Finite Correlation Time Effects in Kinematic Dynamo Problem

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One-point statistics of the magnetic fluctuations in kinematic régime with large Prandtl number and non δ-correlated in time advecting velocity field are studied. A perturbation expansion in the ratio of the velocity correlation time to the dynamo growth time is constructed in the spirit of the Kliatskin-Tatarskii functional method and carried out to first order. The convergence properties are improved compared to the commonly used van Kampen-Terwiel method. The zeroth-order growth rate of the magnetic energy is estimated to be reduced (in three dimensions) by approximately 40%. This reduction is quite close to existing numerical results.

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I. INTRODUCTION

The work on the kinematic dynamo problem, understood as the study of the statistics of magnetic fluctuations excited by a random advecting velocity field with given Gaussian statistics, also known as the passive vector problem, was pioneered by Kazantsev [1], and the dynamo growth rates have in recent years been found exactly for various parameter régimes by a number of authors [2–7]. In all of these works a paradigm involving a short-time correlated velocity field was employed, so the two-time correlation function of the velocities could be approximated by a delta function, $\langle v(t)v(t')\rangle \propto \delta(t-t')$. This Markov property of the velocity greatly simplifies matters: the equations that relate magnetic correlation functions and various response functionals take closed form and yield themselves to exact solution. On the other hand, allowing for a finite velocity correlation time leads to an infinite open hierarchy of integrodifferential equations involving time-history integrals. This is explained in more detail in what follows.

The short correlation time approximations we have just mentioned effectively amount to truncating the hierarchy by retaining only terms that are zeroth-order in the expansion parameter $\lambda_0 \sim \tau_{\rm corr}\gamma_0 \sim (\tau_{\rm corr}/\tau_{\rm eddy})^2$, where $\tau_{\rm corr}$ is the velocity correlation time, γ_0 is the dynamo (magnetic energy) growth rate, and $\tau_{\rm eddy} \sim (\nabla v)^{-1}$ is the "eddy-turnover" time.

We will see that, in terms of the spectral characteristics of the velocity field, λ_0 typically turns out to be $\lambda_0 \propto \log(\Lambda/\tilde{\Lambda})$, where Λ and $\tilde{\Lambda}$ are the ultraviolet and infrared cutoff wave numbers of the model velocity spectrum. One would therefore expect that, depending on the exact form of λ_0 and convergence criteria, only a relatively narrow band of velocity modes can technically be treated by perturbative methods. However, physical considerations allow one to introduce an effective infrared cutoff $\tilde{\Lambda}_{\rm eff}$ which essentially limits the active part of the velocity spectrum to "one smallest eddy." In order for the perturbation expansion to have practical value, it should not be too sensitive to deviations of $\Lambda/\tilde{\Lambda}_{\rm eff}$ from unity.

The extant evidence as to the nature of the finite correlation time effect on the dynamo action is twofold. On the one hand, Chandran's numerical simulation [8] suggests that a growth rate reduction of about 50% takes place. On the other hand, it was demonstrated by Gruzinov, Cowley, and Sudan in a rather compelling argument [9] (which unfortunately fell short of a conclusive proof) that introduction of a non-zero correlation time could not fully suppress the growth of the small-scale magnetic fluctuations.

Quantitative estimation of λ_0 and calculation of the higher-order terms in the expansion are necessary for specific understanding of how the finiteness of the velocity correlation time affects the well-known results valid for zero correlation time. Theoretical efforts in this direction were undertaken by Knobloch [10] and Chandran [8], who used the van Kampen-Terwiel perturbative method for solving stochastic differential equations [11,12]. Rather forbiddingly cumbersome calculations were involved, and a first-order reduction of the dynamo (magnetic energy) growth rate was derived that was valid only for extremely small values of $(\Lambda/\tilde{\Lambda})-1$. Plausible quantitative estimate of this reduction therefore remained elusive.

In this letter, we construct a straightforward perturbation theory in λ_0 for the kinematic dynamo problem in the spirit of the Kliatskin-Tatarskii method [13–15] (used by its authors in a different context and form). We find that the zeroth-order dynamo growth rate γ_0 is reduced (in three dimensions) by $\frac{6}{5}\tau_{\rm corr}\gamma_0^2 \approx \frac{2}{5}\gamma_0$ (the exact meaning of $\tau_{\rm corr}$ in this formula is explained below). The 40% reduction of the growth rate is quite close to Chandran's numerical estimate [8].

