# A Direct Calculation of Current Drive in Toroidal Geometry

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The magnitude and radial profiles of noninductive currents driven by fast magnetosonic waves in tokamaks have been calculated directly from the wave-induced quasilinear flux in a toroidal geometry and a Green's function for the current. An expression for the quasilinear flux has been derived which accounts for coupling between modes in the spectrum of waves launched from the antenna. A Fokker–Planck code for the Green's function and a full wave code for the electric field in the quasilinear flux are used to evaluate the current in a specified toroidal geometry.

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Radio frequency (RF) generated currents may play a critically important role in achieving steady state operation of devices and for current profile control in magnetically confined toroidal plasma such as those found in tokamaks. The RF driven currents in tokamaks are presently modeled with one and two dimensional codes [1] which use Ehst–Karney's parameterization [2] of current drive efficiencies – defined as the ratio of the current, $J$, to the power, $P$, – to calculate fast wave [3] driven currents. This parameterization assumes a model quasilinear diffusion tensor ($\mathbb{D}_{\mathrm{ql}}$) and requires *a priori* knowledge of the wave polarizations. These approximations may be avoided by directly calculating the quasilinear diffusion tensor from the Kennel–Englemann form [4] using electric field polarizations calculated by a full wave code, thereby eliminating the need to use the approximation inherent in the parameterization. Current profiles are then calculated using the adjoint formulation [5]. This approach has been implemented in the FISIC code [6], and the accuracy of the parameterization of the current drive efficiency, $\eta = J/P$, judged by a comparison of the current from $J = \eta P$ with the direct calculation. Results indicate that in the approximation where trapped electron effects may be ignored, the Ehst–Karney parameterization is in excellent agreement with the direct calculation.

In the Ehst–Karney model, the efficiency, $\eta$, was parameterized by combining a ray–tracing solution for the wave fields with the quasilinear flux, $\mathbf{\Gamma}_{\mathrm{ql}}$, derived from a homogenous model. Since the rays are assumed to be uncorrelated, it is unclear how one should employ the efficiency in calculating the current from a spectrum of waves [7]. As we shall show, the current produced by each mode in the spectrum is best given by the product of the efficiency for that mode times the cross spectrum power (the product of one mode with the entire spectrum.) We say "best" because the efficiency, being the ratio of two expressions which are quadratic in the field amplitudes, is not a linear operator. The effect of this approximation will be negligible if the wave spectrum is sufficiently narrow that the efficiency is constant with respect to the phase velocity of the modes. Another consequence of the ray–tracing picture is that one cannot account for phase correlations between multiple wave–particle interactions. We do not address this issue in this paper, though, our model may be readily generalized to include such effects. The central results of this paper are a validation in high aspect ratio ($A \equiv R/r \gtrsim 3$) , where $R$ and $r$ are the major and minor radii of the toroidal surface) regimes of the efficiency when applied properly to a spectrum of waves, and a direct calculation of the current using a quasilinear flux model for a spectrum of parallel wavenumbers, $k_{\|} \equiv \mathbf{k} \cdot \mathbf{B_0}/B_0$, derived from first principles ($B_0$ is the equilibrium magnetic field.)

If the wave–induced electric fields in the plasma are known, one can in principle determine the steady state current from the balance of the quasilinear diffusion, $\mathbb{D}_{\mathrm{ql}}$, against the collisional drag, $C_c(f)$:

$$C_c(f) = -C_w(f) \equiv \frac{\partial}{\partial \mathbf{v}} \cdot \mathbb{D}_{\mathrm{ql}} \cdot \frac{\partial f}{\partial \mathbf{v}} \equiv -\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{\Gamma}_{\mathrm{ql}} \qquad (1)$$

Rather than solve Eq. (1) for the perturbed distribution for each $\mathbf{\Gamma}$ (we will drop the $ql$ subscript at this point), an equivalent solution may be found through the use of the adjoint formulation [8]:

$$J_{\|} = \int d^3v \; \mathbf{\Gamma} \cdot \frac{\partial \chi}{\partial \mathbf{v}}, \quad C_c(f_{em}\chi) = -qv_{\|} f_{em}, \qquad (2)$$
$$P = \int d^3v \; \mathbf{\Gamma} \cdot \frac{\partial \varepsilon}{\partial \mathbf{v}}, \quad \epsilon = \frac{1}{2} m_e v^2,$$

where the adjoint function, $\chi$, is the solution to the Spitzer–Härm problem [9]. $\chi$ measures the incremental current deposited in the plasma by an impulse in velocity space. It serves as a Green's function [5] for the steady state Fokker–Planck equation, Eq. (1), which determines the equilibrium current. Eqs. (2) give the wave induced current, $J_{\|}$, and power, $P$, as moments of the quasilinear flux. The problem is therefore reduced to a calculation of $\mathbf{\Gamma}$ from the wave fields.

