Cusp and Y-type Magnetic Structures and Velocity Fields at the Endpoint of the Reconnection Layer.

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We study the two-dimensional global scale magnetic field structure for a system of two merging cylindrical plasmas in a steady state. In the limit of very large magnetic Reynolds numbers the reconnection process is slow, and the plasma almost everywhere finds itself in magnetostatic equilibrium. We show that under certain conditions the classical Syrovatskii-type Y-point configuration, with surface current concentrated only in the reconnection layer, is not possible. Instead, a cusp configuration is formed, with finite surface current in the separatrix. The equilibrium condition, together with constraints on the volume per flux, enables us to determine the shape of the separatrix and the magnetic field in the vicinity of the cusp point. Our solution is characterized by a singular power law dependence of current density on the flux coordinate Ψ near the separatrix: $j(\Psi) \sim |\Psi|^{-1/2}$. This solution gives us the boundary conditions that are needed to find the flow in the reconnection and the separatrix regions.

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I. INTRODUCTION

It is generally accepted that magnetic reconnection is important in laboratory and space plasmas. In order to understand the mechanism of magnetic reconnection in the limit of large magnetic Reynolds numbers, it is necessary to understand the dynamic behavior of the plasma in thin layers. A closer examination of these layers reveals that the reconnection layer often ends in a cusp-like structure. It is the purpose of this paper to investigate when such a cusp structure appears and to determine the magnetic and the velocity fields in the cusp region. We restrict ourselves to the two dimensional (2D) quasi steady state resistive magnetohydrodynamics (MHD).

While most of our conclusions are rather general, we keep in mind the geometry of two merging cylindrical plasmas relevant to the Magnetic Reconnection eXperiment (MRX) [1]. The general configuration in the middle of the reconnection process is presented very schematically in Fig. 1. Regions I and II are ideal-MHD regions: regions I, which we call the *upstream* regions, represent unreconnected flux and region II (the *downstream* region) represents reconnected or common flux. The two regions I are separated by the very narrow *reconnection layer*, lying on the *midplane* $y = 0$. The poloidal magnetic field reverses across this layer, resulting in very high current density. Because of this, one must take into account resistive effects to describe plasma in this region. Regions I and II are separated by the *separatrix* region. In general, the poloidal magnetic field can have a discontinuity across the separatrix, so that the separatrix region also requires resistive description. Overall symmetry with respect both to the midplane and to the vertical y-axis is assumed.

In many astrophysical situations, the magnetic Reynolds number (or, rather, Lundquist number) R_m is very high [2]. In laboratory experiments this number, though still much greater than one, is much lower than in space (for example, $R_m \sim 10^3$ in the MRX experiment [1]). Therefore, in order to connect the physics of the experiments to that of the space plasmas, we discuss the problem in the limit of very large R_m.

In this limit, the reconnection velocity and the thickness of the resistive current layer are small compared with the Alfvén speed and the length of the layer, respectively. Thus, we have two different scales for both distances and velocities in our problem:

- *the global* (or macroscopic) scale is represented by the half-length L of the layer and by the Alfvén speed V_A. These are determined by the global solution in regions I and II, where ideal MHD is valid, and are, therefore, independent of the details of the narrow reconnection layer. In particular, they remain finite in the limit $R_m \to \infty$.

- *the local* (or microscopic) scale is represented by the thickness δ of the layer and by the reconnection velocity V_{rec}. These are determined by the solution of the local resistive MHD problem considering the reconnection layer with the boundary conditions given by the global ideal MHD solution. These quantities vanish in the limit $R_m \to \infty$, so we shall sometimes call them *infinitesimal*.

These two different scales allow us to break up the whole problem into two separate ones [3]: *the global* problem involving the two ideal regions I and II, and *the local* problem concerning the very thin resistive reconnection region and the separatrix region.

If the boundary conditions for the *global* problem change slowly compared with Alfvén time, the global ideal MHD problem becomes that of the magnetostatic equilibrium, and the whole reconnection process can be described by a one-parameter *sequence of magnetostatic equilibria* [3]. The plasma velocity is much slower than the Alfvén speed almost everywhere, with the exception of the infinitesimally thin reconnection layer, and the separatrix region[1].

At any given moment, once the global magnetostatic equilibrium is found, one can set up the appropriate boundary conditions for the local problem. These boundary conditions

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[1]Indeed, V_{perp} is small because of the Ohm's law: $V_{perp}B = E \ll B_0 V_A$. As for the parallel component of velocity, it is small because of the following argument. The maximum distance the plasma has to move along a line of force is L. The time it takes for the field line to move a distance $\Delta x \gg \delta$ away from the separatrix in the perpendicular direction is $\Delta t \simeq \Delta x / V_{rec} \simeq (\Delta x/\delta)(L/V_A)$. The parallel velocity during this time can be estimated as $v_{||} \simeq L/\Delta t \simeq V_A \delta/\Delta x$. Thus, for $\Delta x \gg \delta$ the parallel velocity is small compared to V_A.

are obtained from the global solution in the vicinity of the reconnection layer and of the separatrix. The term "in the vicinity of the reconnection layer" here means at distances from the layer much larger than its thickness δ, but still much shorter than the global size of the system, for example, the length $2L$ of the layer. Since the velocity on the global scale is zero, the only boundary condition we need to specify for the local problem is the magnetic field as a function of the distance along the reconnection layer and along the separatrix as seen on this global scale (and therefore the current density integrated across these surfaces). Note that both the reconnection layer and the separatrix are infinitesimally thin flux surfaces on the global scale. Therefore, what we are really interested in is the magnetic field structure around the system of current sheets consisting of a singular reconnection current layer of length $2L$ lying on the midplane, and the separatrix branching off somewhere near the endpoints of the reconnection layer. The global solution should also give us the shape of the separatrix. Despite the fact that the global magnetostatic equilibrium is different in different situations, we can draw some general conclusions about the magnetic field structure near an *endpoint* of the reconnection layer. This region is very important for understanding the transition between the flow inside the reconnection region and the flow in the separatrix. The analysis of the neighborhood of an endpoint is the main goal of this paper.

To determine the asymptotic behavior near the endpoint, one has to know the global distribution of currents in the system, including both the surface current density in the reconnection layer and the other global currents. The role of these other global currents is different in the following two cases:

Case 1. There are no additional current sheets attached to the reconnection layer. All the currents are either located at some global distance from the endpoints (like external coils), or distributed over large 2D regions (like the current in the plasma cylinders themselves). These currents do not change the nature of the solution near the endpoints, and thus we are lead to the *Syrovatskii*-like solution.

Case 2. More realistic situation with current sheets along each separatrix. In this case the behavior near the endpoints is changed dramatically, leading to the *cusp* solution, first suggested by Low and Wolfson [4], and then studied in more detail by Vekstein and Priest [5–8] in the context of the evolution of coronal arcades in response to a slow photospheric footpoint motions.

Current in the separatrix is generally caused by a slow (compared with the Alfvén time) discontinuous change in the the global magnetostatic equilibrium, which can be attributed to various reasons. In the situation considered by Low, Wolfson, Vekstein, and Priest, such change of equilibrium occurs even before reconnection starts, and is caused by the change in the global boundary conditions, namely, by the sheared motion of the footpoints on the sun surface [4–8]. In this paper we consider a different case, when the global boundary conditions are held static, and the current in the separatrix appears due to the natural gradual change in the global equilibrium caused by the reconnection process itself.

In Section II we describe briefly the Syrovatskii solution and show how it is affected by other global currents. We also show in this section how in the incompressible case one can determine the velocity field in the downstream region in the vicinity of an endpoint. In Section III we describe the cusp solution. These two sections are logically independent from each other. In section III-A we explain how the reconnective evolution of two merging plasmas leads to the current in the separatrix, and why this current leads to a cusp-like magnetic configuration near the endpoint. In section III-B we formally set up the problem for the magnetic field near the separatrix. In section III-C we repeat the elegant calculation due to Vekstein and Priest [6,7] concerning the downstream region in the vicinity of the cusp point. In section III-D we consider carefully the volume per flux in order to obtain the constraints necessary to uniquely determine the solution in the downstream region. In section III-E we consider the upstream region and show that the solution suggested by Vekstein and Priest for this region in Ref. [7] is not suitable for our geometry of two merging plasmas, and we find another solution which matches properly with the downstream solution. In section III-F we return to the downstream region and give analytical expressions for the magnetic field and for the plasma velocity near the endpoint. In section III-G we briefly discuss the incompressible case. Finally, in section III-H we discuss the relation between our work and that of Vekstein and Priest. We present our conclusions in section IV.

II. THE SYROVATSKII SOLUTION.

On the global scale, the reconnection current layer looks like a singular current sheet of zero thickness and of width $2L$. The current sheet is described in terms of the surface current density $\sigma(x)$, $|x| \leq L$, as a function of the distance x along the midplane $y = 0$.

Syrovatskii [9] gave a one-parameter family of solutions for the magnetic field surrounding a single current sheet in two dimensions. In terms of the surface current density $\sigma(x)$, these solutions can be written as

$$\sigma(x) = \sigma_0 \frac{1 - x^2/a^2}{\sqrt{1 - x^2/L^2}} \tag{1}$$

Unless $a = L$, the current density develops a singularity at the endpoints $x = \pm L$. Such solutions do not seem to be physically possible [10], and we shall not discuss them here. Instead, we concentrate on the special Syrovatskii solution with $a = L$, obtained by requiring the current density to vanish at both endpoints:

$$\sigma(x) = \sigma_0 \sqrt{1 - \frac{x^2}{L^2}} \tag{2}$$

This solution (as well as the general solutions (1)) is obtained as a solution of Laplace's equation on the plane with a single branch cut representing the current sheet. It is assumed that the normal to the midplane component of the magnetic field produced by the other global currents in the system (such as external coils or plasma currents) is a linear function along the entire current sheet: $B_{y,ext}(x) \sim x, |x| \leq L$. This assumption can be justified only if the current sheet's length $2L$ is much shorter than the size of the whole system, i.e. than the distances to these other global currents (which in this section we shall call the *external currents*). In a more general situation, when L is of the same order as these distances to the external currents, the normal magnetic field due to these external sources can be an arbitrary function of x, so that the function $\sigma(x)$ is different from (2). We here show how to obtain the solution for this general case.