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II. THE HIERARCHY OF EQUATIONS FOR THE PDF AND RESPONSE FUNCTIONS

The magnetic field (or passive vector) passively advected by the incompressible velocity field $\mathbf{v}(t,\mathbf{x})$ evolves according to the following (induction) equation:

$$\partial_t B^i + v^k B^i_{,k} = v^i_{,k} B^k + \eta \Delta B^i, \tag{1}$$

where $v^i_{,k} = \partial v^i/\partial x^k$, $B^i_{,k} = \partial B_i/\partial x^k$, and the Einstein summation convention is used throughout. Spacial homogeneity is assumed, and we restrict ourselves to one-point statistics, i. e. to studying integral average characteristics of the magnetic field such as the total magnetic energy. The one-point statistics of fields passively advected by an incompressible velocity field are the same in both Lagrangian and Eulerian framework [7], so the convective derivative in the left-hand side of Eq. (1) can be replaced with simple local derivative with respect to time. In the following derivation, we also omit the resistive term, which does not play an important part during the initial period of advection for large Prandtl numbers [3]. Thus, we study the statistics of the following equation:

$$\partial_t B^i = v^i_{,k} B^k, \tag{2}$$

where the velocity gradients' statistical properties are determined on the assumption that $v^i(t,\mathbf{x})$ is a homogeneous and isotropic Gaussian random field:

$$\left\langle v^i_{,k}(t,\mathbf{x}) v^j_{,l}(t',\mathbf{x}) \right\rangle = \varkappa^{ij}_{kl}(t-t'),$$
$$\varkappa^{ij}_{kl}(\tau) = \varkappa(\tau)\left(\delta^{ij}\delta_{kl} + a\delta^i_k\delta^j_l + a\delta^i_l\delta^j_k\right) = \varkappa(\tau)\, T^{ij}_{kl} \tag{3}$$

where $a = -1/(d+1)$ due to incompressibility of the flow, d being the dimension of space (while $d=3$ is, of course, of most practical interest, carrying the dimensional dependences is instructive). T^{ij}_{kl} as defined by the above formula contains the tensor structure of $\varkappa^{ij}_{kl}$.

In order to determine the one-point statistics of the magnetic field, we follow a standard procedure [16,7] and introduce the characteristic function of $\mathbf{B}(t,\mathbf{x})$ at an arbitrary fixed spacial point $\mathbf{x}$:

$$Z(t;\sigma) = \left\langle \tilde{Z}(t,\mathbf{x};\sigma) \right\rangle = \left\langle \exp\left\{ i\sigma_i B^i(t,\mathbf{x}) \right\} \right\rangle. \tag{4}$$

Here and everywhere in this letter the angle brackets denote ensemble average. This function is the Fourier transform of the PDF of the vector elements B^i. Clearly, Z is independent of $\mathbf{x}$ due to spacial homogeneity. Taking the time derivative of $\tilde{Z}$ and using Eq. (2) we get

$$\partial_t \tilde{Z} = v^i_{,j}\, \sigma_i \frac{\partial}{\partial \sigma_j} \tilde{Z}. \tag{5}$$

Averaging this equation with the aid of the Novikov ("Gaussian integration") formula [17,14], we find that Z satisfies

$$\partial_t Z(t;\sigma) = T^{ib}_{ja}\, \sigma_i \frac{\partial}{\partial \sigma_j} \int_0^t \mathrm{d}\tau\, \varkappa(\tau)\, G^a_b(t, t-\tau;\sigma), \tag{6}$$
$$\text{where} \quad G^a_b(t,t';\sigma) = \left\langle \frac{\delta \tilde{Z}(t;\sigma)}{\delta v^b_{,a}(t')} \right\rangle$$

is the average first-order response function. Clearly, it satisfies causality: $G^a_b(t,t';\sigma) = 0$ for $t' > t$. Integrating Eq. (5) from 0 to t, taking the functional derivative $\delta/\delta v^b_{,a}(t')$, averaging, and setting $t'=t$, we find the response function at $t'=t$:

$$G^a_b(t,t;\sigma) = \frac{1}{2}\, \sigma_b \frac{\partial}{\partial \sigma_a} Z(t;\sigma). \tag{7}$$

