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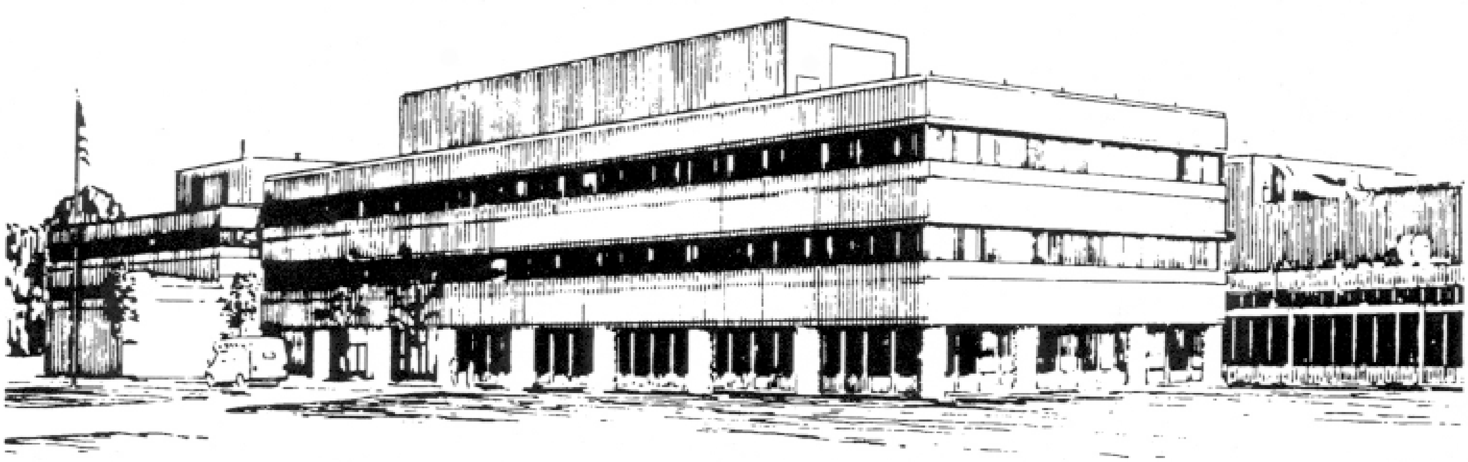

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On Resonant Heating Below the Cyclotron Frequency

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Abstract

Resonant heating of particles by an electrostatic wave propagating perpenducular to a confining uniform magnetic field is examined. It is shown that, with a sufficiently large wave amplitude, significant perpendicular stochastic heating can be obtained with wave frequency at a fraction of the cyclotron frequency.

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Resonant heating of particles in a magnetic field has been examined by many authors and is of importance in the heating of magnetically confined laboratory as well as extraterrestrial plasmas. For a review see Lieberman and Lichtenberg[1]. The transition from adiabatic to stochastic heating was first examined by Nekrasov[2], Jaeger and Lichtenberg[3], and Lieberman and Lichtenberg[4]. Stochastic heating by a lower hybrid wave in a tokamak was investigated by Karney[5] for wave frequency much larger than the cyclotron frequency. It has also been noted[6,7] that heating with ion Bernstein waves (IBW) can be obtained at frequencies above the cyclotron frequency Ω but below the second harmonic, at frequencies of $\omega/\Omega = 3/2, \;\; 4/3$, etc. To our knowledge however, a breaking of the invariance of the magnetic moment at frequencies at a fraction of the cyclotron frequency has never been reported in a theoretical work. Neither have we been able to find a clear experimental search for such heating. In this Letter we wish to demonstrate that, at sufficiently large wave amplitude, low-frequency wave heating is indeed possible. To this end we consider the simplest problem possible; that of a particle gyrating in a constant magnetic field acted upon by an electrostatic plane wave propagating perpendicular to the field.

