermore, $\mathbf{F}'_{foc} = e_b(\mathbf{E}'_{foc} + c^{-1}\mathbf{v}' \times \mathbf{B}'_{foc})$ is the applied focusing force on a particle in the beam frame. Here, the applied electric and magnetic fields transform according to $\mathbf{E}'_{foc} = [E_z \hat{\mathbf{e}}_z + \gamma_b(E_x \hat{\mathbf{e}}_x + E_y \hat{\mathbf{e}}_y) + \gamma_b c^{-1} V_b \hat{\mathbf{e}}_z \times \mathbf{B}]_{foc}$ and $\mathbf{B}'_{foc} = [B_z \hat{\mathbf{e}}_z + \gamma_b(B_x \hat{\mathbf{e}}_x + B_y \hat{\mathbf{e}}_y) - \gamma_b c^{-1} V_b \hat{\mathbf{e}}_z \times \mathbf{E}]_{foc}$, which determines $\mathbf{E}'_{foc}$ and $\mathbf{B}'_{foc}$ directly in terms of the applied focusing fields in the laboratory frame and the Lorentz transformation in Eq. (6).

Maxwell's equations in the beam frame of course are similar in form to Eqs. (2)–(5), and relate the self-generated electric and magnetic fields, $\mathbf{E}'_s(\mathbf{x}', t')$ and $\mathbf{B}'_s(\mathbf{x}', t')$, to the distribution function $f'_b(\mathbf{x}', \mathbf{p}, t')$. For present purposes, it is convenient to introduce the scalar and vector potentials, $\phi'(\mathbf{x}', t')$ and $\mathbf{A}'(\mathbf{x}', t')$, and express

$$\mathbf{E}'_s = \mathbf{E}'_L + \mathbf{E}'_T = -\frac{\partial}{\partial \mathbf{x}'}\phi' - \frac{1}{c}\frac{\partial}{\partial t'}\mathbf{A}' \ ,$$

$$\mathbf{B}'_s = \frac{\partial}{\partial \mathbf{x}'} \times \mathbf{A}' \ , \tag{8}$$

where $\mathbf{E}'_L = -\nabla'\phi'$ is the longitudinal electric field, $\mathbf{E}'_T = c^{-1}\partial\mathbf{A}'/\partial t'$ is the transverse electric field, and the Coulomb gauge condition with $\nabla' \cdot \mathbf{A}' = 0$ is assumed. From Eq. (8), the Maxwell equations $\nabla' \cdot \mathbf{B}'_s = 0$ and $\nabla' \times \mathbf{E}'_s = -c^{-1}\partial\mathbf{B}'_s/\partial t'$ are automatically satisfied, and Poisson's equation and the $\nabla' \times \mathbf{B}'_s$ Maxwell equation in the beam frame are readily expressed as [24]

$$\nabla'^2\phi' = -4\pi e_b \int d^3p' f'_b \ , \tag{9}$$

$$\nabla'^2\mathbf{A}' = -\frac{1}{c}4\pi e_b \int d^3p' \mathbf{v}' f'_b + \frac{1}{c^2}\frac{\partial^2\mathbf{A}'}{\partial t'^2} + \frac{1}{c}\nabla'\frac{\partial\phi'}{\partial t'} \ , \tag{10}$$