At $t' < t$, we take the functional derivative $\delta/\delta v^b_{,a}(t')$ of Eq. (5) and find that G^a_b satisfies the following equation (averaged by the same method that was employed in deriving Eq. (6)):

$$\partial_t G^a_b(t,t';\sigma) = T^{kn}_{lm}\, \sigma_k \frac{\partial}{\partial \sigma_l} \int_0^t \mathrm{d}\tau\, \varkappa(\tau)\, G^{am}_{bn}(t,t',t-\tau;\sigma), \tag{8}$$
$$\text{where} \quad G^{am}_{bn}(t,t',t'';\sigma) = \left\langle \frac{\delta \tilde{Z}(t;\sigma)}{\delta v^b_{,a}(t')\delta v^n_{,m}(t'')} \right\rangle$$

is the second-order average response function. At $t''=t$, it is, analogously to Eq. (7),

$$G^{am}_{bn}(t,t',t;\sigma) = \frac{1}{2}\, \sigma_n \frac{\partial}{\partial \sigma_m} G^a_b(t,t';\sigma). \tag{9}$$

An infinite hierarchy can thus be obtained by further iterating this procedure and introducing response functions of ever-increasing orders. This hierarchy can be closed if assumptions are made about the smallness of the correlation time. Indeed, we can formally expand the velocity correlator up to first order in τ_{corr} as follows:

$$\varkappa(\tau) = 2\overline{\varkappa}\delta(\tau) - 2\overline{\tau}\delta'(\tau) + \ldots, \tag{10}$$
$$\text{where} \quad \overline{\varkappa} = \int_0^\infty \mathrm{d}\tau\, \varkappa(\tau) \quad \text{and} \quad \overline{\tau} = \int_0^\infty \mathrm{d}\tau\, \tau\varkappa(\tau).$$

Here $\overline{\tau}$ is the dimensionless expansion parameter. In the upper integration limits, t has been replaced by ∞, which is valid for $t \gg \tau_{\mathrm{corr}}$, $\tau_{\mathrm{corr}} = \overline{\tau}/\overline{\varkappa}$ being the width of $\varkappa(\tau)$. The above expansion lies at the base of the Kliatskin-Tatarskii method [13–15], which is essentially employed below. We now construct a perturbation theory by using expansion (10).

III. THE PERTURBATION EXPANSION

To first order in the expansion (10), Eq. (6) becomes

$$\partial_t Z(t;\sigma) = T^{ib}_{ja}\,\sigma_i \frac{\partial}{\partial\sigma_j}\Big\{\overline{\varkappa}\, G^a_b(t,t;\sigma) + + \overline{\tau}\Big[\partial_\tau G^a_b(t,t-\tau;\sigma)\Big]_{\tau=0}\Big\} \tag{11}$$

The equal-time response function in the first term on the right-hand is given by Eq. (7), while the derivative of the response function with respect to τ in the second term clearly need only be calculated to zeroth order with the aid of Eq. (8). Using Eq. (9), we find that to zeroth order, Eq. (8) is, of course, closed:

$$\partial_t G^a_b(t,t';\sigma) = \frac{\overline{\varkappa}}{2}\, T^{kn}_{lm}\,\sigma_k \frac{\partial}{\partial\sigma_l}\,\sigma_n \frac{\partial}{\partial\sigma_m}\, G^a_b(t,t';\sigma), \tag{12}$$

and Eq. (7) serves as initial condition for this equation at $t = t'$.

Let us now inverse-Fourier transform all statistical quantities involved, back to $\mathbf{B}$ dependence. The inverse Fourier transform of $Z(t;\sigma)$ is the one-point PDF $P(t;\mathbf{B})$. We will denote the inverse Fourier transform of $G^a_b(t,t';\sigma)$ by $G^a_b(t,t';\mathbf{B})$. Our system of equations becomes (suppressing $\mathbf{B}$ in the arguments):

$$\partial_t P(t) = \frac{\overline{\varkappa}}{2}\,\hat{L}P(t) - \overline{\tau}\, T^{ib}_{ja}\frac{\partial}{\partial B^i}B^j\Big[\partial_\tau G^a_b(t,t-\tau)\Big]_{\tau=0}, \tag{13}$$

$$\partial_t G^a_b(t,t') = \frac{\overline{\varkappa}}{2}\,\hat{L}G^a_b(t,t'), \tag{14}$$

$$G^a_b(t',t') = -\frac{1}{2}\frac{\partial}{\partial B^b}B^a P(t'), \tag{15}$$

where the operator $\hat{L}$, which will turn up repeatedly in this calculation, is