The Hamiltonian for this system is

$$H = \frac{(\vec{v} - \vec{A})^2}{2} + \Phi(x,t) \tag{1}$$

with the magnetic field given by the vector potential $\vec{A} = -By\hat{x}$. Take the units of time to be given by Ω, the cyclotron frequency, let the electrostatic wave be given by a single harmonic, $\Phi = \Phi_0 cos(kx - \omega t)$, and assume zero velocity parallel to the field, $v_z = 0$. There are then three dimensionless parameters characterizing the heating problem. Define $\rho = v/\Omega$ to be the instantaneous cyclotron radius. Then $k\rho$ characterizes the ratio of cyclotron radius to wave length, $k^2\Phi_0$, characterizing the ratio of particle displacement caused by the wave to wave length, is the nonlinearity parameter, and ω is the ratio of the wave frequency to the cyclotron frequency. The initial particle distribution is also characterized by $k\rho_0$ with ρ_0 a mean cyclotron radius for the distribution.

The equations of motion become $\dot{v}_x = v_y + k\Phi_0 sin(kx - \omega t), \quad v_y = -x + x_0$, giving

$$\frac{d^2x}{dt^2} + x = x_0 + k\Phi_0 sin(kx - \omega t). \tag{2}$$

For small wave amplitude near the cyclotron frequency it is possible to describe the particle response to the wave in terms of oscillation at the cyclotron frequency with a slowly varying cyclotron radius, or energy. In the case of interest here wave amplitudes are large and wave frequencies different from, but comparable to, the cyclotron frequency, so response of the particle at additional frequencies must be retained. To treat the full problem it is necessary to include particle motion at fractions of the cyclotron frequency, sidebands, harmonics, etc. The particle motion must be written $x = x_0 + \lambda cos(t) - \mu sin(t) + \sum_m [\alpha_m cos(\nu_m t) + \beta_m sin(\nu_m t)]$ with λ, μ, α_m, β_m slowly varying in compared to $1, \nu_m$, with ν_m giving the set of frequencies necessary to describe the motion. A full analytic treatment is not possible, but analytic approximations can give insight into the nature of the solutions.

First consider Eq. 2 for $ks \equiv k(x - x_0) \ll 1$. Letting $2T = kx_0 - \omega t$ and keeping only lowest order in ks we have

$$\frac{d^2s}{dT^2} + \left[\frac{4}{\omega^2} - \frac{4k^2\Phi_0}{\omega^2} cos(2T)\right] s = k\Phi_0 \omega sin(2T) \tag{3}$$

ie, a driven Mathieu equation with unstable solutions for $\omega \simeq 2/n$. Of course this equation is valid only for small ks, but it indicates the existence of large amplitude solutions for these values of ω.

Now consider a Poincaré section of $k\rho$, $\psi = kx - \omega t$, by taking points when $v_y = 0, \dot{v}_y > 0$. This gives $\psi = \psi_0 - \omega t_j$, with $\psi_0 = kx_0$, and t_j given by the times at which $x = x_0$ and $\dot{x} < 0$. Given $\lambda(t)$, $\mu(t)$, $\alpha_m(t)$, $\beta_m(t)$ one can solve for the Poincaré times t_j. Without loss of generality at $t = 0$ we take x random, v_x random negative and $v_y = 0$; giving $x = x_0$, $\psi(0) = \psi_0$. The values at $t = 0$ then determine one Poincaré point. Others are given by $k\rho(t_j)$, $\psi(t_j) = \psi_0 - \omega t_j$. Fixed points are given by $dv/dt = 0$ and constant phase, or $\dot{\lambda} = \dot{\mu} = \dot{\alpha_m} = \dot{\beta_m} = 0$.

In general these equations are very complicated and the Poincaré section must be examined numerically. For significant heating there must exist resonances. A complete analysis would consist of a determination of all fixed points and then the calculation of the widths of the islands occuring around the elliptic points, followed by an estimate of stochastic threshold due to island overlap. Unfortunately this approach is not feasible, and to make any progress analytically one must be guided by numerical results. A numerical Poincaré plot using Eq. 2 is shown in Fig. 1 for $k^2\Phi_0 = 0.1$, $\omega = 1/2$, showing period two fixed points occuring at small wave amplitude.