Suppose that from the global MHD equilibrium we know the global distribution $j(\vec{r})$ of the external currents in the 2D region Ω surrounding the current sheet. Then we can calculate the normal component of the magnetic field due to this current distribution at any point on the midplane:

$$B_{y,ext}(x,0) = \frac{2}{c}\int_{\Omega} j(\vec{r'})\ \frac{x-x'}{|\vec{r}-\vec{r'}|^2}\ d^2\vec{r'} \tag{3}$$

The normal component of the magnetic field due to the current sheet itself is

$$B_{y,cs}(x,0) = \frac{2}{c}\int_{-L}^{+L} \sigma(x')\frac{dx'}{x-x'} \tag{4}$$

Since the magnetic field immediately above and below the current sheet must be tangential to the midplane, we can write

$$B_y(x,0) = B_{y,ext}(x,0) + B_{y,cs}(x,0) = 0, \qquad |x|<L, \tag{5}$$

which gives us the following linear integral equation for $\sigma(x)$ in terms of the known function $B_{y,ext}(x,0)$:

$$\int_{-L}^{+L} \sigma(x')\frac{dx'}{x-x'} = g(x) \equiv -\frac{c}{2}B_{y,ext}(x,0) \qquad |x|<L \tag{6}$$

This is a singular integral equation of the first kind with a Cauchy kernel. The exact solution of this equation corresponding to $\sigma(\pm L)=0$ is available [11] for any function $g(x)$ satisfying the orthogonality condition $\int\limits_{-1}^{1}\frac{g(x)}{\sqrt{1-x^2}}dx = 0$. The symmetry with respect to the vertical y-axis assures that $g(x)$ is an odd function, so this condition is satisfied. We get

$$\sigma(x) = \frac{1}{\pi^2}\sqrt{1-x^2/L^2}\int_{-L}^{+L}\frac{g(x')}{\sqrt{1-x'^2/L^2}}\frac{dx'}{x'-x} \tag{7}$$

In this section we consider the case when the external global currents are remote sources, so that the function $g(x)$ is a finite regular function. The case when this is not so will be considered in Section III.

The integral in (7) is then a slow function of x which is finite everywhere in the layer including the endpoints $x=\pm L$. Therefore, the current density in the current sheet can in general be described as

$$\sigma(x) = \sqrt{1-\frac{x^2}{L^2}}\ f(x) \tag{8}$$

where $f(x)$ is a smoothly varying function, the particular form of which depends on the particular problem, i.e. on the global distribution of plasma currents and on the location of external coils, etc.

This expression reveals an important universal feature of the current sheet, namely, the square-root behavior of $\sigma(x)$ near the endpoints $x = \pm L$.

Formula (7) can be illustrated by the following example of a possible current distribution in region Ω. Suppose that all the current is concentrated in two singular wires, located symmetrically above and below the midplane. Let a be the distance from each of the wires to the current sheet, and let each of the wires carry current I_0. Then, one can easily see that $g_a(x) = -2I_0 \frac{x}{a^2+x^2}$, and formula (7) gives:

$$\sigma_a(x) = -\frac{2I_0}{\pi} \frac{a}{a^2+x^2} \sqrt{\frac{L^2-x^2}{L^2+a^2}}, \tag{9}$$

which is in agreement with the result obtained by Green [12].

Now we can investigate the magnetic field structure in the vicinity of the endpoint. It is more convenient to work here in polar coordinates with the origin at the endpoint, and with angle ϕ measured from the midplane (see Fig. 2).

This endpoint is a Y-point, so the magnetic field must go to zero at the origin. This means that in the vicinity of this point, $r \ll L$, the zero-order magnetic field produced by the current sheet is canceled by the zero-order magnetic field produced by all the other currents in the system. The next order correction to the magnetic field due to these other currents should be linear in r, while the next order correction to the magnetic field due to the current sheet is of the order $\sqrt{r}$, and thus, this contribution dominates in this region. Then, to the leading order in r/L, the magnetic field can be written as

$$B_r = B_0 \sqrt{r/L} \sin \frac{3\phi}{2} \tag{10}$$

$$B_\phi = B_0 \sqrt{r/L} \cos \frac{3\phi}{2} \tag{11}$$

The separatrix makes a 60° angle with the midplane (see Fig. 2).

Now let us consider the slow velocity field in the downstream region in the vicinity of the endpoint. While the discussion of the magnetic field structure was independent of the plasma dynamics, in order to find the velocities we need to make some assumptions. For example, we assume that the fluid is incompressible. Also we use the fact that ideal MHD is valid in this region (which is outside the resistive current layer). Then we get the following two equations for the two unknown components of velocity:

$$\nabla \cdot \vec{v} = 0 \tag{12}$$

$$\vec{v} \times \vec{B} = -c\vec{E} = \text{const}, \tag{13}$$

with the boundary condition $v_\phi(\phi = 0) = 0$.

The solution of this system is:

$$v_\phi = \frac{cE_z}{3B_0\sqrt{r/L}} \, (\cos\frac{3\phi}{2})^{1/3} I(\cos\frac{3\phi}{2}) \tag{14}$$

$$v_r = -\frac{cE_z}{B_0\sqrt{r/L}\cos\frac{3\phi}{2}} \, \left[1 - \frac{1}{3}(\cos\frac{3\phi}{2})^{1/3} I(\cos\frac{3\phi}{2}) \, \sin\frac{3\phi}{2}\right], \tag{15}$$

where

$$I(\zeta) = \int_\zeta^1 \frac{d\zeta}{\zeta^{4/3}\sqrt{1-\zeta^2}} \tag{16}$$

The asymptotic behavior of $I(\zeta)$ as ζ goes to zero (i.e. $\phi \to 60^\circ$) is $I(\zeta) \simeq 3\zeta^{-1/3} - 2.2405$.

The bulk of the plasma flowing out of the reconnection region is diverted from the midplane and flows along the separatrix (here $E_z < 0$, and so $v_r > 0$). One can easily see that, as we approach the separatrix line, v_r goes to infinity. This singular behavior near the separatrix (where ideal MHD is expected to break down) must be asymptotically matched with the very fast (of order V_A) flow in the separatrix, which requires a local scale analysis taking into account dissipative effects.

The solution for the velocity in the upstream region can not be found as easily as in the downstream region, because, even though the equations are the same, the boundary conditions for the flow in the upstream region can only be set up on the vertical axis $x = 0$, far from the endpoint. Then the solution will depend on the magnetic field structure everywhere along the current layer.

III. THE CUSP SOLUTION.

In this section we consider the magnetic configuration with finite surface current in the separatrix, which leads to the cusp solution. We concentrate our discussion on the closed field line geometry corresponding to two merging cylindrical plasmas. Our approach is in a way an extension of Vekstein and Priest's treatment [6,7] of the solar corona problem in which the field lines are open. However, careful consideration of volume per flux in our analysis allows us to uniquely determine the magnetic field structure in the vicinity of the endpoint, and also to calculate the velocity field in the downstream region.

III.A The Need for a Cusp-like Configuration.

In the Syrovatskii-like solutions it is assumed that there are no current sheets attached to the reconnection layer, in particular, that there is no current in the separatrix. Such solutions do not involve the actual plasma dynamics, and are, therefore, of limited physical interest. More relevant is the situation when the separatrix itself is a current sheet with integrated current density of the same order as that in the reconnection layer. Then the function $g(x)$ introduced in the previous section is not regular near the endpoint, so that the square-root behavior of the current density in the reconnection layer breaks down. As a result, the magnetic field structure in the vicinity of the endpoint changes dramatically, with the Y-point becoming the cusp-point.

Current in the separatrix can emerge even before the reconnection process starts, if there is a discontinuous change of the global boundary conditions. This situation for a force-free compressible plasma was studied by Low and Wolfson [4] and by Vekstein, Priest and Amari [5] in the case of open-field-line geometry, where this change of the global boundary conditions is represented by sheared displacement of the foot-points on the sun surface. We consider a rather different physical situation, which nevertheless is characterized by very similar behavior. In our case of two merging plasmas, the magnetic field lines are closed, and the current in the separatrix arises very naturally due to the reconnective evolution of the system, even with static global boundary conditions.

First, we use the following very crude argument to show how the transfer of plasma from the unreconnected region I into the reconnected region II gives rise to the finite current in the separatrix. Consider a thin flux layer $\Delta\Psi$ before and after reconnection. In the case of compressible plasma (the incompressible case will be discussed in section III-G), we can use energy conservation. Assuming that there are no energy losses (radiation, etc.), the amount of magnetic energy destroyed in the reconnection process is finally converted into thermal energy, plus the work done by the flux layer under consideration during its expansion. Therefore, the pressure on the field line after reconnection P_{II} is increased by a finite amount $P_{II} - P_I$ over the pressure P_I on the field line before reconnection. The pressure balance across the separatrix then requires that the magnetic field strength have a finite jump, meaning finite current in the separatrix.

To see, what this finite current means for the global magnetic structure in the vicinity of the endpoint, we use the following argument, which is very similar to the arguments in Ref. [4,5] for the case of solar corona.

Consider two field lines, one before reconnection, the other after reconnection, but both very close (on the global scale) to the separatrix. Both magnetic surfaces are in equilibrium, so that the pressure is constant along each of them. The difference $P_{II} - P_I$ is finite, which corresponds to finite surface current in the separatrix. There is also a pressure balance across the separatrix (we can neglect global curvature of the magnetic field lines, because the two surfaces are very close to the separatrix):

$$\frac{B_I^2(l)}{8\pi} - \frac{B_{II}^2(l)}{8\pi} = P_{II} - P_I = \mathrm{const} > 0 \tag{17}$$

where l is the distance from the endpoint measured along the separatrix, I represents the magnetic field line before reconnection, and II after reconnection. Applying Eq. (17) at some cross-section very close to the endpoint, at $r \ll L$ (but still $r \gg \Delta x$, where Δx is the distance between the two magnetic field lines at this cross-section, so that Eq. (17) is still valid),we see that, even as $B_{II} \to 0$ as $r \to 0$, $B_I(r)$ must remain finite. At the endpoint $r = 0$, $B_I(r)$ reaches its minimum value $B_{I,min} = 8\pi(P_{II} - P_I)$, which is finite and is determined by the whole global solution. This is in contradiction with the classical Syrovatskii solution $B_I(r) = B_{II}(r) \sim r^{1/2} \to 0$. Moreover, in any solution with the separatrix making a finite angle with the midplane, both $B_I(r)$ and $B_{II}(r)$ go to zero as $r \to 0$. Thus, we have to conclude that the only plausible configuration of magnetic field near the endpoint is *cusp-like*, with the separatrix tangent to the midplane at the endpoint of the reconnection layer (see Fig. 3). A possible hint of the cusp can be seen in numerical simulations by Biskamp [13].