where use has been made of $\nabla' \cdot \mathbf{A}' = 0$. In Eqs. (9) and (10), note that the electrostatic potential $\phi'(\mathbf{x}', t')$ is determined self-consistently in terms of the beam charge density $\rho'_b(\mathbf{x}', t') = e_b n'_b(\mathbf{x}', t') = e_b \int d^3p' f'_b(\mathbf{x}', \mathbf{p}', t')$ from Eq. (9), and $\mathbf{A}'(\mathbf{x}', t')$ is determined in terms of the beam current density $\mathbf{J}'_b(\mathbf{x}', t') = e_b n'_b(\mathbf{x}', t')\mathbf{V}'_b(\mathbf{x}', t') = e_b \int d^3p' \mathbf{v}' f'_b(\mathbf{x}', \mathbf{p}', t')$ from Eq. (10). Here, $n'_b(\mathbf{x}', t')$ is the local number density and $\mathbf{V}'_b(\mathbf{x}', t')$ is the local average flow velocity of particles in the beam frame, and $\mathbf{v}' = \mathbf{p}'/m_b$ is the (nonrelativistic) particle velocity. Note that Poisson's equation (9) can be viewed as an initial condition to Eq. (10), which if true at $t' = 0$ remains true at all subsequent t'. This follows upon taking the divergence of Eq. (10) and making use of the Coulomb gauge condition $\nabla' \cdot \mathbf{A}' = 0$. This readily gives

$$\frac{1}{c}\frac{\partial}{\partial t'}\nabla'^2\phi' - \frac{1}{c}4\pi e_b \nabla' \cdot (n'_b \mathbf{V}'_b) = \frac{1}{c}\frac{\partial}{\partial t'}[\nabla'^2\phi' + 4\pi e_b n'_b] = 0 \ , \tag{11}$$

where use has been made of $\partial n'_b/\partial t' + \nabla' \cdot (n'_b \mathbf{V}'_b) = 0$. It follows trivially from Eq. (11) that if Poisson's equation (9) is satisfied initially at $t' = 0$, then it remains true at all subsequent times t'.

For the boundary conditions at the perfectly conducting cylindrical wall at radius $r = r' = r_w$, we impose the requirements that the tangential electric field and the normal

magnetic field vanish. That is, $[E_z]_{r=r_w} = [E_\theta]_{r=r_w} = [B_r]_{r=r_w} = 0$, where E_z, E_θ and B_r denote field components in cylindrical polar coordinates (r, θ, z) in the laboratory frame. In the beam frame, the corresponding field components are given by $E'_z = E_z$, $B'_r = \gamma_b(B_r + V_b E_\theta/c)$, and $E'_\theta = \gamma_b(E_\theta + V_b B_r/c)$. Therefore, the corresponding boundary conditions at the conducting wall $r' = r_w$ in the beam frame are also given by $[E'_z]_{r'=r_w} = [E'_\theta]_{r'=r_w} = [B'_r]_{r'=r_w} = 0$. Expressed in terms of the scalar and vector potentials, $\phi'(\mathbf{x}', t')$ and $\mathbf{A}'(\mathbf{x}', t')$, these boundary conditions can be expressed as

$$\phi'(r' = r_w, \theta', z', t') = A'_z(r' = r_w, \theta', z', t') = A'_\theta(r' = r_w, \theta', z', t') = const. \ , \tag{12}$$

where (r', θ', z') are cylindrical polar coordinates in the beam frame, with $x' = r' \cos\theta'$ and $y' = r' \sin\theta'$.

The nonlinear Vlasov-Maxwell equations (7), (9) and (10) in the beam frame, subject to the boundary conditions in Eq. (12), are fully equivalent to the Vlasov-Maxwell equations (1)–(5) in the laboratory frame, and provide a complete description of the collective processes and nonlinear dynamics of intense beam propagation. Equations (7), (9) and (10) can be used to investigate detailed equilibrium and stability properties in the beam frame for a wide range of system parameters and choices of applied field configurations. Moreover, as noted earlier, because the particle motion in the beam frame is nonrelativistic, a detailed investigation of Eqs. (7), (9) and (10) is more tractable analytically and numerically than the corresponding Vlasov-Maxwell equations (1)–(5) in the laboratory frame. Furthermore, once the solutions for $f'_b(\mathbf{x}', \mathbf{p}', t')$, $\phi'(\mathbf{x}', t')$ and $\mathbf{A}'(\mathbf{x}', t')$ are obtained in the beam frame, the corresponding solutions are readily obtained in the laboratory frame. For example, the variables $(\mathbf{x}', \mathbf{p}', t')$ are related to $(\mathbf{x}, \mathbf{p}, t)$ by the Lorentz transformation in Eq. (6). Furthermore, the scalar and vector potentials $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ in the laboratory frame are related to the potentials $\phi'(\mathbf{x}', t')$ and $\mathbf{A}'(\mathbf{x}', t')$ in the beam frame by the transformation [24,36]