Guided by numerical results, including a Fourier analysis of the fixed point trajectories, we illustrate the nature of the solutions for this case by considering only the cyclotron motion and the particle response at the wave frequency of $\omega = 1/2$. Employing multiple time scales, express the solution to the equations of motion as $x = x_0 + \lambda cos(t) - \mu sin(t) + \alpha cos(\omega t) - \beta sin(\omega t)$ with $\lambda, \mu, \alpha, \beta$ slowly varying with respect to $1, \omega$. We then find, keeping only leading order in the slow time scale and using $e^{\pm ia sin(b)} = \sum_m J_m(a) e^{\pm imb}$,

$$-2\frac{d\mu}{dt}cos(t) - 2\frac{d\lambda}{dt}sin(t) + (1-\omega^2)\alpha cos(\omega t) - (1-\omega^2)\beta sin(\omega t) =$$

$$k\Phi_0 \sum_{jklm} J_j(k\lambda)J_k(k\mu)J_l(k\alpha)J_m(k\beta)sin[(j-k+l\omega-m\omega-\omega)t]cos[\psi_0+(j+l)\pi/2]$$

$$+k\Phi_0 \sum_{jklm} J_j(k\lambda)J_k(k\mu)J_l(k\alpha)J_m(k\beta)cos[(j-k+l\omega-m\omega-\omega)t]sin[\psi_0+(j+l)\pi/2] \quad (4)$$

Taking the frequency to be a fraction of the cyclotron frequency, $\omega = 1/q$ with q an integer and integrating over the short time scales we have

$$(1-\omega^2)\alpha = k\Phi_0 \sum_{jklm} J_j(k\lambda)J_k(k\mu)J_l(k\alpha)J_m(k\beta)sin(\psi_0+(j+l)\pi/2)\Delta_{\omega+} \quad (5)$$

$$(1-\omega^2)\beta = k\Phi_0 \sum_{jklm} J_j(k\lambda)J_k(k\mu)J_l(k\alpha)J_m(k\beta)cos(\psi_0+(j+l)\pi/2)\Delta_{\omega-} \quad (6)$$

$$2\frac{d\mu}{dt} = -k\Phi_0 \sum_{jklm} J_j(k\lambda)J_k(k\mu)J_l(k\alpha)J_m(k\beta)sin(\psi_0+(j+l)\pi/2)\Delta_{1+} \quad (7)$$

$$2\frac{d\lambda}{dt} = -k\Phi_0 \sum_{jklm} J_j(k\lambda)J_k(k\mu)J_l(k\alpha)J_m(k\beta)cos(\psi_0+(j+l)\pi/2)\Delta_{1-} \quad (8)$$

with $\Delta_{\zeta\pm} = \delta_{j-k+(l-m-1)\omega,\zeta} \pm \delta_{j-k+(l-m-1)\omega,-\zeta}$.

To gain an intuitive understanding of the occurences of the nonlinear resonances which permit heating at frequencies well below the cyclotron frequency we can examine the limit of small wave amplitude, $k^2\Phi_0 \ll 1$ analytically. Then we have $k\alpha \ll 1$, $k\beta \ll 1$.

Now set $\omega = 1/2$. To the leading orders in $k\alpha$ and $k\beta$ we then find the first delta function of $\Delta_{\omega\pm}$ is limited to the values $(j,k,l,m) = (s, s-1, 0, 0)$ and the second delta function to $(s, s, 0, 0)$, with s integer. Similarly, for $\Delta_{1\pm}$, the first delta function is limited to the values $(j,k,l,m) = (s, s-1, 1, 0)$, $(s, s-1, 0, -1)$, $(s, s-2, -1, 0)$, $(s, s-2, 0, 1)$, with s integer, and the second delta function to the values $(j,k,l,m) = (s, s+1, 1, 0)$, $(s, s+1, 0, -1)$, $(s, s, -1, 0)$, $(s, s, 0, 1)$, with s integer. Note that these values are peculiar to the case of $\omega = 1/2$, which is a special degenerate case since $q = 2$ is the only solution to