Fast wave current drive efficiency is maximized under conditions where there are no ion resonance layers in the plasma, so all of the wave's power is ultimately deposited in the electrons. Therefore, we limit our scope to the calculation of the quasilinear response of the electrons to the wave fields. If $f$ is the equilibrium distribution, and $\tilde{f}$ the small perturbation of the distribution by the wave fields, the quasilinear flux is determined from the Vlasov–Maxwell system of equations in the form:

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\frac{q}{m}\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial\tilde{f}}{\partial\mathbf{v}} \\ &= \frac{\partial}{\partial\mathbf{v}}\cdot\mathbb{D}_{\mathrm{ql}}\cdot\frac{\partial f}{\partial\mathbf{v}} = -\mathbf{\Gamma}\cdot\frac{\partial f}{\partial\mathbf{v}} \\ \text{and}\quad \tilde{f} &= -\frac{q}{m}\int_0^\infty d\tau\, e^{i\omega\tau}\left(\mathbf{E}(\mathbf{r}')+\frac{\mathbf{v}'\times\mathbf{B}(\mathbf{r}')}{c}\right)\cdot\frac{\partial f}{\partial\mathbf{v}'}\end{aligned}$$

where $\tau$ measures the history of the particle's motion up till the present, and primed quantities are evaluated along the past orbit [10].

The expression for $\mathbf{\Gamma}$ is made explicit using a spectral representation in which the wave fields are expanded in toroidal and poloidal Fourier modes:

$$\mathbf{E} = \sum_{m,n}\mathbf{E}_{m,n}(\psi)e^{i(m\theta+n\zeta)}e^{-i\omega t} \tag{3}$$

where $\psi$ is a flux surface quantity, $\zeta$ is the toroidal angle, $\theta$ is the poloidal angle measured from the center of the flux surface, and $\omega=\omega_r+i\nu$. After averaging over the toroidal angle of symmetry, $\zeta$, and over the wave period, the RF diffusion is:

$$\begin{aligned}\langle\mathbf{\Gamma}\rangle_{\omega,\zeta} = e^{2\nu t}\left(\frac{-e^2}{2m}\right)\Re\Bigg\{&\sum_{m_1 m_2}\sum_n e^{i(m_2-m_1)\theta} \\ &\times\int_0^\infty d\tau e^{i\omega\tau}e^{im_2(\theta'-\theta)+in(\zeta'-\zeta)} \\ &\left[\mathbf{E}^*_{m_1 n}(\psi)+\frac{\mathbf{v}}{c}\times\mathbf{B}^*_{m_1 n}(\psi)\right] \\ &\left(\mathbf{v}'\frac{\partial f}{\partial\varepsilon}\right)\cdot\left[\mathbf{E}_{m_2 n}(\psi')+\frac{\mathbf{v}'}{c}\times\mathbf{B}_{m_2 n}(\psi')\right]\Bigg\}\end{aligned} \tag{4}$$

The complex exponential terms act as a resonance term that contributes strongest when the variation of the sum of their arguments is near zero. This resonance term depends on only one of the poloidal indices. Both poloidal mode numbers remain because of the inhomogeneity in $\theta$ introduced by the equilibrium magnetic field. The poloidal modes are coupled together by the inhomogeneity of $B_0$ in $\theta$ and prevent a spatial averaging as was done with the toroidal angle. In fact, an average over $\theta$ would lead to performing a bounce–averaged calculation that accounts for the trapping of electrons in the magnetic well of the poloidal field.