$$\phi = \gamma_b(\phi' + V_b A'_z/c) \ ,$$

$$A_x = A'_x \ , \ A_y = A'_y \ ,$$

$$A_z = \gamma_b(A'_z + V_b\phi'/c) \ , \tag{13}$$

where the arguments $(\mathbf{x}', t')$ are related to $(\mathbf{x}, t)$ by Eq. (6).

III. IMPLICATIONS OF ELECTROSTATIC APPROXIMATION IN THE BEAM FRAME

The introduction of the Coulomb gauge condition ($\nabla' \cdot \mathbf{A}' = 0$) and the resulting forms of Maxwell's equations (9) and (10) also make ancillary approximations more transparent in the beam frame. In the following analysis, we consider such a case corresponding to the electrostatic approximation in the beam frame. In particular, it is assumed that the electromagnetic field components with *transverse* polarization, $\mathbf{E}'_T = -c^{-1}\partial\mathbf{A}'/\partial t'$ and $\mathbf{B}'_s = \nabla' \times \mathbf{A}'$, are negligibly small in comparison with the *longitudinal* electric field, $\mathbf{E}'_L = -\nabla'\phi'$. In this case, we approximate

$$\mathbf{A}' = 0 \ ,$$

$$\mathbf{E}'_T = 0 = \mathbf{B}'_s \ , \tag{14}$$

and the nonlinear Vlasov equation (7) in the beam frame becomes

$$\frac{\partial f'_b}{\partial t'} + \mathbf{v}' \cdot \frac{\partial f'_b}{\partial \mathbf{x}'} + [\mathbf{F}'_{foc} - e_b\nabla'\phi'] \cdot \frac{\partial f'_b}{\partial \mathbf{p}'} = 0 \ . \tag{15}$$

Of course the scalar potential $\phi'(\mathbf{x}', t')$ occurring in Eq. (15) is determined self-consistently in terms of the charge density $e_b \int d^3p' f'_b(\mathbf{x}', \mathbf{p}', t')$ from Poison's equation (9). By virtue of Eq. (14), we have neglected any fast electromagnetic processes in the beam frame with transverse polarization (e.g., light waves), and it is assumed that the current carried by the particles in the beam frame is sufficiently small that the self-generated transverse field components $\mathbf{E}'_T$ and $\mathbf{B}'_s$ can be neglected. Equations (9) and (15) of course include the full influence of space-charge effects in the beam frame through the longitudinal electric field $\mathbf{E}'_L = -\nabla'\phi'$.

The nonlinear Vlasov-Poisson equations (9) and (15) constitute a closed description of the collective processes and nonlinear dynamics of the distribution function $f_b'(\mathbf{x}', \mathbf{p}', t')$ and space-charge potential $\phi'(\mathbf{x}', t')$ in the beam frame, valid in the electrostatic approximation. As such Eqs. (9) and (15) can be used to investigate detailed equilibrium and electrostatic stability properties for a wide range of system parameters and choices of focusing field configurations. The purpose of this paper is not to solve Eqs. (9) and (15) in detail. Rather, let us assume that the solutions to Eqs. (9) and (15) have been obtained in the beam frame (these could be analytical or numerical solutions), and pose the question: What are the properties of the corresponding solutions in the laboratory frame? The variables $(\mathbf{x}', \mathbf{p}', t')$ and $(\mathbf{x}, \mathbf{p}, t)$ of course transform according to Eq. (6). Furthermore, once $\phi'(\mathbf{x}', t')$ is determined in the beam frame, field quantities in the laboratory frame are readily obtained by making use of Eqs. (13) and (14). Substituting $\mathbf{A}' = 0$ into Eq. (13) readily gives