$$\hat{L} = T^{ib}_{ja}\frac{\partial}{B^i}B^j\frac{\partial}{B^b}B^a = \frac{d-1}{d+1}\left(B\frac{\partial}{\partial B}+d\right)B\frac{\partial}{\partial B}, \tag{16}$$

the latter expression being its isotropic form. Due to isotropy, we may further write:

$$P(t;\mathbf{B}) = P(t;B), \tag{17}$$

$$G^a_b(t,t';\mathbf{B}) = \delta^a_b H(t,t';B) + \frac{B^aB^b}{B^2}\,G(t,t';B), \tag{18}$$

where P, H, and G are scalar functions of B. Note that the term $\delta^a_b H$ in (18) will be annihilated by T^{ib}_{ja} in Eq. (13) due to incompressibility ($T^{ia}_{ja} = 0$). Therefore, Eq. (13)-(15) now reduce to

$$\partial_t P(t) = \frac{\overline{\varkappa}}{2}\,\hat{L}P(t) - \overline{\tau}\,\frac{d-1}{d+1}\left(B\frac{\partial}{\partial B}+d\right)\Big[\partial_\tau G(t,t-\tau)\Big]_{\tau=0}, \tag{19}$$

$$\partial_t G(t,t') = \frac{\overline{\varkappa}}{2}\left(\hat{L}-2d\right)G(t,t'), \tag{20}$$

$$G(t',t') = -\frac{1}{2}\,B\frac{\partial}{\partial B}P(t';B). \tag{21}$$

The extra term in Eq. (20) arises because operators $\hat{L}$ and B^aB^b/B^2 do not commute.

The solution of Eq. (20)-(21) can be formally written as $G(t,t') = \hat{\mathcal{G}}(t-t')G_0(t')$, where $G_0(t') = G(t',t')$ denotes the initial condition at $t = t'$, and $\hat{\mathcal{G}}$ is the Green integral operator with the properties (i) $\hat{\mathcal{G}}(0) = 1$ and (ii) $\partial_\tau\hat{\mathcal{G}}(\tau)G(t,t) = \partial_\tau G(t+\tau,t)$. We have therefore

$$\begin{aligned}\Big[\partial_\tau G(t,t-\tau)\Big]_{\tau=0} &= \\ &= \Big[\partial_\tau\hat{\mathcal{G}}(\tau)\Big]_{\tau=0}G_0(t) + \hat{\mathcal{G}}(0)\Big[\partial_\tau G_0(t-\tau)\Big]_{\tau=0} \\ &= \frac{\overline{\varkappa}}{2}\left(\hat{L}-2d\right)G_0(t) - \partial_t G_0(t).\end{aligned} \tag{22}$$

Substituting Eq. (21) into Eq. (22), then Eq. (22) into Eq. (19), and noting that all the relevant operators now commute, we obtain a closed equation for the PDF:

$$\left(1+\frac{\overline{\tau}}{2}\,\hat{L}\right)\partial_t P = \frac{\overline{\varkappa}}{2}\left\{\left(1+\frac{\overline{\tau}}{2}\,\hat{L}\right) - \overline{\tau}d\right\}\hat{L}P. \tag{23}$$

Acting on both sides of this equation left to right with the operator $\left(1+\frac{\overline{\tau}}{2}\,\hat{L}\right)^{-1} \simeq 1-\frac{\overline{\tau}}{2}\,\hat{L}$, we get

$$\partial_t P = (1-\overline{\tau}d)\,\frac{\overline{\varkappa}}{2}\,\hat{L}P. \tag{24}$$

This is the desired equation for the one-point PDF of the magnetic field (passive vector) up to first order in $\overline{\tau}$.

IV. THE DYNAMO GROWTH RATES

The evolution of all moments of B can be determined from Eq. (24). The n-th moment is defined by

$$M_n(t) = \langle B^n\rangle = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty \mathrm{d}B\, B^{d+n-1}P(t;B). \tag{25}$$

Multiplying both sides of Eq. (24) by B^{d+n-1} and integrating over B, we find that M_n satisfies:

$$\partial_t M_n = (1-\overline{\tau}d)\,(n+d)\,n\,\frac{d-1}{d+1}\,\frac{\overline{\varkappa}}{2}\,M_n. \tag{26}$$

We note that Eq. (3) implies

$$\overline{\varkappa} = \int_0^\infty \mathrm{d}\tau\,\varkappa(\tau) = \frac{C_d}{d}\int_0^\infty \mathrm{d}\tau\,\varkappa^{ii}_{jj}(\tau), \tag{27}$$

where $C_d = (d+1)/(d-1)(d+2)$, and and that, on the other hand, from Eq. (26), the zeroth-order growth rate of the magnetic energy M_2 is $\gamma_0 = C_d^{-1}\overline{\varkappa}$. Expressing $\overline{\tau}$ in terms of $\varkappa^{ii}_{jj}$ as we did $\overline{\varkappa}$ in Eq. (27), we see that γ_0 is reduced by

$$\overline{\tau}d = C_d \tau_{\rm corr}\gamma_0 d = C_d\left(\frac{\tau_{\rm corr}}{\tau_{\rm eddy}}\right)^2, \tag{28}$$

where, by definition, the correlation time is $\tau_{\rm corr} = \overline{\tau}/\overline{\varkappa}$, and we have introduced the "eddy-turnover" time:

$$\tau_{\rm eddy} = \left(\frac{1}{\tau_{\rm corr}}\int_0^\infty {\rm d}\tau\, \varkappa^{ii}_{jj}(\tau)\right)^{-1/2} \sim (\nabla v)^{-1}. \tag{29}$$

When the ratio $\mu = \tau_{\rm corr}/\tau_{\rm eddy}$ varies in the interval $0 < \mu \leq 1$, the profiles of the growth rate corresponding to zeroth- and first-order approximations are

$$\gamma_0(\mu) = \frac{\mu}{\tau_{\rm eddy} d} \quad \text{and} \quad \gamma(\mu) = \frac{\mu(1 - C_d\mu^2)}{\tau_{\rm eddy} d}. \tag{30}$$

If $\tau_{\rm corr} = \tau_{\rm eddy}$ ($\mu = 1$), we find that $\overline{\tau}d = C_d$. Incidentally, the fact that $\tau_{\rm eddy}\gamma_0 \sim 1/d$ when $\tau_{\rm corr} \sim \tau_{\rm eddy}$, also follows from a simple physical argument, which consists in observing that the eddy only streches the magnetic field line in one of the d available directions during one turnover time.

Thus, the correction $\overline{\tau}d$ is certainly less than unity for $d = 2$ and certainly less than a half for $d \geq 3$. In three dimensions, for example, the growth rate is reduced by 2/5, or 40%, which, as we have already mentioned in the Introduction, is in qualitative agreement with Chandran's numerical result [8].

In order compare our result with those of previous authors [10,8], let us calculate $\overline{\tau}d$ in terms of the spectral characteristics of the velocity field $\mathbf{v}$. If, by definition, $\langle v^2\rangle = \int {\rm d}(\log k)\, v_k^2$, the correlation/eddy-turnover time at scale k^{-1} is $\tau_k \sim (kv_k)^{-1}$. Hence,

$$\overline{\tau}d \propto \int_0^\infty {\rm d}\tau\, \tau\varkappa^{ii}_{jj}(\tau) \sim \int_{\tilde\Lambda}^{\Lambda} {\rm d}k\, kv_k^2\tau_k^2 \sim \log\frac{\Lambda}{\tilde\Lambda}, \tag{31}$$

where we have introduced the infrared cutoff $\tilde\Lambda$, which corresponds to the largest scale in the system, and the ultraviolet cutoff Λ, which can be interpreted as "the scale of the smallest eddy." On physical grounds, the infrared cutoff $\tilde\Lambda$ in this expression should be replaced by some larger effective value $\tilde\Lambda_{\rm eff} \sim \Lambda$. This is because the dominant contribution to the dynamo growth rate is from the small scales $\sim \Lambda^{-1}$, and the large scales are not expected to significantly affect the evolution of magnetic fluctuations [3]. Formally speaking, a perturbation theory with renormalized expansion parameter should be developed. Such pursuits fall beyond the scope of this work and will be undertaken elsewhere.

To summarize, we have obtained the correction to the kinematic dynamo growth rate γ_0 [2,3], which is first order in $\lambda_0 = \tau_{\rm corr}\gamma_0 d = (\tau_{\rm corr}/\tau_{\rm eddy})^2$ and well-behaved even as λ_0 approaches unity. In three dimensions, the latter case corresponds to a reduction of the zeroth-order growth rate by 40%. On the practical note, in the protogalactic setting, which is the principal application of the kinematic dynamo theory currently in focus, the magnetic energy growth rate is very large [18] and is not expected to be much reduced as far as the order of magnitude estimates go.

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