Since the fields have been Fourier transformed in directions orthogonal to $\nabla\psi$, the wave–vector components parallel to the flux surface are representable as algebraic quantities. In order to simplify the phase integral we introduce the equations for electron motion on the flux surface, Eqs. (5). We neglect motion normal to the flux surface which is due to gradient induced drifts of order $\rho/L$, where $\rho = v_\perp/\Omega$ is the Larmor radius, $\Omega$ is the gyrofrequency, and $L$ is the equilibrium gradient scale length. Hence, we use:

$$\begin{aligned}\theta'-\theta &= -\int_0^\tau dt'\,\frac{1}{r'}\mathbf{v}'\cdot\hat{\boldsymbol{\theta}} \\ \zeta'-\zeta &= -\int_0^\tau dt'\,\frac{1}{R'}\mathbf{v}'\cdot\hat{\boldsymbol{\zeta}} \\ \mathbf{v}' &= v'_\parallel\hat{\mathbf{b}}+v'_\perp\left(\hat{\boldsymbol{\psi}}\cos\phi'+\hat{\boldsymbol{\eta}}\sin\phi'\right)\end{aligned} \tag{5}$$

yielding:

$$\begin{aligned}\Psi(\tau) &\equiv \omega\tau+m_2(\theta'-\theta)+n(\zeta'-\zeta) \\ &= \int_0^\tau d\tau'\,\left\{\omega-{k'}^{m_2}_\parallel v'_\parallel-{k'}^{m_2}_\eta v'_\perp\sin(\phi+h(\tau'))\right\}\end{aligned} \tag{6}$$

where we have grouped terms explicitly in $k_\parallel$ and $k_\eta$ such that

$$\begin{aligned}k^{m,n}_\parallel g(\psi)e^{i(m\theta+n\zeta)} &\equiv \left(\hat{\mathbf{b}}\cdot\nabla\right)g(\psi)e^{i(m\theta+n\zeta)} \\ &= \left(\frac{B_\theta}{B}\frac{m}{r}+\frac{B_\zeta}{B}\frac{n}{R}\right)g(\psi)e^{i(m\theta+n\zeta)}, \\ k^{m,n}_\eta g(\psi)e^{i(m\theta+n\zeta)} &\equiv \left(\hat{\boldsymbol{\eta}}\cdot\nabla\right)g(\psi)e^{i(m\theta+n\zeta)} \\ &= \left(-\frac{B_\theta}{B}\frac{n}{R}+\frac{B_\zeta}{B}\frac{m}{r}\right)g(\psi)e^{i(m\theta+n\zeta)}, \\ \hat{\boldsymbol{\eta}} &= \hat{\mathbf{b}}\times\hat{\boldsymbol{\psi}}, \\ h(\tau) &\equiv \phi'-\phi = \int_0^\tau \Omega(t)\,dt,\end{aligned}$$

and where $\phi$ is the gyroangle and $(\hat{\boldsymbol{\psi}},\hat{\boldsymbol{\eta}},\hat{\mathbf{b}})$ are the local Stix coordinates [11]. The direction gradients serve to highlight the fact the the wavenumbers are related to the scale lengths of variation of the fluctuating quantities. The $k_\eta v_\perp$ term in Eq. (6) is small for fast waves and has an average contribution of zero over the wave period except when the frequency is near a cyclotron harmonic. The drifts of the electron's gyrocenters are assumed to be negligible, and so the particle trajectories lie on a flux surface. At this point, we will drop the toroidal harmonic index, $n$, because toroidal symmetry makes the response of the system independent for each $n$, and expand in a power series about the guiding center, to find:

$$\begin{aligned}\mathbf{\Gamma} = e^{2\nu t}\left(\frac{-e^2}{2m}\right)\Re\Bigg\{&\sum_{m_1 m_2} e^{i(m_2-m_1)\theta} \\ &\left(1+\frac{\mathbf{v}_\perp\times\hat{\mathbf{b}}}{\Omega}\cdot\nabla\right)\left[\mathbf{E}^*_{m_1}(\psi)+\frac{\mathbf{v}}{c}\times\mathbf{B}^*_{m_1}(\psi)\right] \\ &\int_0^\infty d\tau e^{i\Psi(\tau)}\left(1+\frac{{\mathbf{v}_\perp}'\times\hat{\mathbf{b}}'}{\Omega'}\cdot\nabla\right)\mathbf{E}_{m_2}(\psi)\cdot\mathbf{v}'\Bigg\}\frac{\partial f}{\partial\varepsilon}\end{aligned} \tag{7}$$