$$\begin{aligned} \phi &= \gamma_b \phi' \ , \\ A_x &= 0 = A_y \ , \\ A_z &= \gamma_b \beta_b \phi' = \beta_b \phi \ , \end{aligned} \tag{16}$$

where $\beta_b = V_b/c$. In Eq. (16), the arguments $(\mathbf{x}', t')$ and $(\mathbf{x}, t)$ are related by Eq. (6), so Eq. (16) gives directly

$$\phi(x, y, z, t) = \gamma_b \phi'[x, y, \gamma_b(z - V_b t), \gamma_b(t - V_b z/c^2)] \ , \tag{17}$$

as well as $A_x = A_y = 0$, and

$$A_z(x, y, z, t) = \beta_b \phi(x, y, z, t) \ . \tag{18}$$

Note from Eqs. (16) and (17) that the (x', y') and (x, y) dependences of ϕ' and ϕ are identical, whereas the (z', t') and (z, t) dependences transform according to Eq. (6). In terms of Fourier-Laplace variables (k_z, ω) and (k_z', ω'), Eq. (17) leads directly to the familiar relations [36,37]

$$k_z = \gamma_b\Big(k'_z + \frac{V_b}{c^2}\omega'\Big) \ ,$$

$$\omega = \gamma_b(\omega' + k'_z V_b) \ . \tag{19}$$

That is, if the potential $\phi'(\mathbf{x}', t')$ has axial wavenumber k'_z and frequency ω' in the beam frame, then the corresponding axial wavenumber k_z and frequency ω in the laboratory frame are given by Eq. (19). Of course the inverse transformation to Eq. (19) is obtained by interchanging (k_z, ω) and (k'_z, ω'), and making the replacement $V_b \to -V_b$.

Equations (16)–(18) allow us to determine the self-generated fields, $\mathbf{E}_s = -\nabla\phi - c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B}_s = \nabla \times \mathbf{A}$, in the laboratory frame, consistent with the electrostatic approximation, $\mathbf{E}'_s = \mathbf{E}'_L = -\nabla'\phi'$, in the beam frame. It follows directly that $\mathbf{B}_s = \nabla \times A_z\hat{\mathbf{e}}_z$ has components

$$B_x = \frac{\partial A_z}{\partial y} = \beta_b \frac{\partial \phi}{\partial y} \ ,$$

$$B_y = -\frac{\partial A_z}{\partial x} = -\beta_b \frac{\partial \phi}{\partial x} \ ,$$

$$B_z = 0 \ . \tag{20}$$

Furthermore, the self-generated electric field $\mathbf{E}_s = -\nabla\phi - c^{-1}(\partial A_z/\partial t)\hat{\mathbf{e}}_z$ can be expressed as

$$E_x = -\frac{\partial \phi}{\partial x} \ , \quad E_y = -\frac{\partial \phi}{\partial y} \ ,$$

$$E_z = -\frac{\partial \phi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} = -\frac{\partial \phi}{\partial z} - \frac{\beta_b}{c}\frac{\partial \phi}{\partial t} \ . \tag{21}$$

As would be expected, even though there is no magnetic field in the beam frame ($\mathbf{B}'_s = 0$) by assumption, in the laboratory frame the beam carries an axial current, and there is a transverse magnetic field generated with components, $B_x = \partial A_z/\partial y$ and $B_y = -\partial A_z/\partial x$, where $A_z = \beta_b\phi$ according to Eq. (18). Furthermore, in addition to the longitudinal electric field $\mathbf{E}_L = -\nabla\phi$, it follows from Eq. (21) that there is an inductive electric field $E_{Tz} = -c^{-1}\partial A_z/\partial t$ in the laboratory frame, where $A_z = \beta_b\phi$. As noted earlier, the nonlinear Vlasov-Poisson equations (9) and (15) can be solved in the beam frame, and then Eqs. (6),

(16), (20) and (21) can be used to determine the corresponding distribution function and self-generated fields in the laboratory frame.