$1-1/q=1/q$. There are fewer but different lowest order terms for other fractions. Now in each term replacing the sum $s=-\infty,\infty$ with $s=2n,2n+1$, $n=-\infty,\infty$, denoting $C_0=cos(\psi_0)$, $S_0=sin(\psi_0)$, and defining $F_{a,b}(\lambda,\mu)=\sum_{n=-\infty}^{\infty}(-1)^n J_{2n+a}(k\lambda)J_{2n+b}(k\mu)$, we have

$$(1-\omega^2)\alpha = k\Phi_0[C_0(F_{1,0}+F_{1,1})+S_0(F_{0,0}+F_{0,-1})], \tag{9}$$

$$(1-\omega^2)\beta = k\Phi_0[C_0(F_{0,0}-F_{0,-1})+S_0(F_{1,0}-F_{1,1})], \tag{10}$$

$$\begin{aligned}-2\frac{d\mu}{dt} &= k\Phi_0 J_1(k\alpha)[C_0(F_{0,-1}+F_{0,-2}+F_{0,1}+F_{0,0})-S_0(F_{1,0}+F_{1,-1}+F_{1,2}+F_{1,1})]\\ &+k\Phi_0 J_1(k\beta)[C_0(-F_{1,0}+F_{1,-1}-F_{1,2}+F_{1,1})+S_0(-F_{0,-1}+F_{0,-2}-F_{0,1}+F_{0,0})],\end{aligned} \tag{11}$$

$$\begin{aligned}2\frac{d\lambda}{dt} &= k\Phi_0 J_1(k\alpha)[S_0(F_{0,-1}+F_{0,-2}+F_{0,1}-F_{0,0})+C_0(F_{1,0}+F_{1,-1}+F_{1,2}+F_{1,1})]\\ &+k\Phi_0 J_1(k\beta)[S_0(-F_{1,0}+F_{1,-1}+F_{1,2}-F_{1,1})+C_0(F_{0,-1}-F_{0,-2}-F_{0,1}+F_{0,0})].\end{aligned} \tag{12}$$

These equations determine the motion of a Poincaré point in the $k\rho$, ψ plane for small $k^2\Phi_0$. To determine the existence of resonances first look for fixed points of the Poincaré map, with $k^2\Phi_0 \ll 1$. In this case α and β are small, and since $\lambda(0)=-\alpha(0)$ and for a fixed point λ must be constant, it remains small. Keeping only up to first order in λ we find for the existence of a fixed point in the case $\omega=1/2$ either $C_0=0$ or $S_0=0$.

For $C_0=1, S_0=0$ we have

$$(1-\omega^2)\alpha = k\Phi_0 J_1(k\lambda)[J_2(k\mu)+J_0(k\mu)], \tag{13}$$

$$(1-\omega^2)\beta = -k\Phi_0[J_1(k\mu)+J_0(k\mu)], \tag{14}$$

$$0 = J_1(k\lambda)[J_0(k\mu) + J_2(k\mu)][J_2(k\mu) + J_0(k\mu)]$$

$$-J_1(k\lambda)[J_1(k\mu) + J_0(k\mu)][2J_0(k\mu) + 2J_2(k\mu) + J_1(k\mu) + J_3(k\mu)], \quad (15)$$

$$0 = J_1^2(k\lambda)[J_0(k\mu) + J_2(k\mu)][2J_0(k\mu) + 2J_2(k\mu) - J_1(k\mu) - J_3(k\mu)]$$

$$+[J_1(k\mu) + J_0(k\mu)][-2J_1(k\mu) - J_2(k\mu) + J_0(k\mu)]. \quad (16)$$

To lowest order in λ, μ is given by $0 = [J_1(k\mu) + J_0(k\mu)][-2J_1(k\mu) - J_2(k\mu) + J_0(k\mu)]$. The first root is from $J_0 = 2J_1 + J_2$ giving $k\mu = 0.825$. Equation 15 then gives $\lambda = 0$, and we find $\alpha = 0$, $\beta = -1.62k\Phi_0$. Since $\lambda = \alpha = 0$, the Poincaré points are given by $t_j = 2j\pi$. The fixed points are then $k\rho = k(\mu - \omega\beta)$, $\psi = 0$ and $k\rho = k(\mu + \omega\beta)$, $\psi = \pi$, as seen in Fig. 1.