After averaging over the gyroangle in velocity space and dropping harmonic terms:

$$
\begin{aligned}
\mathbf{\Gamma} = e^{2\nu t}\left(\frac{-e^2}{2m}\right)\Re\Bigg\{&\sum_{m_1 m_2} e^{i(m_2-m_1)\theta} \\
&\left[E_{\|}^{*m_1}(\psi)+\frac{v_\perp^2}{2\Omega}(\nabla_\perp\times\mathbf{E})_{\|}^{*m_1}(\psi)\right]\hat{\mathbf{b}} \\
&\int_0^\infty d\tau e^{i\Psi(\tau)}\left[E_{\|}^{m_2}(\psi)+\frac{v_\perp'^2}{2\Omega'}(\nabla_\perp\times\mathbf{E})_{\|}^{m_2}(\psi)\right]v_{\|}'\frac{\partial f}{\partial\varepsilon}\Bigg\}
\end{aligned} \tag{8}
$$

The method of stationary phase in the limit, $\nu \to 0$, leads to the usual $\delta$–function resonance behavior.

$$
\begin{aligned}
\mathbf{\Gamma} &= \sum_{m_1,m_2}\frac{\pi e^2}{m_e^2}a_{m_1}^* a_{m_2}\mathbf{v}_{\|}\frac{f}{T_e/m_e}\delta(\omega-k_{\|}^{m_2}v_{\|}) \\
&\equiv \sum_{m_1,m_2}\Gamma_{m_1,m_2}\delta(\omega-k_{\|}^{m_2}v_{\|}) \\
a_m &= \frac{k_{\|}^m v_{\|}}{\omega}E_{\|}^m+\frac{k_{\|}^m v_\perp^2}{2\omega\Omega}(\nabla_\perp\times\mathbf{E})_{\|}^m
\end{aligned} \tag{9}
$$

Note that if $\nabla_\perp \to k_\perp$, Eq. (9) is identical to the Kennel–Englemann form [4] in which only the $n = 0$ terms are retained and the Bessel functions are expanded to lowest order in $k_\perp\rho$ (here $n$ is the cyclotron harmonic.) Expanding Eq. (9) out, leads to our final expression for $\mathbf{\Gamma}$:

$$
\begin{aligned}
\mathbf{\Gamma} = \sum_{m_1,m_2}\frac{\pi e^2}{m_e^2}\Bigg[&\frac{k_{\|}^{m_1}k_{\|}^{m_2}v_\perp^4}{(2\Omega\omega)^2}(\nabla_\perp\times\mathbf{E})_{\|}^{*m_1}(\nabla_\perp\times\mathbf{E})_{\|}^{m_2} \\
&-\Re(i(\nabla_\perp\times\mathbf{E})_{\|}^{*m_1}E_{\|}^{m_2})\frac{k_{\|}^{m_1}k_{\|}^{m_2}v_{\|}v_\perp^2}{\Omega\omega^2} \\
&+\frac{k_{\|}^{m_1}v_{\|}}{\omega}\frac{k_{\|}^{m_2}v_{\|}}{\omega}E_{\|}^{*m_1}E_{\|}^{m_2}\Bigg]\mathbf{v}_{\|}\frac{f}{T_e/m_e}\delta(\omega-k_{\|}^{m_2}v_{\|})
\end{aligned} \tag{10}
$$

In this form, the first term contains the magnetic pumping contribution whose strength is related to the parallel magnetic field, the third term is the usual Landau damping term, and the second is the cross term. Note also that for these Landau resonant terms, the flux is purely in the parallel direction in velocity space. In Fig. 1, Eq. (10) is plotted for several poloidal locations on a given flux surface. The profile shows a null at $v_\perp = \sqrt{2}v_{te}$, where $v_{te} \equiv \sqrt{T_e/m_e}$ is the electron thermal velocity. At this point there is a balance between magnetic pumping and Landau damping, since these two are out of phase for electrons. The general shape of the curves is $||\mathbb{D}_{\rm ql}|| \approx (2-x^2)^2$, where $x \equiv v_\perp/v_{te}$. This is the result expected in a homogenous plasma for Alfvén wave polarization and is the assumed analytic form for the quasilinear diffusion used in the Ehst–Karney parameterization.