While Eqs. (9) and (15) provide a complete description of the system in the electrostatic aproximation in the beam frame, it is nonetheless interesting to examine the implications of the electrostatic approximation in the beam frame for the corresponding Vlasov-Maxwell equations in the laboratory frame. First, making use of Eqs. (20) and (21), it follows that the force on a beam particle in the laboratory frame due to the self-generated fields, $\mathbf{F} = e_b(\mathbf{E}_s + c^{-1}\mathbf{v}\times\mathbf{B}_s)$, can be expressed as $\mathbf{F} = \mathbf{F}_\perp + F_z\hat{\mathbf{e}}_z$, where

$$\begin{aligned}\mathbf{F}_\perp &= e_b\Big(-\nabla_\perp\phi + \frac{1}{c}\Big[\mathbf{v}\times\Big(\nabla A_z\times\hat{e}_z\Big)\Big]_\perp\Big)\\ &= -e_b\nabla_\perp\Big(\phi - \frac{1}{c}v_zA_z\Big)\end{aligned} \tag{22}$$

denotes the perpendicular force in the $x-y$ plane, and

$$\begin{aligned}F_z &= e_b\Big(-\frac{\partial\phi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} - \frac{1}{c}\mathbf{v}_\perp\cdot\nabla_\perp A_z\Big)\\ &= -e_b\Big[\frac{\partial}{\partial z}\Big(\phi - \frac{1}{c}v_zA_z\Big) + \frac{1}{c}\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)A_z\Big]\end{aligned} \tag{23}$$

is the axial force. Here, $\nabla_\perp = \hat{\mathbf{e}}_x\partial/\partial x + \hat{\mathbf{e}}_y\partial/\partial y$ is the perpendicular spatial gradient, and $A_z = \beta_b\phi$ follows from Eq. (18). Equations (22) and (23) can be substituted into the laboratory-frame Vlasov equation (1). One important simplification occurs in this regard. The characteristics of the Vlasov equation (1) are the single-particle orbits in the self-generated fields. For example, the coefficient of $\partial f_b/\partial p_z$ is $dp_z/dt = F_z$. From Eq. (23), introducing the axial canonical momentum $P_z = p_z + (e_b/c)A_z$, and making use of $(d/dt)A_z = (\partial/\partial t + \mathbf{v}\cdot\nabla)A_z$, it follows that $dP_z/dt = -e_b(\partial/\partial z)(\phi - v_zA_z/c)$. Therefore, if we change variables from $(x, y, z, p_x, p_y, p_z, t)$ to $(x, y, z, P_x, P_y, P_z, t)$, where $P_x = p_x$ and $P_y = p_y$ (because $A_x = 0 = A_y$) and $P_z = p_z + (e_b/c)A_z$, it follows from Eqs. (1), (22) and (23) that the nonlinear Vlasov equation for the distribution function $f_b(\mathbf{x}, \mathbf{P}, t)$ in the laboratory frame can be expressed in the compact form

$$\frac{\partial f_b}{\partial t} + \mathbf{v}\cdot\frac{\partial f_b}{\partial\mathbf{x}} + \Big[\mathbf{F}_{foc} - e_b\Big(1 - \frac{v_z}{c}\beta_b\Big)\nabla\phi\Big]\cdot\frac{\partial f_b}{\partial\mathbf{P}} = 0\ . \tag{24}$$

In obtaining Eq. (24), use has been made of $A_z = \beta_b \phi$ to express $\nabla(\phi - v_z A_z/c) = (1 - \beta_b v_z/c)\nabla\phi$. Furthermore, from Eqs. (2) and (21), Poisson's equation can be expressed as

$$\nabla_\perp^2 \phi + \frac{\partial}{\partial z}\Big(\frac{\partial \phi}{\partial z} + \frac{1}{c}\beta_b \frac{\partial \phi}{\partial t}\Big) = -4\pi e_b \int d^3P f_b(\mathbf{x}, \mathbf{P}, t) \ , \tag{25}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The Vlasov-Poisson equations (24) and (25), valid in the laboratory frame, are fully equivalent to the Vlasov-Poisson equations (9) and (15) obtained in the beam frame in the electrostatic approximation. Of course when the two frames coincide ($\beta_b = 0$, $\gamma_b = 1$, $\phi = \phi'$ and $A_z = A_z' = 0$), Eqs. (24) and (25) are identical in form to Eqs. (9) and (15), as expected.