Note that the relative amount of the magnetic energy destroyed in the reconnection layer is roughly proportional to $L/(L+L_1)$, where L is the half-length of the reconnection layer, and L_1 is the length of the separatrix, from the endpoint of the reconnection layer to the top point A (see Fig. 1). In general, L and L_1 are expected to be of the same order of magnitude, so that the relative jump of magnetic field strength across the separatrix is finite. However, if $L \ll L_1$, we recover a separatrix without current, leading to the transition to the Syrovatskii solution: $P_{II} - P_I \ll B_I^2/8\pi \Rightarrow B_I(0) \ll B_I(l \sim L_1)$. The cusp region becomes very small, and the separatrix turns rather sharply.

III.B Formulation of the Problem

Now let us investigate the magnetic field structure near the cusp point. This is more difficult than in the Syrovatskii solution. For one thing, the exact shape of the separatrix is not known and must be determined self-consistently. Also, as we shall see, the contribution from the global distributed currents can not be neglected.

We choose to work in polar coordinates with the origin at the cusp point and with the midplane lying along the x-axis. We assume symmetry with respect to the midplane. On the separatrix $\Psi = 0$, and we choose the convention that $\Psi > 0$ in the upstream region I, and $\Psi < 0$ in the downstream region II (see Fig. 3).

The magnetic field is determined from the solution of the Poisson equation

$$\nabla^2 \Psi(r, \phi) = -\frac{4\pi}{c} j(\Psi) \tag{18}$$

separately in regions I and II. (Since plasma is in magnetostatic equilibrium, the current density is constant along the field lines: $j = j(\Psi)$.) The boundary conditions are given on the midplane and on the separatrix of some yet unknown shape $\phi = \phi_s(r)$. For region I the boundary conditions are:

$$\Psi_I(r, \pi) = 0, \qquad \text{and} \qquad \Psi_I(r, \phi_s(r)) = 0, \tag{19}$$

and for region II they are:

$$\Psi_{II}(r, \phi_s(r)) = 0, \qquad \text{and} \qquad \frac{\partial \Psi_{II}}{\partial \phi}|_{\phi=0} = 0 \tag{20}$$

The shape of the separatrix $\phi_s(r)$ is fixed by imposing the condition of pressure balance across the separatrix:

$$\frac{B_{sI}^2(l)}{8\pi} - \frac{B_{sII}^2(l)}{8\pi} = P_{II} - P_I = \text{const} > 0, \tag{21}$$

where $B_{sI}(l)$ and $B_{sII}(l)$ are the magnetic fields on the two sides of the separatrix as functions of the length measured along the separatrix.

While the complete solution of this problem requires the knowledge of the entire global magnetostatic equilibrium, it turns out that one can make some universal conclusions about the asymptotic behavior near the endpoint which are valid for a variety of global equilibria.

In the next two sections we consider only the downstream region II. As we show in the Appendix, the downstream current density as a function of flux must be singular near the separatrix $\Psi = 0$. As will be justified *a posteriori*, we may assume that this is a power law singularity:

$$j(\Psi) = -\frac{c}{4\pi} D\, (-\Psi)^{-n}, \quad D > 0, \quad n > 0 \tag{22}$$

(we include the "-" sign here because in the reconnection layer and in the separatrix the current density is negative, and we want to be able to match the global divergent $j(\Psi)$ to the local current density in the separatrix continuously). Since the magnetic field does not diverge at the separatrix, the inequality $0 < n < 1$ must be satisfied.

Thus, we obtain the following nonlinear Poisson equation:

$$\nabla^2 \Psi = D(-\Psi)^{-n}, \tag{23}$$

with the boundary conditions given by (20).

III.C The Vekstein and Priest Solution for the Downstream Region.

The basic approach to the analysis of the cusp region was set forth by Vekstein and Priest in connection with the solar corona problem [6,7]. Although the global geometry in the case of two merging cylindrical plasmas is rather different, a significant part of their analysis still applies. In this sub-section we present, in a slightly different notation, that part of Vekstein and Priest's analysis of the downstream region, which is relevant to our problem.

In the vicinity of the endpoint ($r \ll L$) the asymptotic expression for the shape of the separatrix can be written as

$$\phi_s(r) = K r^{\beta}, \tag{24}$$

where $\beta > 0, K > 0$.

For $r \ll L$ we expect that Eq. (23) has a scaling solution of the following form:

$$\Psi = -r^{\alpha} f(\xi) \tag{25}$$

where $\xi = \phi/\phi_s = \phi/Kr^{\beta}, \quad 0 < \xi < 1$.

One can write down the expression for magnetic fields in terms of $f(\xi)$:

$$B_r = \frac{\partial \Psi}{r \partial \phi} = -\ \frac{r^{\alpha-1-\beta}}{K}\ f'(\xi) \tag{26}$$

$$B_\phi = -\frac{\partial \Psi}{\partial r} = \alpha r^{\alpha-1} f(\xi) - \beta r^{\alpha-1} f'(\xi) \xi \tag{27}$$

The requirement that B_r, B_ϕ go to zero as $r \to 0$ gives

$$\alpha > 1 + \beta > 1 \tag{28}$$

Substituting (25) into Eq. (23) we get

$$\nabla^2\Psi = -r^{\alpha-2}\left[f''(\xi)\frac{1}{K^2\ r^{2\beta}} + f''(\xi)\xi^2\beta^2 + \alpha^2 f(\xi) - (2\alpha-\beta)\beta\xi f'(\xi)\right] =$$

$$= Dr^{-\alpha n} f^{-n}(\xi) \tag{29}$$

For very small r, such that $\phi_s(r) \ll 1$, the first term in the brackets is much greater than all the other terms; the function $f(\xi)$ and its derivative are finite (or small), and $f''(\xi)$ changes, as can be seen from Eq. (29), from a finite constant at $\xi = 0$ to infinity at $\xi = 1$. Thus, in the limit $r \ll L$, Eq. (23) is indeed satisfied by the scaling solution (25) with

$$n = \frac{2\beta + 2 - \alpha}{\alpha} \tag{30}$$

and with $f(\xi)$ satisfying the following second order ODE:

$$f''(\xi) = -\frac{\chi}{2}\, f^{-n}(\xi), \qquad \chi = 2DK^2 > 0 \tag{31}$$

The boundary conditions for $f(\xi)$ follow from Eq.(20):

$$f(1) = 0 \qquad f'(0) = 0 \tag{32}$$

Taking into account that $f'(\xi) \leq 0$ everywhere (so that $B_r \geq 0$, see Eq. (26)), and defining $f_0 = f(0)$, we obtain from Eq. (31):

$$f'(\xi) = -\sqrt{\frac{\chi}{1-n}}\quad \sqrt{f_0^{1-n} - f^{1-n}(\xi)} \tag{33}$$

The solution of (33) is given implicitly by integration:

$$\int_f^{f_0} \frac{df}{\sqrt{f_0^{1-n} - f^{1-n}}} = \sqrt{\frac{\chi}{1-n}}\quad \xi \tag{34}$$

The boundary condition $f(1) = 0$ can be used to determine the value of $f(\xi)$ on the midplane $\xi = 0$ in terms of χ and n:

$$f_0 = \left(\frac{\chi}{1-n}\right)^{\frac{1}{1+n}} \quad \left[\frac{1}{1-n}\ B\left(\frac{1}{1-n}, \frac{1}{2}\right)\right]^{-\frac{2}{1+n}} \tag{35}$$

From Eq. (33) and again using $f(1) = 0$ we get:

$$f'(1) = -\sqrt{\frac{\chi}{1-n}} \quad f_0^{\frac{1-n}{2}} \tag{36}$$

Magnetic field components on the separatrix are:

$$B_{rs}(r) = -\frac{1}{K}\, r^{\alpha-1-\beta}\ f'(1) > 0 \tag{37}$$

$$B_{\phi s}(r) = -\beta r^{\alpha-1}\ f'(1) > 0 \tag{38}$$

Since $\beta > 0$, $B_{rs} \gg B_{\phi s}$ for $r \to \infty$.

III.D The Volume per Flux.

Our main goal is to determine the three power exponents α, β, and n describing the solution in the vicinity of the endpoint. Eq. (30) gives us one relationship between the exponents. In this section we show that in the case of two merging plasmas (not considered by Vekstein and Priest), it is possible, under certain conditions, to derive a second relationship between the power exponents. Finally, in section III-E, the matching with the upstream solution will give us the third relationship (which will differ from that obtained by Vekstein and Priest), thus fixing the values of α, β, and n.

The total volume per flux on field line Ψ is:

$$V(\Psi) = \int_0^{L_1} \frac{dl}{B(l,\Psi)}, \tag{39}$$

where the integral is taken along a quarter of the field line in region II, namely, from the cusp region up to the y-axis near point A[2].

We are looking at the volume per flux on a field line close to the separatrix, corresponding to small Ψ. To zeroth order in Ψ, the volume per flux is equal to the value on the separatrix $V(0)$. Our goal is to estimate the corrections to $V(0)$ of the lower than linear order in Ψ.

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[2]The symmetry with respect to the x- and y-axes allows us to consider only the upper right quadrant of our system, so that the actual total volume per flux should be four times the value in Eq. (39).

For compressible plasma, we show that the leading correction to the volume per flux should be proportional to $(-\Psi)^{1-n}$. Vekstein and Priest discussed this correction for the case of open field lines in a force-free equilibrium produced by the footpoint shearing displacement [7,8]. In the case of the reconnective evolution of two merging cylindrical plasmas such a non-regular correction can be explained as follows.