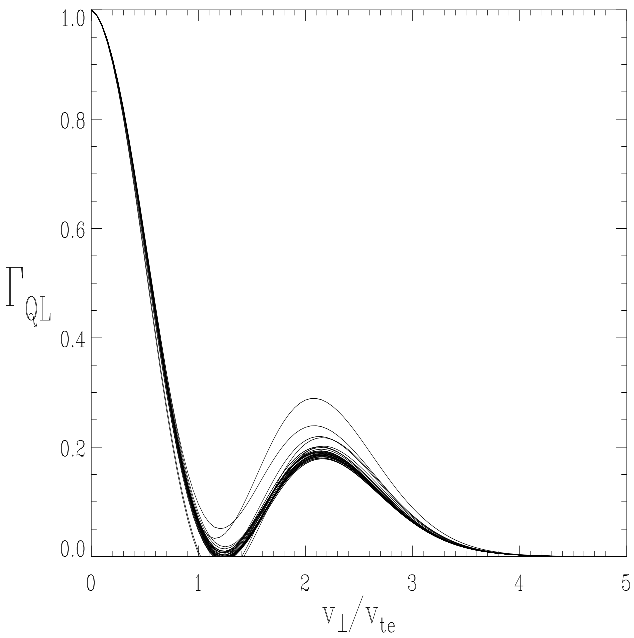


FIG. 1. $\mathbf{\Gamma}_{\rm ql}$ plotted for various $\theta$ on a given flux surface.

We proceed to calculate the power and current deposited in the plasma via Eqs. (2). In order to make meaningful comparisons between the parameterized model, $J_{\|} = P\eta$, and our direct calculation, $J_{\|} = \int d^3v\ \mathbf{\Gamma}\cdot\partial\chi/\partial\mathbf{v}$, we check the accuracy of our calculation of $\mathbf{\Gamma}$. This is done by comparing the power calculated from the plasma conductivity, $P = \mathbf{E}\cdot\overleftrightarrow{\sigma}_A\cdot\mathbf{E}^*$, with the power calculated from the quasilinear flux, $P = \int d^3v\ \mathbf{\Gamma}\cdot\partial\epsilon/\partial\mathbf{v}$, which are analytically equivalent [12]. Excellent agreement is found, as displayed in Fig. 2.

With the accuracy of $\mathbf{\Gamma}$ verified, the current flowing on a flux surface is now calculated from both the Ehst–Karney parameterization and the adjoint moment of the quasilinear flux for a high aspect ratio tokamak. The parameterized efficiency is available only for a single mode. It is not be correct to calculate the current from the power for each mode multiplied by the efficiency for each mode, as has been done by others [7]. Instead of a simple product, the correct procedure is a convolution of efficiency with the power over the modal spectrum. This is motivated by Eqs. (2) and (10) which may be combined and written in a more suggestive form that emphasizes the convolution over $m$'s for the current:

$$
\begin{aligned}
J_{\|} &= \int_0^\infty v_\perp\, dv_\perp \sum_{m_1,m_2}\left(\Gamma_{m_1,m_2}v_{\|}\right)\frac{1}{v_{\|}}\frac{\partial\chi}{\partial v_{\|}}\bigg|_{v_{\|}=\omega/k_{\|}^{m_2}} \\
&\equiv \int_0^\infty v_\perp\, dv_\perp \sum_{m_1,m_2}P_{m_1,m_2}\eta_{m_2}(v_\perp) \\
&\equiv \sum_{m_2}\int_0^\infty v_\perp\, dv_\perp P_{m_2}\eta_{m_2}
\end{aligned} \tag{11}
$$

Thus, the current is calculated from the parameterization for a $k_{\|}$ spectrum using the paradigm: $J_{\|} = \sum_{m_2}P_{m_2}\eta(k_{\|}^{m_2})$ where $P_{m_2} = \sum_{m_1}P_{m_1,m_2}$. Figure 3 shows that the two current profiles are in excellent agreement. Given the results shown in Fig. 1, this may have been expected.

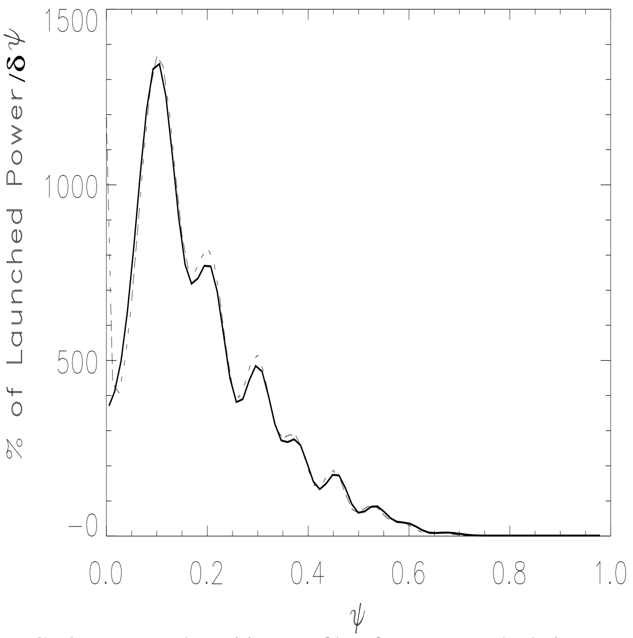


FIG. 2. Power deposition profiles from two calculations are equivalent. The dashed curve is a plot of power from the plasma conductivity, $P = \mathbf{E} \cdot \overleftrightarrow{\sigma}_A \cdot \mathbf{E}^*$ and the solid curve is from the quasi-linear flux, $P = \int d^3v\ \mathbf{\Gamma} \cdot \partial\epsilon/\partial\mathbf{v}$.

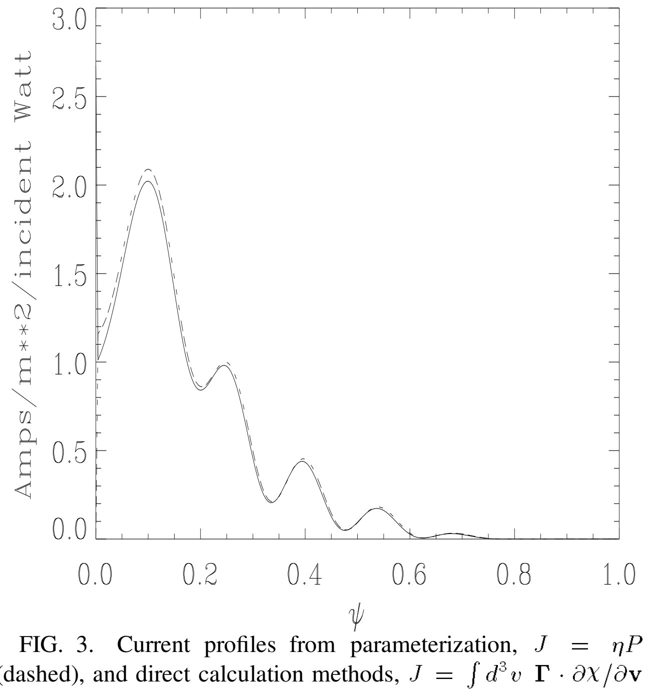


FIG. 3. Current profiles from parameterization, $J = \eta P$ (dashed), and direct calculation methods, $J = \int d^3v\ \mathbf{\Gamma} \cdot \partial\chi/\partial\mathbf{v}$ (solid).

These calculations were performed in a high aspect ratio geometry where the local approximation is good. As indicated above, others have chosen to ignore the cross–spectrum correlations and set $m_1 = m_2$ in Eq. (11) [7]. Numerical experiments show that in the high aspect ratio limit, neglecting these correlations can introduce deviations of $\sim 10\%$ from the direct calculation, Eq. (2). It is anticipated that the increased variation of $k_{\|}$ with the poloidal field at lower aspect ratios will enhance the deviations of the direct model from the parameterization because of increased correlations between the spectral modes. At lower aspect ratios, the trapped particle fraction, which cannot contribute to the net parallel current, becomes significant, causing the parameterization fit to break down [2]. Trapped particle effects are currently being included by generalizing to a bounce–averaged version of the method outlined in this paper and low aspect ratio equilibrium are being considered. A discussion of these effects will be reported in detail elsewhere.


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