As noted earlier, because the particle motion is nonrelativistic in the beam frame, it is often advantageous to solve Eqs. (9) and (15) directly, rather than Eqs. (24) and (25). Nonetheless, with some ancillary approximations, the laboratory-frame Vlasov-Poisson equations also simplify further. For example, if the axial velocity spread around $v_z = V_b = \beta_b c$ is very small ($|\Delta v_z| \gamma_b^2 \beta_b \ll c$), then we approximate $1 - v_z \beta_b c \simeq 1 - \beta_b^2 = 1/\gamma_b^2$, and Eq. (24) reduces to

$$\frac{\partial f_b}{\partial t} + \mathbf{v} \cdot \frac{\partial f_b}{\partial \mathbf{x}} + \Big[\mathbf{F}_{foc} - \frac{e_b}{\gamma_b^2}\nabla\phi\Big] \cdot \frac{\partial f_b}{\partial \mathbf{P}} = 0 \ . \tag{26}$$

Equation (26) shows clearly that the magnetic pinching force due to the self-magnetic field in the laboratory frame reduces the electric force by the factor $1/\gamma_b^2$ [1]. A further simplification occurs in Eq. (25) in circumstances corresponding to an axially continuous beam or very long charge bunch. In Eq. (25), we denote $\partial/\partial x \sim 1/a$ and $\partial/\partial y \sim 1/b$ where $a \sim b$ are the transverse beam dimensions; $\partial/\partial z \sim 1/L$ where L is the characteristic length scale of axial variations; and $\partial/\partial t \sim \omega$ where ω^{-1} is the characteristic time scale of variations in $\phi(\mathbf{x}, t)$. Then, for $a \sim b$, the terms on the left-hand side of Eq. (25) stand in the ratio

$$1 : \frac{a^2}{L^2} : \frac{a^2}{L^2}\frac{\omega L}{c}\beta_b \ . \tag{27}$$

Then for $a^2 \ll L^2$, and even for moderately high frequency with $|\omega L/c| \sim 1$ or $|\omega L/c| \sim \beta_b$, the second and third terms on the left-hand side of Eq. (25) are negligibly small, and Poisson's equation can be approximated by

$$\nabla_\perp^2 \phi = -4\pi e_b \int d^3P f_b(\mathbf{x}, \mathbf{P}, t) \ , \tag{28}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Equations (26) and (28) are similar to the laboratory-frame Vlasov-Poisson equations widely used in the literature [29] to describe thin beam propagation in the paraxial approximation. Approximate Vlasov-Maxwell equations similar to Eqs. (26) and (28) can also be derived using the Darwin-approximation model [31–35] developed by Lee, et al. [30] for intense beam propagation.

IV. CONCLUSIONS

In summary, in this paper we have developed a clear procedure for solving the nonlinear Vlasov-Maxwell equations for a one-component intense charged particle beam or finite-length charge bunch propagating through a cylindrical conducting pipe (radius $r = r_w = const.$), and confined by an applied focusing force $\mathbf{F}_{foc}$. In particular, the nonlinear Vlasov-Maxwell equations were Lorentz-transformed to be beam frame ('primed' variables) moving with axial velocity $V_b = \beta_b c = const.$ relative to the laboratory. In the beam frame, the particle motions are nonrelativistic for the applications of practical interest, already a major simplification. Then, in the beam frame, we made the electrostatic approximation ($\mathbf{E}'_s = -\nabla'\phi'$, $\mathbf{E}'_T \simeq 0 \simeq \mathbf{B}'_s$) which fully incorporates beam space-charge effects, but neglects any fast electromagnetic processes with transverse polarization (e.g., light waves). The resulting Vlasov-Maxwell equations were then Lorentz-transformed back to the laboratory frame, and properties of the self-generated fields and resulting nonlinear Vlasov-Maxwell equations in the laboratory frame were discussed.