First, let us estimate the difference between the magnetic field $B(r,0)$ on the separatrix $\Psi = 0$ and the magnetic field $B(r,\Psi)$ on a field line $\Psi < 0$ close to the separatrix. This is possible due to the fact that j is constant along magnetic field. At some finite distance r from the endpoint the magnetic field line is almost parallel to the separatrix, and we can write, denoting the distance from the separatrix by x':

$$B(r,\Psi) = B(r,0) + \frac{4\pi}{c}\int_0^{x'} j dx' = B(r,0) - D\int_0^{x'} (-\Psi)^{-n} dx'$$

If Ψ is small enough, the magnetic field does not change significantly, and we can estimate: $dx' = \frac{d(-\Psi)}{B(r,0)}$. Then,

$$B(r,\Psi) = B(r,0) - \frac{D}{B(r,0)}\frac{(-\Psi)^{1-n}}{1-n} \tag{40}$$

plus higher order terms, which we neglect here. Notice that we can regard the difference between $B(r,\Psi)$ and $B(r,0)$ as a small correction only if $D(-\Psi)^{1-n} \ll B^2(r,0)$. The smallest value of $B(r,0)$ is at distances $r \ll L$, where we can estimate $B(r,0) \simeq -f'(1)/K r^{\alpha-1-\beta}$. Thus, for the expression (40) to be valid we need

$$r \gg R_1(\Psi), \tag{41}$$

where $R_1(\Psi) = (-\Psi/f_0)^{1/\alpha}$ is the distance from the cusp point to the point where the field line Ψ crosses the midplane (see Fig. 3).

Eq. (40) and pressure balance across the magnetic field then give:

$$\Delta P(\Psi) = P(\Psi) - P(0) = \frac{D}{4\pi}\frac{(-\Psi)^{1-n}}{1-n} > 0 \tag{42}$$

Now, assume that in region II plasma density is constant along each field line. This can be justified by observing that due to the dissipative effects in the infinitesimally thin separatrix region, such as parallel thermal conductivity, the entropy density is equalized along each newly reconnected field line. After reconnection, the quantity $s \equiv P/\rho^{\gamma_0}$ (here we denote the adiabatic constant by γ_0 to distinguish it from the power exponent γ defined later in this section) remains to be constant along each field line, since plasma evolves adiabatically in region II: $s = s(\Psi)$. Since the pressure is also constant along magnetic field, then so is the plasma density: $\rho = \rho(\Psi)$.

Now, during the adiabatic plasma evolution after reconnection, the value of s on a given field line in region II does not change. This value is equal to its initial value on the same field line before reconnection, plus a change due to the entropy production which occurred inside the reconnection layer and the separatrix at the time when the given field line underwent reconnection. In general, both the initial value of s and its change are regular smooth functions of the field line label Ψ. Therefore, $s(\Psi)$ can be Taylor-expanded at any value of Ψ in region II. In particular, for a field line Ψ sufficiently close to the separatrix $\Psi = 0$ we can write:

$$s(\Psi) = s(0) + s'(0)\Psi$$

This equation means that there is no deviation of $s(\Psi)$ from $s(0)$ of lower than linear order in Ψ.

Thus, using Eq. (42) and the definition of $s(\Psi)$, we can estimate $\Delta\rho(\Psi) = \rho(\Psi) - \rho(0)$ as

$$\Delta\rho(\Psi) = \frac{D}{4\pi\gamma_0 P(0)} \frac{(-\Psi)^{1-n}}{1-n} \rho(0) \tag{43}$$

The mass $M(\Psi)$ on the given flux surface is also conserved, which means that, just as we did for $s(\Psi)$, we can write: $M(\Psi) = M(0) + O(\Psi)$. On the other hand, $M(\Psi) = \rho(\Psi)V(\Psi)$, so, to lowest order in $(-\Psi)$,

$$\Delta V(\Psi) = V(\Psi) - V(0) = -V(0) \cdot \frac{D}{4\pi\gamma_0 P(0)} \frac{(-\Psi)^{1-n}}{1-n} < 0 \tag{44}$$

We have thus shown that the leading correction to the volume per flux is negative and is of order $(-\Psi)^{1-n}$, as stated above. (The transition to the incompressible case can be obtained by taking the limit $\gamma_0 P(0) \to \infty$, in which case the $(-\Psi)^{1-n}$-correction vanishes, see section III-G.)

Since, according to (40), $B(r,\Psi) < B(r,0)$, the fact that $\Delta V(\Psi) < 0$ can be attributed only to the shortening of field lines with increased $(-\Psi)$. This means that the contribution from the vicinity of the cusp-point (where, as can be seen from Fig. 3, the field lines do shorten) must play an important role. In order to isolate the role of this contribution, we shall divide the whole volume per flux on a given field line into two parts (see Fig. 3):

$$V(\Psi) = V_<(R,\Psi) + V_>(R,\Psi), \tag{45}$$

where $V_<(R,\Psi)$ corresponds to $r < R$, and $V_>(R,\Psi)$ corresponds to $R < r < R_{max}$. Here R_{max} is the distance from the origin to the point A at the top, and R is chosen so that $R \ll L$, hence $\phi_s(R) \ll 1$, and Eq. (25) is still valid. On the other hand, we take R large enough: $R \gg R_1(\Psi)$. In other words, for given small $R \ll L$ we consider field lines that are sufficiently close to the separatrix.

Let us first estimate the correction due to $V_>(R,\Psi)$. For $r > R \gg R_1(\Psi)$, magnetic field lines are almost parallel to the separatrix, and we can use Eq. (40) for the magnetic field. This equation enables us to estimate:

$$V_>(R,\Psi) = V_>(R,0) + \frac{D(-\Psi)^{1-n}}{1-n} \int\limits_{l(R)}^{L_1} \frac{dl}{B^3(r,0)} \tag{46}$$

Note that the correction is always positive: $V_>(R,\Psi) > V_>(R,0)$. Since, according to (44), the total $\Delta V(\Psi)$ is of order $(-\Psi)^{1-n}$ and negative, this means that $\Delta V_<(R,\Psi)$ must also be of the same order and negative. The condition for this to be possible will give us an additional relationship between α and β.

Let us now consider $V_<(R,\Psi)$. Using Eq. (26) for B_r, we get

$$V_<(R,\Psi) = \int_0^{l(R)} \frac{dl}{B(r,\Psi)} = \int\limits_{R_1(\Psi)}^{R} \frac{dr}{B_r(r,\Psi)} = -K \int\limits_{R_1(\Psi)}^{R} \frac{r^{1+\beta-\alpha}dr}{f'(\xi)} \tag{47}$$

Going from the integration over r to the integration over f at fixed Ψ, and using expression

(33) for $f'(\xi)$, we get

$$V_<(R,\Psi) = \frac{K}{\alpha}\sqrt{\frac{1-n}{\chi}}\; f_0^{-1/\alpha}(-\Psi)^{\gamma-1} \quad I(\varepsilon) \tag{48}$$

$$I(\varepsilon) = \int_\varepsilon^1 \frac{x^{-\gamma}dx}{\sqrt{1-x^{1-n}}}, \tag{49}$$

where $x = f/f_0$, $\varepsilon = x(R) = -\Psi R^{-\alpha}/f_0 = \left(\frac{R_1(\Psi)}{R}\right)^\alpha \ll 1$, and

$$\gamma = \frac{2+\beta}{\alpha} \tag{50}$$

We now consider the asymptotic expansion of (49) in $\varepsilon \ll 1$ for different values of γ and n.

It is easy to see that, if $\gamma < 1$, $I(\varepsilon)$ converges as $\varepsilon \to 0$, and $V_<(R,\Psi) \to \infty$ as $\Psi \to 0$. Therefore, in order for the volume per flux on the separatrix $V_<(R,0)$ to be finite, we must require that

$$\gamma > 1 \qquad \Rightarrow \qquad \alpha < 2+\beta \tag{51}$$

Then, the integral (49) diverges as $\varepsilon \to 0$: $I(\varepsilon) \sim \frac{1}{\gamma-1}\varepsilon^{1-\gamma} \to +\infty$, and so, to the lowest order in $(-\Psi)$, we get:

$$V_<(R,\Psi) \simeq -\frac{K}{f'(1)} \;\; \frac{R^{2+\beta-\alpha}}{2+\beta-\alpha} = V_<(R,0) \tag{52}$$

(This result for $V_<(R,0)$ and the condition (51) can also be obtained immediately by using Eq. (37) for $B_{rs}(r)$.)

An expression similar to (48-49) and the result (52) for the volume per flux on the separatrix have been derived by Vekstein and Priest [7], and are valid in either case. However, the next order terms in the expansion of $I(\varepsilon)$, corresponding to the difference between $V_<(R,\Psi)$ and $V_<(R,0)$, must be analyzed differently. We can write

$$\Delta V_<(R,\Psi) = V_<(R,\Psi) - V_<(R,0) = \frac{K}{\alpha}\sqrt{\frac{1-n}{\chi}}\; f_0^{-1/\alpha}\,(-\Psi)^{\gamma-1}\,\Delta I(\varepsilon), \tag{53}$$

where

$$\Delta I(\varepsilon) = \int_\varepsilon^1 x^{-\gamma}\left[\frac{1}{\sqrt{1-x^{1-n}}} - 1\right]\,dx - \frac{1}{\gamma-1} \tag{54}$$

Notice that the first term is always positive and the second term is negative.

Suppose that the integral in (54) goes to infinity as $\varepsilon \to 0$, i.e. that the corresponding integral from 0 to 1 diverges at lower limit. Then, since the second term is finite, the whole $\Delta I(\varepsilon)$ is positive, and so $\Delta V_<(R,\Psi)$ is of the order $(-\Psi)^{1-n}$ and positive. But as we showed above, $\Delta V_>(R,\Psi) > 0$. Thus, the total change in volume per flux, $\Delta V(\Psi) = \Delta V_<(R,\Psi) + \Delta V_>(R,\Psi)$ will have to be positive, and it will be of order $(-\Psi)^{1-n}$. Actually, under the assumption that $\Delta I(\varepsilon)$ diverges, $\Delta V_>(R,\Psi)$ converges as $R \to 0$, so, to lowest order in $(-\Psi)$, $\Delta V(\Psi) = \Delta V_>(0,\Psi) > 0$. However, from Eq. (44) we know that the total $\Delta V(\Psi)$ must be negative. Thus, we obtain a contradiction, and we therefore have to conclude that, in order to get a negative correction of order $(-\Psi)^{1-n}$ to volume per flux, the coefficients γ and n must be such that the integral $\int_0^1 x^{-\gamma}[1/\sqrt{1-x^{1-n}} - 1]\, dx$ converges. This convergence condition can be written as

$$\gamma + n < 2 \qquad \Rightarrow \qquad \alpha > \frac{4}{3} + \beta \tag{55}$$

Assuming that condition (55) is satisfied, we can write:

$$\Delta I(\varepsilon) = G(\gamma, n) - \frac{1}{\gamma - 1} - \frac{\varepsilon^{2-\gamma-n}}{2(2-\gamma-n)}, \tag{56}$$

where

$$G(\gamma, n) = \int_0^1 x^{-\gamma} \left[\frac{1}{\sqrt{1-x^{1-n}}} - 1 \right] \quad dx \tag{57}$$

Going back to $\Delta V_>(R,\Psi)$ we notice that, for $r \sim R \ll L$ we can estimate $B(r,0) = B_r(r,0) \sim r^{\alpha-1-\beta}$, and $dl \sim dr$. Then $\int\limits_R \frac{dr}{B_r^3(r,0)} \sim \int\limits_R r^{-3(\alpha-1-\beta)} dr$, and taking into account condition (55), we see that the main contribution to this integral comes from the lower limit $r = R \ll L$. Isolating this contribution, we can write:

$$\Delta V_>(R,\Psi) = \frac{D(-\Psi)^{1-n}}{1-n} \int\limits_R^{R_{max}} \frac{dr}{B_r^3(r,0)} =$$

$$= \frac{D(-\Psi)^{1-n}}{1-n} \left(\frac{K}{f'(1)} \right)^3 \frac{R^{4-3(\alpha-\beta)}}{4-3(\alpha-\beta)} + \mathrm{const} \cdot (-\Psi)^{1-n} \tag{58}$$

Because of (55), the first term becomes much larger than the independent of R second term as $R \to 0$. Because $f'(1) < 0$, the whole expression is then positive.

One can easily see that the first term in (58) is exactly equal to the last term in (56) with the opposite sign, so these two terms cancel each other in the expression for total $\Delta V(R,\Psi)$. This means that the dependent on R part of the contribution to ΔV of order $(-\Psi)^{1-n}$ vanishes, as it should. Thus, we can write the expression for the deviation of the volume per flux from its value on the separatrix up to terms of lower than linear order in Ψ as:

$$\Delta V(\Psi) = V(\Psi) - V(0) = \Delta V_<(R,\Psi) + \Delta V_>(R,\Psi) =$$
$$= \frac{K}{\alpha}\sqrt{\frac{1-n}{\chi}}\, f_0^{-1/\alpha}\, (-\Psi)^{\gamma-1}\left[G(\gamma,n) - \frac{1}{\gamma-1}\right] + O((-\Psi)^{1-n}) \tag{59}$$

This expression includes terms of order $(-\Psi)^{\gamma-1}$, originating from $\Delta V_<(R,\Psi)$, and also terms of order $(-\Psi)^{1-n}$ (note that because of (55), $(-\Psi)^{1-n} \ll (-\Psi)^{\gamma-1}$). The terms proportional to $(-\Psi)^{1-n}$ are due to the second (finite) term in Eq. (58), and also due to the higher order terms in the expansions (22) for $j(\Psi)$, (24) for $\phi_s(r)$, (25) for $\Psi(r,\phi)$, which we have neglected so far. (For example, a contribution to $\Delta V_<(R,\Psi)$ of order $(-\Psi)^{1-n}$ can be obtained by adding a term of order $(-\Psi)^{\gamma+2n-2} \ll (-\Psi)^{-n}$ to $j(\Psi)$.) These higher order terms can not be determined without the knowledge of the whole global equilibrium. Their role here is to produce the correction to the volume per flux of order $(-\Psi)^{1-n}$, which would be in agreement with (44).

From Eq. (44), the lowest order term in $\Delta V(\Psi)$ should be of order $(-\Psi)^{1-n}$. Since $\gamma - 1 < 1 - n$, this can be true only if the dominant term in Eq. (59) vanishes, that is if the following relationship between γ and n is satisfied:

$$G(\gamma,n) = \int_0^1 x^{-\gamma}\left[\frac{1}{\sqrt{1-x^{1-n}}} - 1\right] dx = \frac{1}{\gamma-1} \tag{60}$$

Defining $\Gamma = \frac{\gamma-n}{1-n}$, $1 < \Gamma < 2$, equation (60) can be written as

$$\frac{1}{\Gamma-1} = \tilde{G}(\Gamma) = \int_0^1 t^{-\Gamma}\left[\frac{1}{\sqrt{1-t}} - 1\right] dt \tag{61}$$

The solution of this equation is $\Gamma = 3/2$. This solution is unique, because the LHS monotonically decreases with Γ, and the RHS monotonically increases with Γ. Recalling the definition of Γ, and that of γ, we can express all the power exponents in terms of β:

$$\gamma = \frac{3-n}{2} \quad \Rightarrow \quad \alpha = \beta + \frac{3}{2} \quad \Rightarrow \quad n = 1 - \frac{1}{\alpha} = 1 - \frac{1}{\beta + \frac{3}{2}} \tag{62}$$

We see that inequalities (28), (51), and (55) are satisfied, and that from the condition $\beta > 0$ it follows that $n > 1/3$.

The power exponent β can not be determined from the local analysis of the downstream region only. In order to determine it, we need to match the downstream solution to the upstream solution. One important observation, however, can be made at this point: from Eqs. (37) and (62) we see that the dependence of the radial magnetic field on the separatrix on r is independent of β and can be written as

$$B_{rs}(r) = -\frac{1}{K}\, f'(1)\, r^{1/2} \tag{63}$$

III.E Solution in the Upstream Region.

Now let us turn to the upstream region. We want to get the additional relationship between α and β from the condition of pressure balance across the separatrix.

Just as in region II, the magnetic field in region I is determined from the Poisson equation (18). This region is also in the state of magnetostatic equilibrium, so $j = j(\Psi)$. What type of behavior can this function possess? Can it be singular at $\Psi = 0$, or is it just a finite function? We show that the latter case must be true.

Indeed, in this region, consider a flux surface $\Psi > 0$, close to the separatrix $\Psi = 0$, and compare the volume per flux $V(\Psi)$ on this surface with that on the separatrix, $V(0)$. Our analysis here is analogous to that in the beginning of the previous section. In the upstream region, the magnetic field on the separatrix $B_s(l) = B(l,0)$ is finite everywhere along the separatrix. The same is true for $B(l,\Psi)$, the magnetic field on the given flux surface. The magnetic field line Ψ is essentially parallel to the separatrix, so we can estimate $B(l,\Psi)$ as

$B(l,\Psi) \simeq B(l,0) - \frac{4\pi}{c}\int_0^\Psi j(\Psi')\frac{d\Psi'}{B(l,\Psi')}$. Since Ψ is small, we can approximate $B(l,\Psi')$ in the second term by $B(l,0)$, which gives $B(l,\Psi) \simeq B(l,0) - \frac{4\pi}{cB(l,0)}\int_0^\Psi j(\Psi')d\Psi'$.

Now if we assume that $j(\Psi)$ has a power-law singularity at $\Psi = 0$: $j(\Psi) = -(cD'/4\pi)\Psi^{-m}, 0 < m < 1$, then,

$$B(l,\Psi) = B(l,0) + \frac{D'}{B(l,0)}\frac{\Psi^{1-m}}{1-m} \tag{64}$$

Then, just as we did in section III-D, we can use pressure balance to find $\Delta P(\Psi)$, the adiabatic law to find $\Delta\rho(\Psi)$, and the mass conservation to finally write

$$\Delta V(\Psi) = V(\Psi) - V(0) = V(0)\frac{D'}{4\pi\gamma_0 P(0)}\frac{\Psi^{1-m}}{1-m} > 0 \tag{65}$$

On the other hand, we can estimate $V(\Psi)$ up to terms of order $\Psi^{1-m} \gg \Psi$ directly using Eq. (64):

$$V(\Psi) = V(0) - \frac{D'}{1-m}\Psi^{1-m}\int_{-L}^{L_1}\frac{dl}{B^3(l,0)} \tag{66}$$

The integrand is a regular positive finite function, and therefore, the integral $\int_{-L}^{L_1}\frac{dl}{B^3(l,0)}$ is just a finite positive constant, independent of Ψ. This means that we get a negative correction to $V(\Psi)$ of order $\Psi^{1-m} \gg \Psi$, in contradiction with Eq. (65). Therefore, we conclude that $j(\Psi)$ can not be singular at $\Psi = 0$ in the upstream region I.

Thus, to the lowest order in Ψ we have $j(\Psi) \simeq j(0) = \text{const}$. The correction to the volume per flux due to this current density will be linear in Ψ.

In this case, the magnetic field structure in region I in the vicinity of the endpoint is determined by the Poisson equation

$$\nabla^2\Psi = C = \text{const}, \tag{67}$$

where $C = -\frac{4\pi}{c}j(0)$, with the boundary conditions

$$\Psi = 0 \quad \text{at} \quad \phi = \pi \qquad \text{and at} \qquad \phi = \phi_s = Kr^\beta \tag{68}$$

The source term on the RHS of this equation turns out to be unimportant, and we can consider Laplace's equation instead.

First, let us discuss the analysis of the upstream solution given by Vekstein and Priest [7]. As we shall see, this will lead to a contradiction in our case. We write the solution of Laplace's equation as

$$\Psi = B_0 r \sin\phi + B_1 r^p \sin[p(\phi - \pi)], \qquad p > 1, B_0 > 0 \tag{69}$$

The boundary condition $\Psi(\phi = \pi) = 0$ is satisfied automatically. In order to satisfy the boundary condition on the separatrix $\phi = K r^\beta$ to the lowest order in r we demand

$$p = \beta + 1 \tag{70}$$

and

$$B_1 \sin\beta\pi = -B_0 K < 0 \tag{71}$$

The radial component of the magnetic field is

$$B_r = \frac{1}{r}\frac{\partial\Psi}{\partial\phi} \simeq B_0 \cos\phi + p B_1 r^\beta \cos[p(\phi - \pi)] \tag{72}$$

On the separatrix we have: $B_{sI}^2 \simeq B_{rsI}^2 \simeq B_0^2 - 2(1+\beta) B_1 B_0 r^\beta \cos\beta\pi$. On the other hand, in region II, $B_{rsII}^2 \simeq \frac{1}{K^2}[f'(1)]^2 r^{2(\alpha-\beta-1)}$. Then the pressure balance, $B_{sI}^2 - B_{sII}^2 = B_0^2 = \text{const}$, gives

$$\alpha = 1 + \frac{3}{2}\beta \tag{73}$$

and

$$B_1 \cos\beta\pi = -\frac{1}{2(1+\beta)B_0} \quad \left(\frac{f'(1)}{K}\right)^2 < 0 \tag{74}$$

The two inequalities (71) and (74) can only be satisfied if $\tan\beta\pi > 0$, i.e. if

$$0 < \beta < \frac{1}{2}, \quad \text{or} \quad 1 < \beta < \frac{3}{2}, \quad \text{etc.} \tag{75}$$

Now one can easily see that this solution is incompatible with our solution for region II. Indeed, in previous section we derived from the volume per flux arguments the relationship between α and β: $\alpha = \beta + \frac{3}{2}$. Combined with equation (73) this equation uniquely determines $\beta = 1$ and $\alpha = 5/2$, which is in contradiction with inequalities (75).

Therefore, solution (69) presented by Vekstein and Priest is not suitable for our case of reconnection of two cylindrical plasmas.

There are two ways out of this situation. The first one applies for a range in $\beta > 1$, and the second one is a special case with $\beta = 1/2$.

First, let us consider $\beta > 1$. In the Vekstein and Priest solution, the correction term $B_1 r^p \sin[p(\phi - \pi)]$ in (69) has to serve two functions: it must make $\Psi = 0$ at $\phi = \phi_s$ to lowest order in r, and it also must provide the variation of the magnetic field on the separatrix needed for the pressure balance. Now, if $\beta > 1,\ p > 2$, we can add the term $B_2 r^2 \sin 2\phi$, which will give a bigger contribution to the magnetic field on the separatrix, and at the same time, it's contribution to $\Psi(\phi_s)$ will be negligible. Thus, we can have a family of solutions with $\beta > 1$:

$$\Psi = B_0\ r \sin\phi + B_1\ r^p \sin[p(\phi - \pi)] + B_2\ r^2 \sin 2\phi \tag{76}$$

and $p = 1 + \beta > 2$, $B_1 \sin\beta\pi = -B_0 K$, and $B_0 B_2 = \left(\frac{f'(1)}{2K}\right)^2 > 0$.

The magnetic field up to the first order in r is given by

$$B^2 = B_0^2 + 4 B_0 B_2\ r \cos\phi \tag{77}$$

Although at first glance this solution appears satisfactory, a more careful look at this expression reveals a potential problem with this solution: the absolute value of the magnetic field along the separatrix increases monotonically as we pass the endpoint from left to the right, i.e. the magnetic field does not have minimum at the endpoint. It seems to be difficult to incorporate such magnetic field into the usual picture of reconnection, where the outside magnetic field is strongest at the middle of the reconnection layer. However, this solution may explain the origin of the O-point configuration which is observed in the MRX experiment in the co-helicity merging [1].

The second case, which we believe is more physical and seems to satisfy all the physical conditions we can impose, is the particular case, $\beta = 1/2$.

Then, the solution satisfying $\Psi(\phi_s) = 0$ can be written as

$$\Psi = B_0\ r \sin\phi - B_0 K\ r^{3/2} \cos\frac{3}{2}\phi + B_2\ r^2 \sin 2\phi \tag{78}$$

The last term gives negligible contribution to $\Psi(\phi_s)$, however, its contribution to the magnetic field on the separatrix is of the same order as that of the second term:

$$B_s^2(r) = B_0^2 \left[1 + \left(\frac{15}{4}K^2 + 4\frac{B_2}{B_0}\right)\ r\right] > B_0^2 \tag{79}$$

(compare with the equilibrium solution by Morozov and Solov'ev for a vacuum magnetic field outside a cusp containing plasma without magnetic field [14].)

The pressure balance across the separatrix gives us the expression for $[f'(1)]^2$ in terms of B_0, B_2 and K:

$$[f'(1)]^2 = \frac{15}{4}K^4 B_0^2 + 4K^2 B_0 B_2 \tag{80}$$

The magnetic field along the reconnection layer, $\phi = \pi$ is also increasing with r:

$$B_r^2(\phi = \pi, r) = B_0^2(1 + 3Kr^{1/2}) > B_0 \tag{81}$$

Thus the cusp-point (0,0) is really the point of minimum of the upstream magnetic field. It is interesting that a change of the entire solution induced by changing the constant K is effectively the same as adding the Syrovatskii solution Eq. (8), which is also proportional to $\sqrt{r}$ near the endpoint, to the solution in the upstream region.

Now, even though we managed to determine the power exponents, we are still left with uncertainty regarding the value of B_2. We think that B_2 is determined by the entire global equilibrium. The only condition we can impose on B_2 is that the RHS of (80) must be positive.

III.F The Magnetic Field Structure and the Velocity Field in the Downstream Region for $\beta = 1/2$.

For the special case $\beta = 1/2$ we can obtain exact analytical expressions for the magnetic flux function in region II. Using Eqs. (62) and (30), we get:

$$\beta = \frac{1}{2} \qquad \alpha = 2 \qquad n = \frac{1}{2} \tag{82}$$

Then, using (35) and (36), we find $f_0 = \left(\frac{3}{8}\right)^{4/3}(2\chi)^{2/3}$, and $f'(1) = -\left(\frac{3}{8}\right)^{1/3}(2\chi)^{2/3}$.

The integral on the LHS of Eq. (34) can be calculated exactly resulting in a cubic equation for $u = -f'(\xi)/(\sqrt{2\chi}\ f_0^{1/4}) = \sqrt{1-\sqrt{f/f_0}}$: $u^3 - 3u + 2\xi = 0$, where $\xi = \phi/K\sqrt{r}$. The solution of this equation is $u = 2\sin\left(\frac{1}{3}\arcsin\xi\right)$, so that

$$f(\xi) = f_0\left[1 - 4\sin^2\left(\frac{1}{3}\arcsin\xi\right)\right]^2, \tag{83}$$

and the components of the magnetic field are given by Eqs. (26-27):

$$B_r = 4\sqrt{Dr}f_0^{1/4}\sin\left(\frac{1}{3}\arcsin\xi\right)$$

$$B_\phi = 2rf(\xi) + \sqrt{2\chi}f_0^{1/4}r\xi\sin\left(\frac{1}{3}\arcsin\xi\right)$$

Now we can find the plasma velocity in the vicinity of the cusp point. In order to find the velocity field for a steady state ideal MHD flow in a given magnetic field configuration, we make use of mass conservation and also of the fact that the density is constant along each field line. Now, consider a certain tiny fluid element. At any given moment, the position of this fluid element can be described by two variables (r, Ψ). The motion of this element in the downstream region can be specified by the integral of motion, the mass per flux $\rho(\Psi)V_<(r,\Psi)$. Using expression (48) for $V_<(R,\Psi)$ and Eq. (43) for $\rho(\Psi)$, we have

$$\rho(\Psi(t))V_<(r(t),\Psi(t)) = \rho(0)\frac{2K}{\sqrt{2\chi f_0}}\left(1 + \frac{D}{4\pi\gamma_0 P(0)}\frac{(-\Psi)^{1-n}}{1-n}\right)\sqrt{f_0^{1/2}\ r - (-\Psi)^{1/2}} = \text{const} \tag{84}$$

The correction in the parentheses due to the variation of the density with Ψ can be neglected for small $(-\Psi)$. For the steady state situation, the motion of a given field line is described by a simple relationship: $\Psi(t) = -cEt$, where E is the magnitude of the constant, uniform electric field. Then, the radial position of the given fluid element as a function of time is

$$r(t) = \frac{1}{f_0}\left(\sqrt{cEt} + C\right), \tag{85}$$

where $C = \left[(2\chi)^{2/3}\left(\frac{3}{8}\right)^{3/8}V_</2K\right]^2 = \text{const}$. The radial velocity is obtained by simple differentiation:

$$v_r = \dot{r} = \frac{cE}{2\sqrt{f_0}}(-\Psi)^{-1/2} = \frac{cE}{2rf_0}\left[1 - 4\sin^2\left(\frac{1}{3}\arcsin\frac{\phi}{Kr^{1/2}}\right)\right]^{-1} \tag{86}$$

An interesting feature of this formula is that the radial velocity is constant along the field line, and decreases away from the separatrix as $(-\Psi)^{-1/2}$, in the same way as the current density does.

III.G The Incompressible Case.

In this section we show that our results are also valid for incompressible plasma, although some of the arguments differ from those used in the compressible case.

For ideal plasma the incompressibility condition can be expressed in terms of the global condition of the volume per flux conservation:

$$V(\Psi) = \int \frac{dl}{B} = \mathrm{const}(t) \tag{87}$$

In each of regions I and II the volume per flux is conserved, because plasma is essentially ideal. But we can make even stronger statement that, despite the tiny slippage of plasma across magnetic field which occurs at the instant of reconnection of a given flux surface, the volume per flux on this surface virtually does not change, i.e. $V_I(\Psi) \simeq V_{II}(\Psi)$. This is because the amount of plasma that is transferred from the given flux surface to the next surface is the same (in leading order) as the amount of plasma that is transferred from the previous surface to the given surface. Thus, at any moment of time the function $V(\Psi)$ is the same as it was initially. In general, we expect $V(\Psi)$ to be a regular smooth function, which can be Taylor-expanded at any value of Ψ. In particular, if we again set $\Psi = 0$ on the separatrix flux surface (undergoing reconnection at this particular moment), then for sufficiently close flux surfaces (on both sides of the separatrix) we can write:

$$V(\Psi) = V(0) + V'(0)\Psi \tag{88}$$

This equation means that any deviation of $V(\Psi)$ from $V(0)$ of the lower than linear order in Ψ is not possible.

First we show that one has to have finite surface current in the separatrix in the incompressible case. Consider a flux layer $\Delta\Psi$ before and after reconnection. Before reconnection, the length of the whole flux layer is $L + L_1$, after reconnection it is just L_1. In the case of incompressible plasma, the volume of the flux layer is conserved; thus, to compensate for the shortening of the field lines, the thickness of the layer is increased by a factor $1 + L/L_1$:

$$\Delta x' = \Delta x \left(1 + \frac{L}{L_1}\right)$$

where Δx is the average thickness before reconnection, and $\Delta x'$ after reconnection.

If B_I is some average magnetic field before reconnection, and B_{II} - after reconnection, then we have:

$$B_{II} = B_I \frac{L_1}{L + L_1} \tag{89}$$

The difference gives us the non-zero surface current density in the separatrix: $\Delta B = B_I \frac{L}{L+L_1}$. Then we can apply the argument described at the end of section III-A to see that in this case we again get a cusp at the end of the reconnection layer.

All the analysis in sections III-B and III-C is independent of the compressibility assumption and applies also for the incompressible case.

A consideration of the volume per flux similar to that in section III-D gives us again relationship (62) between α and β. Indeed, just as in the compressible case, one can easily see from Eqs. (46) and (53-54) that, if $\gamma + n > 2$, the leading corrections to $V_<(R, \Psi)$ and to $V_>(R, \Psi)$ will be of order $(-\Psi)^{1-n}$, and they will be positive. Thus, in this case, the condition (88) that the volume per flux stay constant up to lower than linear orders in Ψ can not be satisfied. We therefore have to conclude that condition (55) has to be satisfied even for the incompressible case. Also we must require that, in order to preserve constant volume per flux, the contribution to $\Delta V(\Psi)$ of order $(-\Psi)^{\gamma-1}$ must vanish, which gives us relationship (62). As for the terms of order $(-\Psi)^{1-n}$, the R-dependent parts of $\Delta V_>(R, \Psi)$ (Eq. (58)) and $\Delta V_<(R, \Psi)$ (Eqs. (53),(56)) cancel each other, just as in the compressible case, and the second (finite) term in expression (58) for $\Delta V_>(R, \Psi)$ must be cancelled by the previously neglected higher order corrections to $\Delta V_<(R, \Psi)$. Thus, $\alpha = 3/2 + \beta$ for the incompressible plasma as well.

The analysis of the upstream region in the incompressible case is also essentially the same as for compressible plasma. The only difference is that, in order to show that $j(\Psi)$ can not have a power-law singularity near the separatrix, one merely has to use equation (66) for the volume per flux. This equation gives a correction of order $(-\Psi)^{1-m}$, which is in contradiction with condition (88). In all other aspects, the solution in the incompressible case is the same as in the compressible case, leading to the same power exponents (82), the same function $f(\xi)$ given by Eq. (83), and the same velocity Eq. (86).

III.H Comparison with the Results of Vekstein and Priest.

The analysis in sections III-A to III-E is very similar to the pioneering analysis of Vekstein and Priest [5–8], even though they considered a different problem with different conditions to determine $V(\Psi)$. It turns out to be possible to evaluate the analogous quantity $V_>(R,\Psi)$ in their problem using the techniques of section III-D. This analysis shows again that $\Delta V_>(R,\Psi)$ is of order $(\Psi)^{1-n}$ and positive. This sign is opposite to that required for the total $\Delta V(\Psi)$ (see Eqs. (17-18), (29) of Ref. 8). Therefore, there must be a larger negative contribution to $\Delta V(\Psi)$ from the neighborhood of the cusp point. This in turn means that inequality (55), which is opposite to the inequality given after Eq. (30) in Ref. 8, must be satisfied. Then, just as in the case considered in the present paper, the dominant order term in $\Delta V_<(R,\Psi)$ will be proportional to $(-\Psi)^{\gamma-1}$, and the condition that this term vanishes again gives $\alpha = \beta + 3/2$. Higher order corrections to $\Delta V_<$ together with $\Delta V_>$ add up to the correct sign and order of ΔV.

Thus, the downstream analysis of the problem considered by Vekstein and Priest should be identical with our analysis, and similarly, their upstream analysis should include the additional r^2-term. In other words, even though the problem itself is different, the solution should be identical with ours.

IV. CONCLUSIONS

In this paper we have studied the magnetic and velocity fields in the neighborhood of the endpoint of the reconnection layer. In particular we considered the 2-D MHD steady state problem in the geometry of two merging cylindrical plasmas relevant to the Magnetic Reconnection eXperiment (MRX).

The magnetic field structure near the endpoint strongly depends on the presence of other current sheets attached to the reconnection current layer. In the case when there are no such attached currents, we present an explicit expression for the surface current in the layer, which forms a generalization of the well-known Syrovatskii solution. In general, the configuration is characterized by a 60^o Y-point with the characteristic square-root dependence of the magnetic field on the distance from the endpoint. In this case, universal expressions for magnetic and velocity fields are obtained.

However, the condition of magnetostatic equilibrium taken together with energy and mass conservation (or volume per flux conservation for incompressible case) unavoidably leads to finite surface current along the separatrix. This surface current then leads to a cusp-like configuration near the endpoint.

To properly investigate the dynamics in the reconnection and separatrix layers, it is necessary to determine the flow through this cusp region. For this it is necessary to find the structure of the magnetic field in the neighborhood of the cusp. Surprisingly, because of the global volume per flux constraints, arising from the constants of motion, such as mass, entropy, and flux, we find that significant contribution to $\Delta V(\Psi)$ must come from the cusp region itself. Together with the matching conditions with the upstream region, this constraint turns out to be strong enough to determine the complete behavior of the magnetic field near the cusp, up to a couple of constants, independent of the global behavior of the equilibrium solution away from the cusp. This solution is given explicitly in section III-F.

We find that an extension of the analysis pioneered by Vekstein and Priest [6–8] enables us to carry out this program, and to arrive to an almost complete determination of these fields.

APPENDIX: Singular behavior of $j(\Psi)$ near the Separatrix in region II.

The first question we need to ask is whether the source term in Eq. (18) is important. If we neglect this term, thereby assuming that the current is concentrated only in the reconnection layer and in the separatrix, and that the current density in regions I and II is exactly zero, we will easily see that the solution in region II is in fact exponentially small ($\Psi \sim e^{-1/r}$), and the volume per flux on the separatrix diverges, which contradicts the constraints given in section III-D. This means that the source term is in fact important in the downstream region.

Now we need to find out, whether $j(\Psi)$ can be a smooth function of Ψ as $\Psi \to 0$. If this were so, then close enough to the separatrix we could replace the source term in Eq. (18) by its value at $\Psi = 0$:

$$\nabla^2 \Psi = C = \text{const} \qquad (A1)$$

Any solution of this equation can be written as the sum of a particular solution $\Psi_1 = (Cr^2/2)\sin^2\phi$ of the Poisson Equation, and a solution Ψ_0 of the corresponding Laplace's equation. Taking into account that Ψ must be even in ϕ, we can write

$$\Psi_0 = \sum_{i=1}^{\infty} A_i r^{p_i} \cos(p_i \phi) \qquad (A2)$$

The boundary condition $\Psi = \Psi_0 + \Psi_1 = 0$ at $\phi = \phi_s$ gives:

$$A_1 = -\frac{CK^2}{2}, \qquad \text{and} \qquad p_1 = 2(1+\beta) > 2 \qquad (A3)$$

Then the magnetic field B_s on the separatrix to the lowest order in r is

$$B_s(r) = B_{rs}(r) = CKr^{1+\beta} \qquad (A4)$$

(the contribution from Ψ_0 is negligible), and we immediately see that the volume per flux on the separatrix diverges:

$$V(0) = \int \frac{dr}{B_{rs}(r)} \sim \int \frac{dr}{r^{1+\beta}} = \infty \qquad (A5)$$

Thus, it is impossible to construct a magnetostatic equilibrium in region II with $j(\Psi)$ staying finite as $\Psi \to 0$ and with convergent volume per flux on the separatrix. We then have to conclude that $j(\Psi)$ must have a singularity at $\Psi = 0$.

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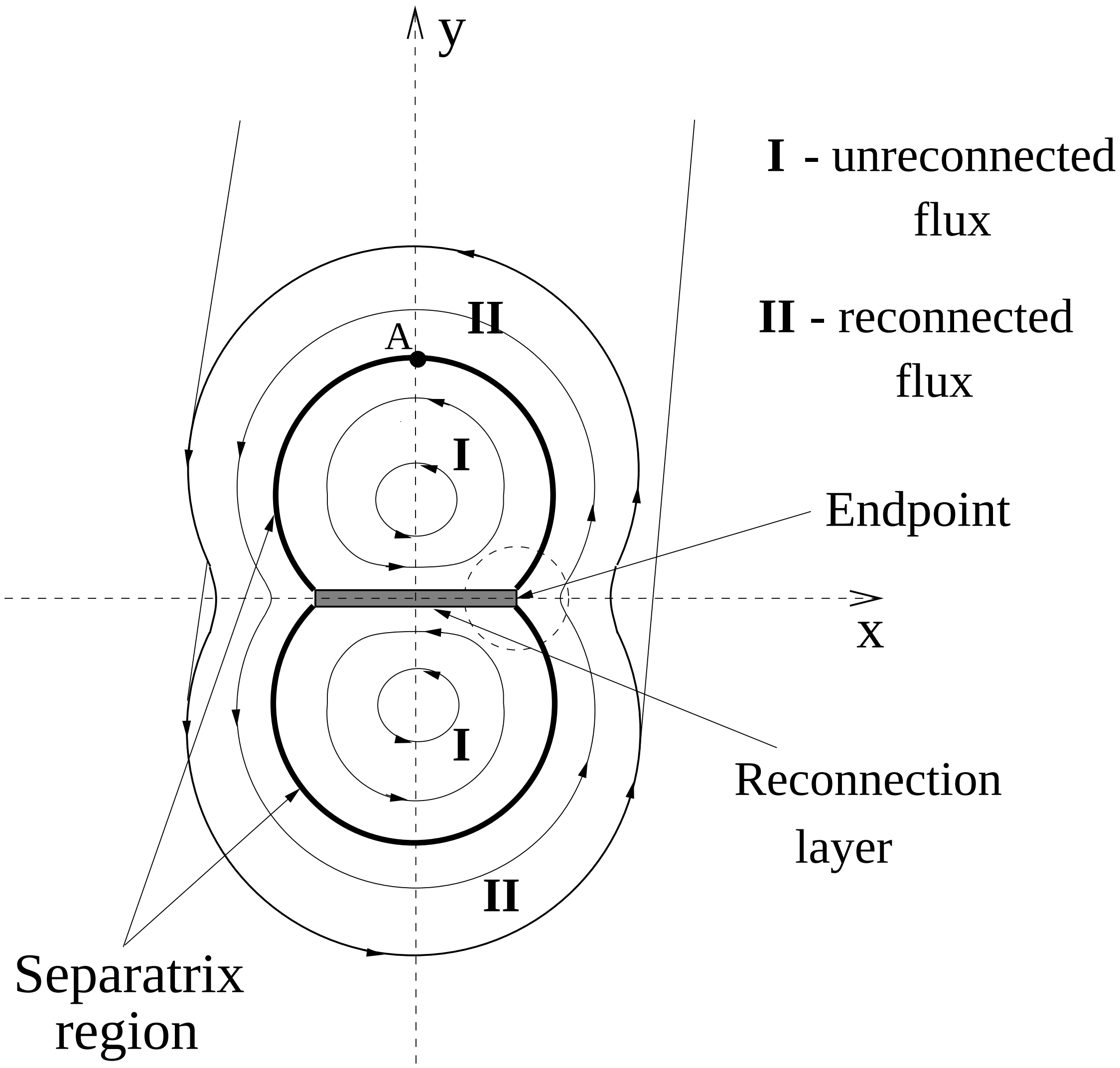

FIG. 1. The global two-dimensional geometry of the problem. Two reconnecting plasma cylinders are shown when they are partially reconnected. Regions I are the unreconnected cylindrical regions, and region II is the region of reconnected flux. The region between the two cylinders is the reconnection layer, while the thin surface between regions I and II is the separatrix layer. The region inside the dashed circle around one of the two endpoints of the reconnection layer is the region of primary interest of this paper. Regions I and II are in near magnetostatic equilibrium, while the flows in the thin layers are fast.

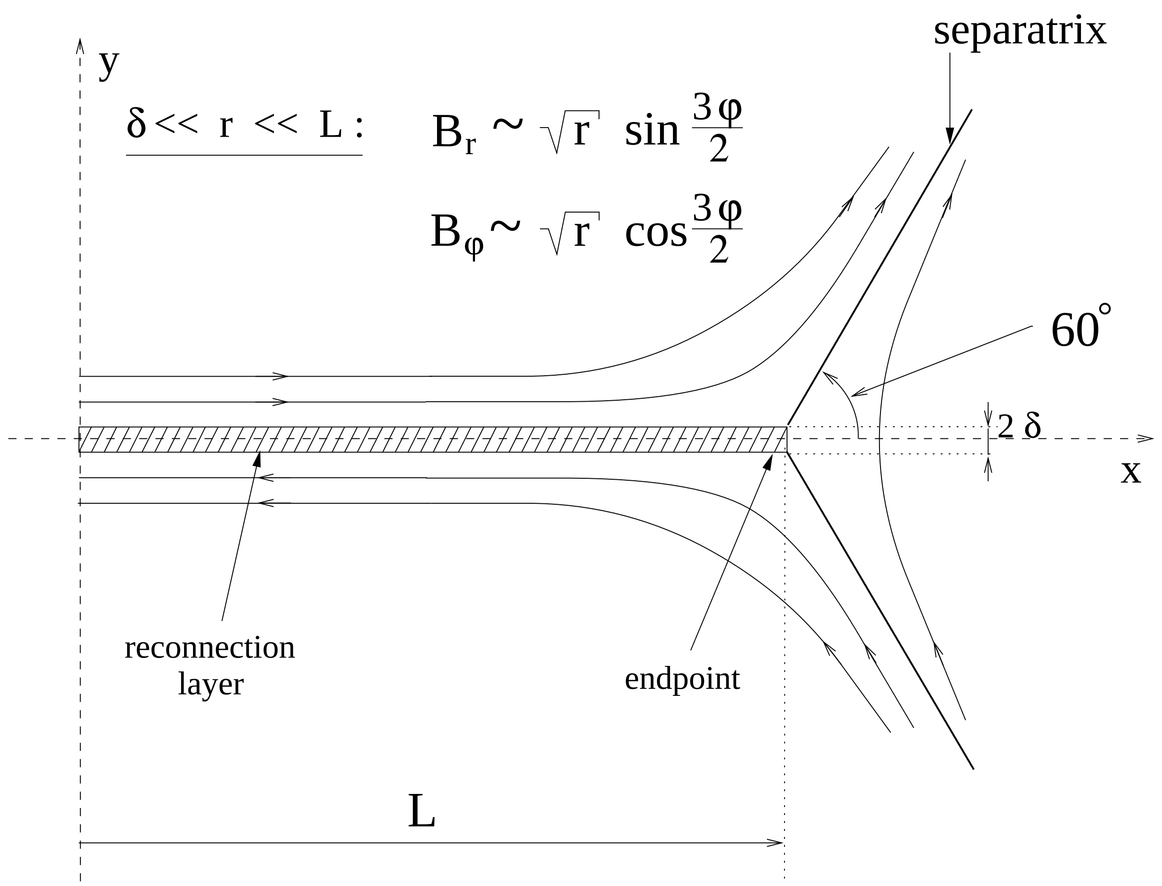

FIG. 2. The region about the endpoint of the reconnection layer for the Syrovatskii case, where there is no surface current in the separatrix. Polar coordinates (r, ϕ) with the origin at the endpoint are introduced to represent the local magnetic field.

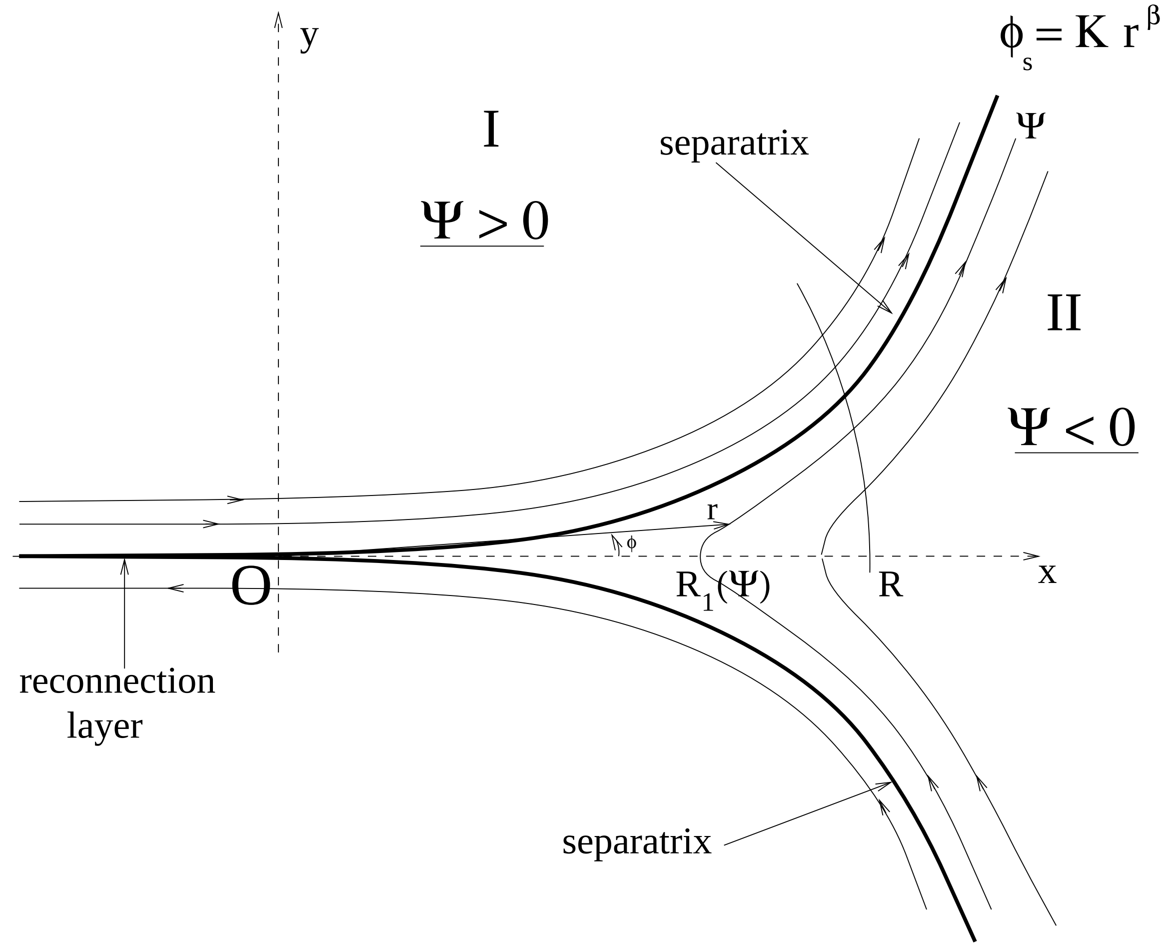

FIG. 3. The neighborhood of the endpoint when there is a surface current in the separatrix. The magnetic field structure is characterized by the *cusp* geometry, with the shape of the separatrix $\Psi = 0$ described in the polar coordinates by $\phi_s(r) = Kr^{\beta}$. Region I is the upstream region, and region II is the downstream region. The point where the field line Ψ crosses the midplane $y = 0$ is $R_1(\Psi)$. The circle of radius R is the dividing line between the two contributions to the volume per flux $V(\Psi) = V_<(R, \Psi) + V_>(R, \Psi)$.