No solution is found for $C_0 = 0, S_0 = 1$. The second pair of fixed points in Fig. 1 at $\psi = \pm\pi/2$ and $k\rho = 1.84$ is more complex, due to a combination of motion at ω and 3ω. It should be obvious from the above that by including particle response at more frequencies, and allowing larger values of $k^2\Phi_0$ the number of fixed points in the map will increase enormously.

For $\omega = 1/q$, with $q > 2$ the situation is qualitatively different. To leading order there do not exist any fixed points; rather the fixed points of the map emerge from $\rho = 0$ as Φ_0 is increased. Nevertheless, such fixed points exist for all integer q, associated with the unstable domains of the associated Mathieu equation, which are much narrower for $q > 2$. A numerical Poincaré plot is shown in Fig. 2 for $k^2\Phi_0 = 0.1$, $\omega = 1/3$, showing period three fixed points which move upward as $k^2\Phi_0$ increases.

Now investigate the approach to chaos and the extent of the chaotic domain, which limits the possible heating obtained. Figure 3 shows an example of the extent of the stochastic domain for $\omega = 1/2$, bounded by good KAM surfaces at large $k\rho$. The initial particle distribution was random with $k\rho < 0.1$. In the domain of the good KAM surfaces the perpendicular energy is only oscillatory, described by the magnetic moment, becomes an adiabatic invariant for large energy and relatively weak wave amplitude[3]. The extent of the stochastic domain increases in discrete jumps as new resonances overlap and domains around them become stochastic, and Fig. 3 shows the occurence of such a step, as the stochastic domain sweeps around a new period three resonance. Heating of an initially cold distribution proceeds to the maximum limit given by the good KAM surfaces in a rather short time; on the order of one to two hundred cyclotron periods. Even at a wave frequency of 1/10 of the cyclotron frequency a Poincaré plot is quite stochastic for $k^2\Phi_0 = 1$. Note that this is a collisionless result.

Figure 4 shows the variation of the extent of the heating domain in $k\rho$ versus $k\sqrt{\Phi_0}$ for three different wave frequencies. Curve a) is heating at the cyclotron frequency, b) at half the cyclotron frequency, and c) at 1/5 the cyclotron frequency. Note that, for small wave amplitude, heating at the cyclotron frequency is clearly more efficient, but that as the amplitude increases, heating at a lower frequency can be almost as effective.

Finally Figure 5 shows the variation of the extent of the heating domain in $k\rho$ versus wave frequency for $k\sqrt{\Phi_0} = 0.6$, 0.9, and 1.6. For smaller wave amplitude some peaking can indeed be seen at low-order (small) integer fractions, as predicted by the Mathieu equation approximation. As the amplitude increases, however, nonlinear generation of many fixed points smooths out the resonance structures and makes the extent of the domain almost linear in ω.

In conclusion, we have demonstrated that significant perpendicular heating can be obtained at a fraction of the cyclotron frequency. Although we have investigated only the case of a longitudinal wave propagating across a constant magnetic field, the nonlinear resonance phenomenon should be more fundamental, and may have application in high power radio-frequency heating schemes in laboratory as well as in astrophysical plasmas. Furthermore, we expect similar heating mechanisms to be operative for large amplitude Alfvèn waves, and will explore this effect in a separate publication.

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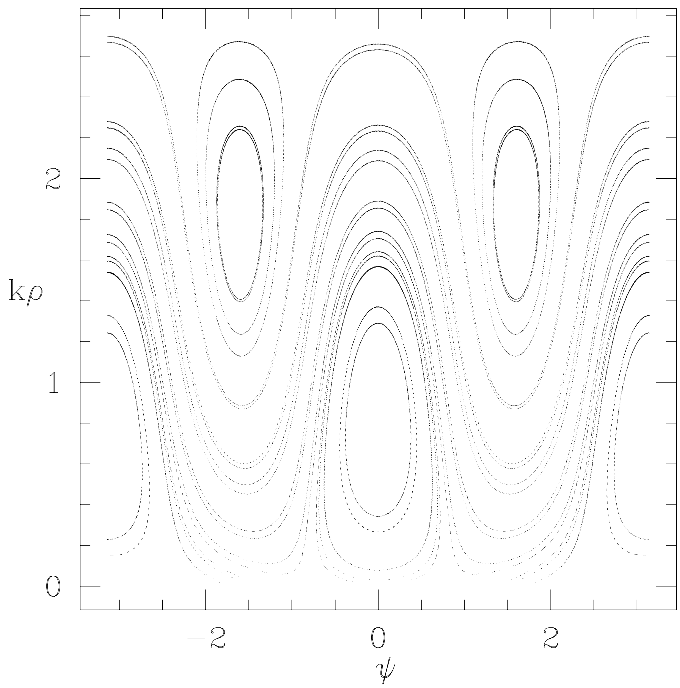

Fig. 1. Poincaré, $k^2\Phi_0 = 0.1$, $\omega = 1/2$

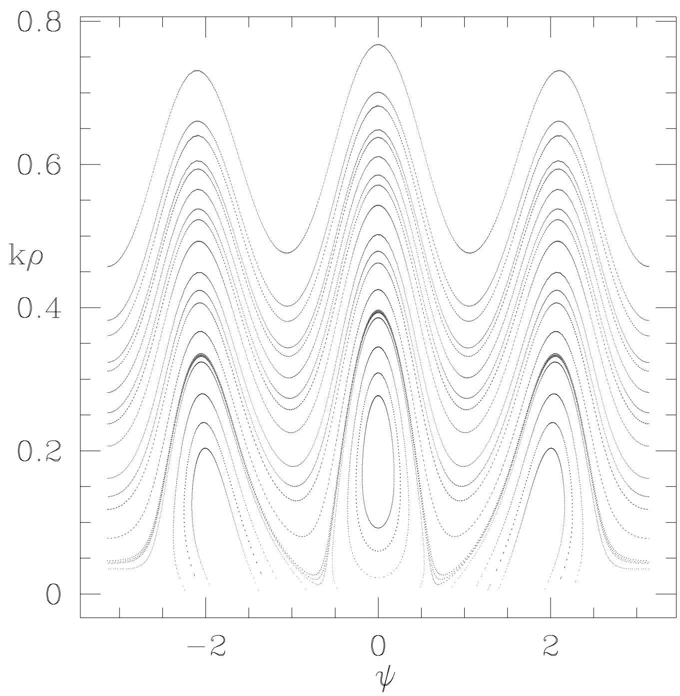

Fig. 2. Poincaré, $k^2\Phi_0 = 0.1$, $\omega = 1/3$

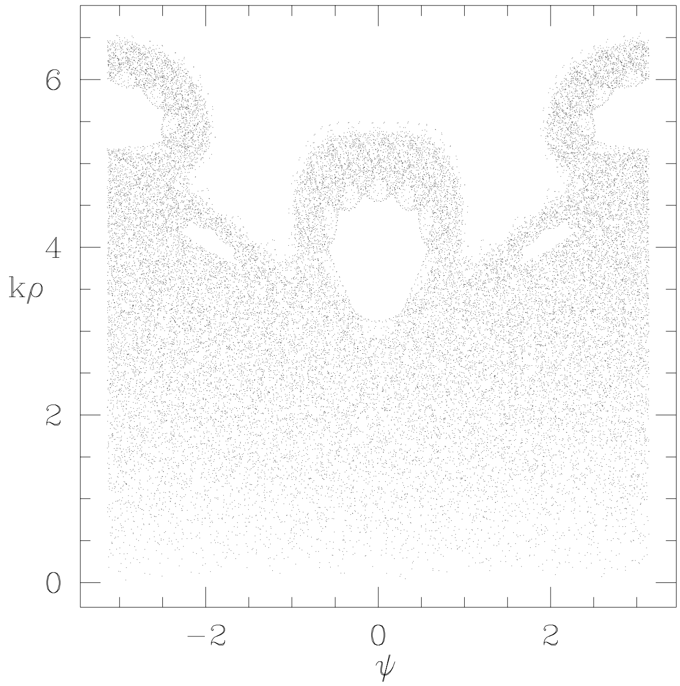

Fig. 3. Poincaré, $k\sqrt{\Phi_0} = 1.75$, $\omega = 1/2$

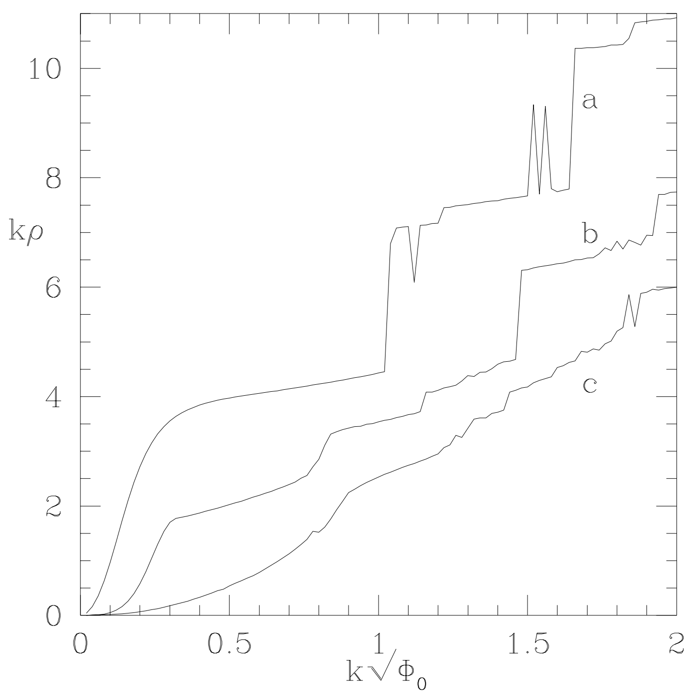

Fig. 4. Heating Domain vs $k\sqrt{\Phi_0}$, a) $\omega = 1$, b) $\omega = 1/2$, c) $\omega = 1/5$.

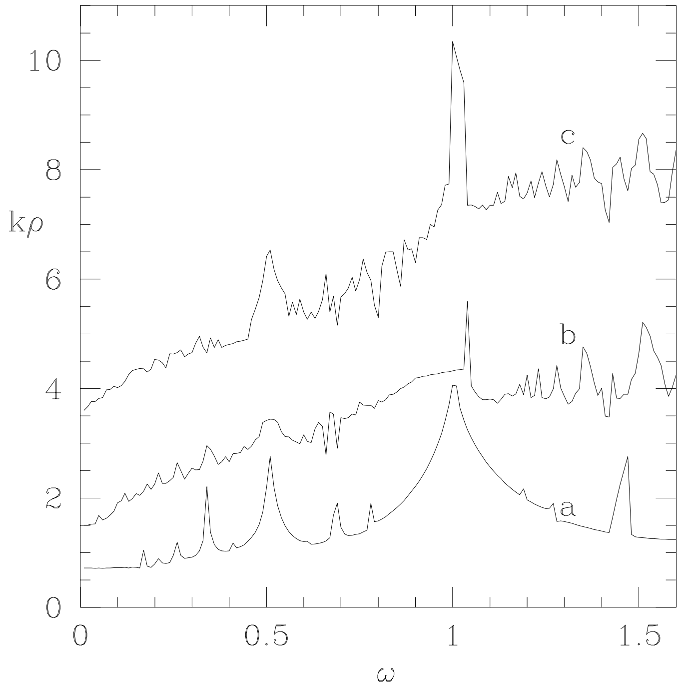

Fig. 5. Heating Domain vs ω for $k\sqrt{\Phi_0}$ = a)0.6, b) 0.9 c) 1.6

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