ACKNOWLEDGMENTS

This research was supported by the U.S. Department of Energy.

REFERENCES

[1] R. C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators* (World Scientific, Singapore, 2001).

[2] S. Y. Lee, *Accelerator Physics* (World Scientific, Singapore, 1999).

[3] A. W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (John Wiley & Sons, Inc., New York, 1993).

[4] M. Reiser, *Theory and Design of Charged Particle Beams* (John Wiley & Sons, Inc., New York, 1994).

[5] T. P. Wangler, *Principles of RF Linear Accelerators* (John Wiley & Sons, Inc., New York, 1998).

[6] H. Wiedemann, *Particle Accelerator Physics I: Basic Principles and Linear Beam Dynamics* (Springer-Verlag, New York, 1999).

[7] H. Wiedemann, *Particle Accelerator Physics II: Nonlinear and Higher-Order Beam Dynamics* (Springer-Verlag, New York, 1998).

[8] See, for example, *Proceedings of the 1999 Particle Accelerator Conference* (IEEE Catalog Number 99CH36366), pp. 1–3778.

[9] See, for example, *Proceedings of the International Heavy Ion Fusion Symposium*, Nuclear Instruments and Methods in Physics Research **A464**, pp. 1–674 (2001).

[10] I. M. Kapchinskij and V. V. Vladimirskij, in *Proceedings of the International Conference on High Energy Accelerators and Instrumentation* (CERN Scientific Information Service, Geneva, 1959), p. 274.

[11] R. L. Gluckstern, in *Proceedings of the 1970 Proton Linear Accelerator Conference*, Batavia, IL, edited by M. R. Tracy (National Accelerator Laboratory, Batavia, IL, 1971), p. 811.

[12] I. Hofmann, L. J. Laslett, L. Smith and I. Haber, Particle Accelerators **13**, 145 (1983).

[13] T. -S. Wang and L. Smith, Particle Accelerators **12**, 247 (1982).

[14] J. Struckmeier, J. Klabunde and M. Reiser, Particle Accelerators **15**, 47 (1984).

[15] I. Hofmann and J. Struckmeier, Particle Accelerators **21**, 69 (1987).

[16] J. Struckmeier and I. Hofmann, Particle Accelerators **39**, 219 (1992).

[17] C. Chen and R. C. Davidson, Phys. Rev. **E49**, 5679 (1994).

[18] R. L. Gluckstern, W. -H. Cheng and H. Ye, Phys. Rev. Lett. **75**, 2835 (1995).

[19] N. Brown and M. Reiser, Phys. Plasmas **2**, 965 (1995).

[20] R. C. Davidson and C. Chen, Particle Accelerators **59**, 175 (1998).

[21] R. C. Davidson and H. Qin, Physical Review Special Topics on Accelerators and Beams **2**, 114401 (1999).

[22] R. C. Davidson, W. W. Lee and P. Stoltz, Phys. Plasmas **5**, 279 (1998).

[23] R. C. Davidson, Phys. Rev. Lett. **81**, 991 (1998).

[24] R. C. Davidson, Phys. Plasmas **5**, 3459 (1998).

[25] P. H. Stoltz, R. C. Davidson, and W. W. Lee, Phys. Plasmas **6**, 298 (1999).

[26] P. J. Channell, Phys. Plasmas **6**, 982 (1999).

[27] H. Qin, R. C. Davidson and W. W. Lee, Physical Review Special Topics on Accelerators and Beams **3**, 084401 (2000); **3**, 109901 (2000).

[28] R. C. Davidson, H. Qin and P. J. Channell, Physical Review Special Topics on Accelerators and Beams **2**, 074401 (1999); **3**, 029901 (2000).

[29] See, for example, Chapters 5–8 of Ref. 1.

[30] W. W. Lee, E. Startsev, H. Qin and R. C. Davidson, Proceedings of the 2001 Particle Accelerator Conference, in press (2001).

[31] C. W. Nielson and H. R. Lewis, *Methods in Computational Physics* (Academic Press, New York, 1976), Vol. 16, p. 976.

[32] R. C. Davidson, D. A. Hammer, I. Haber, and C. E. Wagner, Phys. Fluids **15**, 317 (1972).

[33] Busnardo-Neto, et al., J. Comput. Phys. **23**, 300 (1977).

[34] W. W. Lee, M. S. Chance and H. Okuda, Phys. Rev. Lett. **46**, 1675 (1981).

[35] C. G. Darwin, Phil. Mag. **39**, 537 (1920).

[36] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1971).

[37] J. D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, Inc., New York, 1975).

External Distribution

Plasma Research Laboratory, Australian National University, Australia
Professor I.R. Jones, Flinders University, Australia
Professor João Canalle, Instituto de Fisica DEQ/IF - UERJ, Brazil
Mr. Gerson O. Ludwig, Instituto Nacional de Pesquisas, Brazil
Dr. P.H. Sakanaka, Instituto Fisica, Brazil
The Librarian, Culham Laboratory, England
Library, R61, Rutherford Appleton Laboratory, England
Mrs. S.A. Hutchinson, JET Library, England
Professor M.N. Bussac, Ecole Polytechnique, France
Librarian, Max-Planck-Institut für Plasmaphysik, Germany
Jolan Moldvai, Reports Library, MTA KFKI-ATKI, Hungary
Dr. P. Kaw, Institute for Plasma Research, India
Ms. P.J. Pathak, Librarian, Insitute for Plasma Research, India
Ms. Clelia De Palo, Associazione EURATOM-ENEA, Italy
Dr. G. Grosso, Instituto di Fisica del Plasma, Italy
Librarian, Naka Fusion Research Establishment, JAERI, Japan
Library, Plasma Physics Laboratory, Kyoto University, Japan
Research Information Center, National Institute for Fusion Science, Japan
Dr. O. Mitarai, Kyushu Tokai University, Japan
Library, Academia Sinica, Institute of Plasma Physics, People's Republic of China
Shih-Tung Tsai, Institute of Physics, Chinese Academy of Sciences, People's Republic of China
Dr. S. Mirnov, TRINITI, Troitsk, Russian Federation, Russia
Dr. V.S. Strelkov, Kurchatov Institute, Russian Federation, Russia
Professor Peter Lukac, Katedra Fyziky Plazmy MFF UK, Mlynska dolina F-2, Komenskeho Univerzita, SK-842 15 Bratislava, Slovakia
Dr. G.S. Lee, Korea Basic Science Institute, South Korea
Mr. Dennis Bruggink, Fusion Library, University of Wisconsin, USA
Institute for Plasma Research, University of Maryland, USA
Librarian, Fusion Energy Division, Oak Ridge National Laboratory, USA
Librarian, Institute of Fusion Studies, University of Texas, USA
Librarian, Magnetic Fusion Program, Lawrence Livermore National Laboratory, USA
Library, General Atomics, USA
Plasma Physics Group, Fusion Energy Research Program, University of California at San Diego, USA
Plasma Physics Library, Columbia University, USA
Alkesh Punjabi, Center for Fusion Research and Training, Hampton University, USA
Dr. W.M. Stacey, Fusion Research Center, Georgia Institute of Technology, USA
Dr. John Willis, U.S. Department of Energy, Office of Fusion Energy Sciences, USA
Mr. Paul H. Wright, Indianapolis, Indiana, USA

The Princeton Plasma Physics Laboratory is operated
by Princeton University under contract
with the U.S. Department of Energy.

Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2750
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov